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Numerical solution of nonlinear fractional Volterra-Fredholm integro-differential equations with mixed boundary conditions

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Abstract

The aim of this paper is solving nonlinear Volterra-Fredholm fractional integro-differential equations with mixed boundary conditions. The basic idea is to convert fractional integro-differential equation to a type of second kind Fredholm integral equation. Then the obtained Fredholm integral equation will be solved with Nyström and Newton-Kantorovitch method. Numerical tests for demonstrating the accuracy of the method is included.

Keywords : Fractional integro-differential equations; Boundary mixed Conditions; Nyström method; Newton-Kantorovitch method.

1 Introduction

It is rare to find methods for solving the nonlinear fractional Volterra-Fredholm integro-differential equations with mixed boundary conditions. Motivated by this reason, the purpose of this paper is to discuss the following fractional Volterra-Fredholm integro-differential equation

$$(D^\alpha y)(x) = g(x) + \int_a^x K_1(x,t)H_1(y(t))dt + \int_a^b K_2(x,t)H_2(y(t))dt, \quad (1.1)$$

subject to the mixed boundary conditions

$$\sum_{j=1}^{m} \left[\gamma_{ij} y^{(j-1)}(a) + \eta_{ij} y^{(j-1)}(b)\right] = r_i, \quad i = 1, 2, ..., m \quad (1.2)$$

where $y(x)$ is determined function, The derivative D^α is understood here in the Caputo sense, and $H_1(.)$ and $H_2(.)$ are the continuous nonlinear term.

For solving fractional differential and fractional integro-differential equations different numerical techniques have been proposed. For example, in [1] fractional differential transform method is extended to solve linear and nonlinear fractional integro-differential equations of Volterra type with initial and boundary conditions. Zhu, considered solving nonlinear fractional Fredholm integro-differential equations and nonlinear fractional Volterra integro-differential equations in separated work with initial conditions by using second kind Chebyshev wavelet [12, 13]. In [5] a Chebyshev cardinal operational matrix method described for solving nonlinear Volterra and Fredholm integro-differential equations of fractional order. In [3], the authors used the collocation method to approximate solution of an integro-differential equation by converting it to the corresponding nonlinear system of equations.

For considering existence and uniqueness of the

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solutions of fractional Volterra-Fredholm integro-differential equations we refer the reader to see [9, 14], also existence and uniqueness results for fractional integro-differential equations with boundary conditions considered in [6]. The rest of this paper is organized as follows: In Section **??**, we introduce preliminaries which are used throughout the paper. In Section 3, numerical approach for the fractional integral will be derived. Application of Nyström method is recalled in Section 4. In Section 5, some numerical results are provided to clarify the method. At last, conclusion is given.

2 Preliminaries

Let us recall and prove the following lemmas along with some definitions. For more details see [7].

Definition 2.1 *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function y, is defined as*

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}y(x)dx, \qquad (2.3)$$

where Γ is Gamma function.

Definition 2.2 *The Caputo derivative of fractional order $\alpha > 0$ for a function $y(t)$ is defined by*

$$({}^{C}D^{\alpha}y)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{y^{(n)}(x)}{(t-x)^{\alpha-n+1}}dx,$$
$$n-1 < \alpha \leq n \text{ and } n = [\alpha]+1,$$

where $[\alpha]$ denotes integral part of the real number α.

Lemma 2.1 *Let $\alpha > 0$ and $n = [\alpha]+1$. Then*

$$I^{\alpha}({}^{C}D^{\alpha}y(t)) = y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k.$$

Lemma 2.2 *Problem (1.1)-(1.2) with $m = 2$, $a = 0$ and $b = 1$ is equivalent to the nonlinear Fredholm integral equation*

$$\begin{aligned} y(x) =& f(x) + \int_0^1 N_1(\tau,x)H_1(y(\tau))d\tau \\ &+ \int_0^1 N_2(\tau,x)H_2(y(\tau))d\tau, \end{aligned} \qquad (2.4)$$

where

$$\begin{aligned} f(x) =& A_1 + A_2 x \\ &+ (B_1 + B_2 x)\int_0^1 (1-t)^{\alpha-1}g(t)\,dt \\ &+ (C_1 + C_2 x)\int_0^1 (1-t)^{\alpha-2}g(t)\,dt \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}g(t)\,dt, \\ N_1(\tau,x) =& [D_1L_1(\tau,1) + E_1L_{1_x}(\tau,1)] \\ &+ x[D_2L_1(\tau,1) + E_2L_{1_x}(\tau,1)] \\ &+ H(\tau - x)L_1(\tau,x), \\ N_2(\tau,x) =& [F_1L_1(\tau,1) + G_1L_{1_x}(\tau,1)] \\ &+ x[F_2L_1(\tau,1) + G_2L_{1_x}(\tau,1)] \\ &+ L_2(\tau,x), \\ L_1(\tau,x) =& \frac{1}{\Gamma(\alpha)}\int_\tau^x (x-t)^{\alpha-1}K_1(t,\tau)dt, \\ L_2(\tau,x) =& \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}K_2(t,\tau)dt, \end{aligned}$$

provided

$$\begin{vmatrix} \gamma_{11}+\eta_{11} & \gamma_{12}+\eta_{11}+\eta_{12} \\ \gamma_{21}+\eta_{21} & \gamma_{22}+\eta_{21}+\eta_{22} \end{vmatrix} \neq 0,$$

here H is Heviside function.

Proof. Taking fractional integral of order α from both sides of equation (1.1) and using lemma 2.1, yields

$$\begin{aligned} y(x) =& y(0) + y'(0)x \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}g(t)dt \\ &+ \int_0^x \left[\frac{1}{\Gamma(\alpha)}\int_\tau^x (x-t)^{\alpha-1}K_1(t,\tau)dt\right] \\ &\times H_1(y(\tau))d\tau \\ &+ \int_0^1 \left[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}K_2(t,\tau)dt\right] \\ &\times H_2(y(\tau))d\tau \\ =& y(0) + y'(0)x \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}g(t)dt \\ &+ \int_0^x L_1(\tau,x)H_1(y(\tau))d\tau \\ &+ \int_0^1 L_2(\tau,x)H_2(y(\tau))d\tau \end{aligned} \qquad (2.5)$$

to obtain $y(0)$, $y'(0)$ and in the sequel $y(x)$, we substitute Eq. (2.5) into the boundary condition

(1.2), then by solving resulted system of equations, $y(0)$, $y'(0)$ will be obtained. At the end, by substitution $y(0)$, $y'(0)$ into Eq. (2.5) we will reach to the Fredholm integral equation (2.4).

Lemma 2.3 *Let* $s = \frac{x(t-\tau)}{x-\tau}$. *Then*

$$\begin{aligned} L_1(\tau,x) =& \frac{(x-\tau)^\alpha}{x^\alpha\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \\ &\times K\left(\frac{(x-\tau)s}{x}+\tau,\tau\right)ds \\ =& \frac{(x-\tau)^\alpha}{x^\alpha} I^\alpha K\left(\frac{(x-\tau)s}{x}+\tau,\tau\right). \end{aligned}$$

Proof. The proof is straightforward.

3 Numerical approach for the fractional integral

In this section we describe a numerical approach for the fractional integral (2.3) which will be used for computing $L_1(\tau,x)$, $L_2(\tau,x)$ and $f(x)$ in the next section. To this end, we interpolate the function $y(t)$ at the nodes t_{3i}, t_{3i+1}, t_{3i+2} and t_{3i+3} and integrate over $[t_{3i}, t_{3i+3}]$. Thus

$$\begin{aligned} I^\alpha_{0,t_n} y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^{t_n} (t-x)^{\alpha-1} y(x)\,dx \\ &\approx \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n-1}\int_{t_{3i}}^{t_{3i+3}} (t_n-x)^{\alpha-1} \\ &\quad\times\left(y_{3i}+s\Delta y_{3i}+\frac{s(s-1)}{2}\Delta^2 y_{3i}\right. \\ &\quad\left.+\frac{s(s-1)(s-2)}{6}\Delta^3 y_{3i}\right)dx, \\ =& \frac{h}{\Gamma(\alpha)}\sum_{i=0}^{n-1}\int_0^3 (3n-s-3i)^{\alpha-1}\Big(y_{3i}+s\Delta y_{3i} \\ &+\frac{s(s-1)}{2}\Delta^2 y_{3i}+\frac{s(s-1)(s-2)}{6}\Delta^3 y_{3i}\Big)ds \\ =& \frac{h^\alpha}{\Gamma(\alpha+4)}\sum_{i=0}^{n-1}\Big(W_{0i}y(t_{3i})+W_{1i}y(t_{3i+1}) \\ &+W_{2i}y(t_{3i+2})+W_{3i}y(t_{3i+3})\Big), \end{aligned}$$

where $s=(x-t_{3i})/h$, $\quad h=t_n/3n$,

$$\begin{aligned} W_{0i} =&\Big\{Z(i)[F-G_1(i)+R_1(i)-Q_1(i)] \\ &-Z(i+1)[F-G_1(i+1) \\ &+R_2(i)-Q_2(i)]\Big\} \end{aligned} \tag{3.6}$$

$$\begin{aligned} W_{1i} =&\Big\{Z(i)[G_1(i)-2R_1(i)+3Q_1(i)] \\ &-Z(i+1)[G_1(i+1) \\ &-2R_2(i)+3Q_2(i)]\Big\} \end{aligned} \tag{3.7}$$

$$\begin{aligned} W_{2i} =&\Big\{Z(i)[R_1(i)-3Q_1(i)] \\ &-Z(i+1)[R_2(i)-3Q_2(i)]\Big\} \end{aligned} \tag{3.8}$$

$$W_{3i} =\Big\{Z(i)[Q_1(i)]-Z(i+1)[Q_2(i)]\Big\}, \tag{3.9}$$

$$\begin{aligned} F =&(\alpha+1)(\alpha+2)(\alpha+3) \\ G_1(i) =&(\alpha+2)(\alpha+3)(n-3i), \\ Z(i) =&(n-3i)^\alpha \\ R_1(i) =&(\alpha+3)(18n^2+(-3\alpha-6)n \\ &+3\alpha i-30i+18i^2) \\ R_2(i) =&(\alpha+3)(18n^2+(15\alpha-36i-6)n \\ &+6\alpha^2-15\alpha i+3\alpha+18i^2+6i) \\ Q_1(i) =&162n^3+(-54\alpha-486i-54)n^2 \\ &+(30\alpha+6\alpha^2+486i^2 \\ &+i(108\alpha+324)+36)n \\ &-162i^3+(-54\alpha-162)i^2 \\ &+(-6\alpha^2-30\alpha-36)i \\ Q_2(i) =&162n^3+(108\alpha-486i-162)n^2 \\ &+(33\alpha^2-216\alpha i-51\alpha+486i^2 \\ &+324i+36)n+6\alpha^3-33\alpha^2 i \\ &+3\alpha^2+108\alpha i^2+51\alpha i+9\alpha \\ &-162i^3-162i^2-36i. \end{aligned}$$

Let $y(t)\in C^4[0,b]$. Then

$$\begin{aligned} \Big| I^\alpha_{0,t_n}y(t) &- \frac{h^\alpha}{\Gamma(\alpha+4)}\sum_{i=0}^{n-1}\Big(W_{0i}y(t_{3i}) \\ &+W_{1i}y(t_{3i+1})+W_{2i}y(t_{3i+2})+W_{3i}y(t_{3i+3})\Big)\Big| \\ &= O(h^4) \end{aligned} \tag{3.10}$$

where $W_{j,i}$, $j = 0, 1, 2, 3$ are defined by (3.6)-(3.9). **Proof.** By using interpolation error formula, we have

$$E(t_s) = \frac{s(s-1)(s-2)(s-3)}{4!} h^4 y^{(4)}(c(t_s)),$$
$$s = \frac{t - t_{3i}}{h}, \quad t_s \in (t_{3i}, t_{3i+3}).$$

Thus

$$\begin{aligned} |I^{\alpha}_{0,t_n} y_n(t) - I^{\alpha}_{0,t_n} y(t)| &= |I^{\alpha}_{0,t_n}(y_n(t) - y(t))| \\ &\leq |y_n(t) - y(t)| \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - \tau)^{\alpha-1} d\tau \\ &\leq \frac{b^{\alpha}}{\Gamma(\alpha+1)} |y_n(t) - y(t)| = O(h^4). \end{aligned}$$

For numerical verification of theorem 3, see table 1.

4 Numerical method

The philosophy of Nyström method is to approximate the integral operator by an operator that is derived by numerical integration formula (see [4] for a thorough knowledge of the subject). So, application of this method to the nonlinear operator of Eq. (2.4) leads to a nonlinear system of n equations for the n unknowns functions values $y_n(t_{n,j})$. To solve it, we will use the Newton-Kantorovitch method which is undoubtedly the most popular method for solving nonlinear equations [8]. Now, the Nyström method for solving Eq. (2.4) leads to finding y_n such that

$$\begin{aligned} y_n(x) = &f(x) \\ &+ \sum_{j=1}^{n} \omega_{n,j} \Big(N_1(t_{n,j}, x) H_1(y_n(t_{n,j})) \\ &+ N_2(t_{n,j}, x) H_2(y_n(t_{n,j})) \Big), \end{aligned} \tag{4.11}$$

where $\omega_{n,j}$ are weights of numerical quadrature. Writing Eq. (4.11) at the nodes $t_{n,i}$, we obtain a nonlinear system of equations in $\mathbb{R}^{n+1}$ defined by

$$\begin{aligned} y_n(t_{n,i}) = &f(t_{n,i}) + \sum_{j=1}^{n} \omega_{n,j} \Big(N_1(t_{n,j}, t_{n,i}) \\ &\times H_1(y_n(t_{n,j})) + N_2(t_{n,j}, t_{n,i}) \\ &H_2(y_n(t_{n,j})) \Big), \ 1 \leq i \leq n. \end{aligned} \tag{4.12}$$

Eq. (4.12) can be solved by Newton-Kantorovitch method. Let $y_n^{(k)}$ be the iterate number k. The iterate $y_n^{(k+1)}$ solves $\left(I_n - C_n^{(k)}\right) y_n^{(k+1)} = d_n^{(k)}$, where

$$\begin{aligned} C_n^{(k)}(i,j) = &\omega_{n,j} \Big(N_1(t_{n,j}, t_{n,i}) \\ &\times \frac{\partial}{\partial y} H_1(y_n^{(k)}(t_{n,j})) \\ &+ N_2(t_{n,j}, t_{n,i}) \frac{\partial}{\partial y} H_2(y_n^{(k)}(t_{n,j})) \Big), \end{aligned}$$

$$\begin{aligned} d_n^{(k)} &:= f(t_{n,i}) + \sum_{j=1}^{n} \omega_{n,j} N_1(t_{n,j}, t_{n,i}) \\ &\times \Big(H_1(y_n^{(k)}(t_{n,j})) - \frac{\partial}{\partial y} H_1(y_n^{(k)}(t_{n,j})) y_n^{(k)}(j) \Big) \\ &+ \sum_{j=1}^{n} \omega_{n,j} N_2(t_{n,j}, t_{n,i}) \Big(H_2(y_n^{(k)}(t_{n,j})) \\ &- \frac{\partial}{\partial y} H_2(y_n^{(k)}(t_{n,j})) y_n^{(k)}(j) \Big). \end{aligned}$$

We recover the approximation $y_n^{(k+1)}$ with the natural interpolation formula

$$\begin{aligned} y_n^{(k+1)}(x) = &f(x) + \sum_{j=1}^{n} \omega_{n,j} \Big(N_1(t_{n,j}, x) \\ &\times H_1(y_n^{(k+1)}(t_{n,j})) \\ &+ N_2(t_{n,j}, x) H_2(y_n^{(k+1)}(t_{n,j})) \Big). \end{aligned}$$

With this strategy, whatever we hope is the convergence of the iterates towards the solution y_n of the approximate Equation (2.4) .

Remark 4.1 *Convergence of proposed method have been considered by many authors, we refer the reader to see [2, 8, 10, 11].*

5 Numerical experiments

Example 5.1 *Consider the linear fractional Volterra integro-differential equation*

$$\begin{aligned} (D^{\sqrt{3}} y)(x) = &\frac{2}{\Gamma(3 - \sqrt{3})} x^{2-\sqrt{3}} + 2\sin x - 2x \\ &+ \int_0^x \cos(x-t) y(t) dt, \end{aligned} \tag{5.13}$$

Table 1: The approximated solutions with error analysis obtained from the steepest descent method forThe absolute error of $I^{\alpha}_{0,1}t^4$ for verifying Theorem 3.

N	$\alpha=.8$	$\alpha=1$	$\alpha=1.6$
2	2.58×10^{-4}	2.31×10^{-4}	1.58×10^{-4}
4	1.58×10^{-5}	1.44×10^{-5}	1.00×10^{-5}
8	9.84×10^{-7}	9.04×10^{-7}	6.30×10^{-7}
16	6.11×10^{-8}	5.65×10^{-8}	3.94×10^{-8}
32	3.80×10^{-9}	3.53×10^{-9}	2.46×10^{-9}

Table 2: Numerical results of Example 5.1.

x	$n=40$	$n=80$	$n=160$
.2	2.94×10^{-7}	4.62×10^{-8}	7.31×10^{-9}
.4	2.82×10^{-7}	4.42×10^{-8}	7.00×10^{-9}
.6	2.71×10^{-7}	4.25×10^{-8}	6.74×10^{-9}
.8	2.63×10^{-7}	4.14×10^{-8}	6.55×10^{-9}
1	2.60×10^{-7}	4.09×10^{-8}	6.49×10^{-9}

Table 3: Numerical results of Example 5.2.

x	$n=160$	$n=320$	$n=640$
.1	7.81×10^{-6}	1.92×10^{-6}	4.65×10^{-7}
.3	1.55×10^{-5}	3.84×10^{-6}	9.31×10^{-7}
.5	1.65×10^{-5}	4.10×10^{-6}	9.92×10^{-7}
.7	1.27×10^{-5}	3.15×10^{-6}	7.65×10^{-7}
.9	5.07×10^{-6}	1.25×10^{-6}	3.04×10^{-7}

Table 4: Numerical results of Example 5.3.

	ff = 7/6, k = 5	
x	$n=20$	$n=40$
.1	7.454×10^{-8}	5.110×10^{-10}
.3	8.099×10^{-7}	5.946×10^{-9}
.5	2.461×10^{-6}	1.911×10^{-8}
.7	5.123×10^{-6}	4.181×10^{-8}
.9	8.868×10^{-6}	7.566×10^{-8}
	ff = 19/6, k = 3	
x	$n=20$	$n=40$
.1	4.890×10^{-12}	3.061×10^{-13}
.3	4.756×10^{-10}	2.978×10^{-11}
.5	3.996×10^{-9}	2.502×10^{-10}
.7	1.623×10^{-8}	1.016×10^{-9}
.9	4.627×10^{-8}	2.897×10^{-9}

with the mixed boundary conditions

$$\begin{aligned} y(0)+y(1)-y'(0)-y'(1)&=-1,\\ 3y(0)+4y(1)+y'(0)-3y'(1)&=-2. \end{aligned} \tag{5.14}$$

The exact solution is $y(x)=x^2$. Table 2 shows the numerical results including absolute errors of the approximated solution by using composite trapezoidal rule.

Example 5.2 *Let us consider the nonlinear fractional Volterra-Fredholm integro-differential* *equation*

$$(D^{\frac{\sqrt{7}}{2}}y)(x) = g(x) + \int_0^x \frac{1+2t}{1+y(t)}\,dt + \int_0^1 (1+2t)e^{y(t)}\,dt, \qquad (5.15)$$

with the boundary conditions $y(0) = 0,\ y(1) = 2$. $g(x)$ *is chosen such that the exact solution to be* $y(x) = x^2 + x$. *Table 3 shows the numerical results including absolute errors of the approximated solution by using composite trapezoidal rule and number of iterations* $k = 3$.

Example 5.3 *Consider the nonlinear fractional Fredholm integro-differential equation*

$$(D^{\alpha}y)(x) = g(x) + \int_0^1 xe^t y^2(t)dt, \qquad (5.16)$$

$g(x)$ *is chosen such that the exact solution to be* $y(x) = x - x^3$. *Table 4 shows the numerical results including absolute errors of the approximated solution by using composite trapezoidal rule.*

6 Conclusion

In this study, the approximate solution of the equation (1.1) is obtained by converting it to the equivalent nonlinear Fredholm integral equation and then by numerical approaching the fractional integral the approximate solution of this equation was obtained by Nyström and Newton-Kantorovitch method. Finally we presented three test problems to show the efficiency of the method.

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