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Research Article

Fuzzy Lyapunov stability and exponential stability in control systems

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Abstract

Fuzzy control systems have had various applications in a wide range of science and engineering in recent years. Since an unstable control system is typically useless and potentially dangerous, stability is the most important requirement for any control system (including fuzzy control system). Conceptually, there are two types of stability for control systems: Lyapunov stability (a special case of which is exponential stability) and input-output stability. This paper develops fuzzy Lyapunov stability through investigating the concept of stability for finite-dimensional systems under uncertainty and provides some related theorems. Considering the capability of fuzzy differential systems for modeling uncertainty and processing vague or subjective information in mathematical models, exponential stability and Lyapunov stability of fuzzy differential systems are presented. Also, numerical examples are given to support the theoretical results.

Keywords : Systems theory; Stability; Fuzzy systems; Fuzzy stability.

1 Introduction

Recently, Allahviranloo and Salahshour have investigated the state-space representation of fuzzy linear continuous-time systems under generalized H-differentiability as an application of fuzzy Laplace transforms for the first time [3]. However, they have not considered the stability of such systems. Note that, this problem appears in system theory especially in control theory. The theory of fuzzy logic control originates from Zadeh's pioneering work on fuzzy sets [16]. In 1974, the fuzzy logic technique was first applied successfully to control applications by Mamdani [11]. Fuzzy controllers are rule based nonlinear controllers. Therefore, their main application should be the control of nonlinear systems. However, since linear systems are good approximations of nonlinear systems around the operating points, it is desirable to study fuzzy control of linear systems systematically. Stability is the most important factor of any control system. The aim of this study is to explore the fundamentals of fuzzy stability. Stable fuzzy control of linear systems has been studied by a number of researchers. Ray and Majumder applied the circle criteria to analyze linear systems with fuzzy controllers [14]. Their approach is limited to a special fuzzy controller, which is equivalent to a multi-level relay (it is known nowadays that fuzzy controllers are universal nonlinear controllers [15]). Langari and Tomizuka designed a class of stable fuzzy controllers for linear systems, and Chiu and Chand developed a fuzzy control for a flexible wing aircraft model and analyzed the stability [7, 10]. Allahviranloo et al. discussed input-output stabil-

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ity from a mathematical point of view [2]. Questions about stability arise in almost every control problem. As a consequence, stability is one of the most extensively studied subjects in systems theory, and a great diversity of stability notions has been introduced.

In this paper, the concept of stability for finite-dimensional systems under uncertainty is investigated and some related theorems are presented. We have worked on exponential stability and Lyapunov stability of fuzzy differential systems, because fuzzy differential systems are powerful tools for modeling uncertainty and processing vague or subjective information in mathematical models. Fuzzy differential systems have been applied to a wide variety of real problems, for instance, the golden mean [6], quantum optics and gravity [8], medicine [1] and engineering applications [9]. The fuzzy differential equation, in particular, is a very important topic from the theoretical point of view, e.g., in population models and some other models. This paper is organized as follows: In Section 2, some basic concepts are presented. In Section 3, fuzzy exponential stability is investigated. Fuzzy Lyapunov stability is detailed in Section 4, and some numerical examples are given in Section 5. The final section contains some concluding remarks.

2 Preliminaries

The basic definition of fuzzy numbers introduced in [17]. Let $\Re$ be the set of all real numbers. A fuzzy number is a mapping $u\colon \Re \to [0,1]$ with the following properties:

(i) u is upper semi-continuous.

(ii) u is fuzzy convex i.e. $u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}$ for all $x, y \in \Re, \lambda \in [0,1]$.

(iii) u is normal, i.e. $\exists x_0 \in \Re$ with $u(x_0) = 1$.

(iv) closure $supp\ u, cl(supp u)$, is compact where $supp\ u = \{x \in \Re : u(x) > 0\}$.

Let E be the set of all fuzzy numbers on $\Re$. The α-level set of a fuzzy number $u \in E, 0 \leq \alpha \leq 1$, denoted by $[u]_\alpha$, is defined as

$$[u]_\alpha = \begin{cases} \{x \in \Re : u(x) \geq \alpha\} & \text{if} \quad \alpha > 0 \\ cl(supp\ u) & \text{if} \quad \alpha = 0. \end{cases}$$

It is clear that the α-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$, where $\underline{u}(\alpha)$ denotes the left-hand endpoint of $[u]_\alpha$, and $\overline{u}(\alpha)$ denotes the right-hand endpoint of $[u]_\alpha$. Each $y \in \Re$ can be regarded as a fuzzy number $\tilde{y}$ defined by

$$\tilde{y} = \begin{cases} 1 & \text{if} \quad t = y, \\ 0 & \text{if} \quad t \neq y, \end{cases}$$

and $\tilde{0} = [0]_\alpha = [\underline{0}, \overline{0}]$

Definition 2.1 *[13] The Hausdorff distance between $u, v \in E$ is given by d: $E \times E \to \Re^+ \cup \{0\}$,*

$$d(u,v) = sup\ max\{|\underline{u}(\alpha) - \underline{v}(\alpha))|, |\overline{u}(\alpha) - \overline{v}(\alpha))|\}$$

where $u = (\underline{u}(\alpha), \overline{u}(\alpha))$ and $v = (\underline{v}(\alpha), \overline{v}(\alpha))$ are utilized in [4]. Then, it is easy to see that d is a metric in E and has the following properties:

(i) $d(u+w, v+w) = d(u,v), \forall u, v, w \in E$,

(ii) $d(ku, kv) = |k| d(u,v), \forall k \in \Re, u, v \in E$,

(iii) $d(u + v, w + e) \leq d(u,w) + d(v,e), \forall u, v, w, e \in E$,

(iv) *(d, E) is a complete metric space.*

Definition 2.2 *[13] Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.*

The sign " $\ominus$ " always stands for H-difference. We also mention that $x \ominus y \neq x + (-1)\, y$.

Definition 2.3 *[5] Let $f : (a,b) \longrightarrow E$ and $x_0 \in (a,b)$. The fuzzy function f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in E$ such that*

(i) *for all $h > 0$ sufficiently small, $\exists f(x_0+h) \ominus f(x_0), \exists f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric d)*

$$lim_{h \downarrow 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = lim_{h \downarrow 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0),$$

or

(ii) *for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h), \exists f(x_0-h) \ominus f(x_0)$ and the limits (in the metric d)*

$$lim_{h \downarrow 0} \frac{f(x_0) \ominus f(x_0+h)}{-h} = lim_{h \downarrow 0} \frac{f(x_0-h) \ominus f(x_0)}{-h} = f'(x_0),$$

or

(iii) *for all $h > 0$ sufficiently small, $\exists f(x_0+h) \ominus f(x_0), \exists f(x_0-h) \ominus f(x_0)$ and the limits (in the metric d)*

$$lim_{h\downarrow 0}\frac{f(x_0+h)\ominus f(x_0)}{h} = lim_{h\downarrow 0}\frac{f(x_0-h)\ominus f(x_0)}{-h} = f'(x_0),$$

or

(iv) *for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h), \exists f(x_0) \ominus f(x_0-h)$ and the limits (in the metric d)*

$$lim_{h\downarrow 0}\frac{f(x_0)\ominus f(x_0+h)}{-h} = lim_{h\downarrow 0}\frac{f(x_0)\ominus f(x_0-h)}{h} = f'(x_0).$$

The Jordan canonical form of a matrix gives some insight into the form of the solution of a linear system of differential equations, and it is used to provide some theorems in the paper.

Theorem 2.1 *(The Jordan Normal Form) [12] Let A be a real matrix with real eigenvalues $\lambda_j, j=1,\dots,k$ and complex eigenvalues $\lambda_j = a_j + ib_j, j = k+1,\dots,n$. Then, there exists a basis $\{v_1,\dots,v_k,v_{k+1},u_{k+1},\dots,v_n,u_n\}$ for $\Re^{2n-k}$, where $v_j, j = 1,\dots,k$ and $w_j, j = k+1,\dots,n$ are generalized eigenvectors of $A, u_j = Re(w_j)$ and $v_j = Im(w_j)$ for $j = k+1,\dots,n$, such that the matrix $V = [v_1,\dots,v_k,v_{k+1},u_{k+1},\dots,v_n,u_n]$ is invertible, and*

$$V^{-1}AV = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r \end{pmatrix}$$

where the elementary Jordan blocks $B = B_j, j = 1,\dots,r$, are either of the form

$$B = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}$$

for λ one of the real eigenvalues of A, or of the form

$$B = \begin{pmatrix} D & I_2 & 0 & \cdots & 0 \\ 0 & D & I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & D & I_2 \\ 0 & \cdots & \cdots & 0 & D \end{pmatrix}$$

with $D = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right), I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for $\lambda = a+ib$ one of the complex eigenvalues of A.

3 Fuzzy Exponential Stability

Every fuzzy linear ordinary differential equation with constant coefficients can be written in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3.1}$$

where x is a vector in E^n, y is a vector in E^m and A, B, C and D are crisp matrices of appropriate sizes. The above equations are known as the state space representation, state space system or state space model. The vector x is called the state vector. State space representations with a finite-dimensional state space are also called finite-dimensional systems. Eq. (3.1) is named the state differential equation. For finite-dimensional systems, there are different kinds of stability. In this paper, we restrict our attention to exponential stability and Lyapunov stability.
Stability in the Lyapunov sense: A system is said to be stable if its trajectories can be made arbitrarily close to the origin when the initial starting state is chosen. Stability in the Lyapunov sense (a special case of which is exponential stability) is a widely used definition in the control community originating from the concept of "energy" proposed by the Russian mathematician Alexander Lyapunov in the 19th century [12]. We define fuzzy exponential stability for the fuzzy homogeneous state space equation and consider the system

$$\dot{x}(t) = Ax(t), x(0) = x_0 \in E^n, t \geq 0. \tag{3.2}$$

Here A is a crisp $n \times n$ matrix, and x_0 is a n-dimensional vector. We denote by $\mathcal{K}$ either $\Re$ or $\mathcal{C}$.

Definition 3.1 *Let $A \in \Re^{n\times n}, A > 0$. The differential Eq. (3.2) is called fuzzy exponentially stable, if for every initial condition $x_0 \in E^n$ the solution of Eq. (3.2) satisfies*

$$\lim_{t\to\infty} x(t) = \tilde{0},$$

or $\forall\varepsilon > 0\ \exists M > 0 :\ \ \forall t > M\ d(x(t),\tilde{0}) < \varepsilon$, where d is Hausdorff distance. The unique solution of Eq. (3.2) is given by $x(t) = e^{At}x_0, x_0 \in E^n, t \geq 0$. In the case of $A < 0$, the solutions *of Eq. (3.2) can be found by using strongly generalized differentiability of kind* **(ii)** *in Definition 2.3.*

Remark 3.1 [12] Each coordinate in the solution $x(t)$ of the initial value problem (3.2) is a linear combination of functions of the form $t^k e^{at}\cos bt$ or $t^k e^{at}\sin bt$ where $a+ib$ is an eigenvalue of the matrix A and $0 \le k \le n-1$.

Since A is a real matrix with real and complex eigenvalues, and the solution $x(t)$ of the initial value problem (3.2) is a fuzzy function, all coefficients in the mentioned linear combination in Remark 3.1 are fuzzy numbers.
The following theorem shows that the differential Eq. (3.2) is fuzzy exponentially stable, if and only if all eigenvalues of the matrix A lie in the open left half-plane of $\mathcal{C}$ (the set of complex numbers). By the Theorem 2.1 and Remark 3.1, we have following remark:

Remark 3.2 [12] Let $A \in \Re^{n\times n}, x_0 \in \Re^n, t \ge 0$. If one of the eigenvalues $\lambda = a+ib$ has positive real part, $a > 0$, then there exists an $x_0 \in \Re^n, x_0 \ne 0$, such that $\mid e^{At}x_0 \mid \ge e^{at} \mid x_0 \mid$ and, if all of the eigenvalues of A have negative real part, then there exist positive constants a, c, m and M such that for all $x_0 \in \Re^n, \mid e^{At}x_0 \mid \le Me^{-ct} \mid x_0 \mid$ for $t \ge 0$ and $\mid e^{At}x_0 \mid \ge me^{-at} \mid x_0 \mid$ for $t \le 0$.

Theorem 3.1 *The following statements are equivalent:*

(a) *For all $x_0 \in E^n, \lim_{t\to\infty} e^{At}x_0 = \tilde{0}$ and for $x_0 \ne \tilde{0}, \lim_{t\to\infty} d(e^{At}x_0, \tilde{0}) = \infty$.*

(b) *All eigenvalues of A have negative real part.*

(c) *There are positive constants a, c, m and M such that for all $x_0 \in E^n$*

$$d(e^{At}x_0, \tilde{0}) \le Me^{-ct} d(x_0, \tilde{0})$$

for $t \ge 0$, and

$$d(e^{At}x_0, \tilde{0}) \ge me^{-at} d(x_0, \tilde{0})$$

for $t \le 0$.

Proof. $(a \Rightarrow b)$ If one of the eigenvalues $\lambda = a+ib$ has positive real part, $a > 0$, then by Remark 3.2, there exists an $x_0 \in E^n, x_0 \ne \tilde{0}$, such that $\mid e^{At}\underline{x_0} \mid \ge e^{at} \mid \underline{x_0} \mid$ and $\mid e^{At}\overline{x_0} \mid \ge e^{at} \mid \overline{x_0} \mid$ where $x_0 = (\underline{x_0}, \overline{x_0})$. Thus, by Definition 2.1, we have $d(e^{At}x_0, \tilde{0}) \ge e^{at} d(x_0, \tilde{0})$. Therefore, $d(e^{At}x_0, \tilde{0}) \to \infty$ as $t \to \infty$; i.e. $d(\lim_{t\to\infty} e^{At}x_0, \tilde{0}) \ne 0$. And if one of the eigenvalues of A has zero real part, say $\lambda = ib$, then by Remark 3.1, there exists $x_0 \in E^n, x_0 \ne \tilde{0}$, such that at least one component of the solution is of the form $ct^k\cos bt$ or $ct^k\sin bt$ with $k \ge 0$. And once again $d(\lim_{t\to\infty} e^{At}x_0, \tilde{0}) \ne 0$. Thus, if not all of the eigenvalues of A have negative real part, there exists $x_0 \in E^n$ such that $\lim_{t\to\infty} e^{At}x_0 \ne \tilde{0}$; i.e. $(a \Rightarrow b)$.
$(b \Rightarrow c)$: If all of the eigenvalues of A have negative real part, then it follows from Remark 3.2 that there exist positive constants a, c, m and M such that for all $x_0 \in E^n$,

$$\begin{aligned} \mid e^{At}\underline{x_0} \mid &\le Me^{-ct} \mid \underline{x_0} \mid, \\ \mid e^{At}\overline{x_0} \mid &\le Me^{-ct} \mid \overline{x_0} \mid \end{aligned}$$

for $t \ge 0$, and

$$\begin{aligned} \mid e^{At}\underline{x_0} \mid &\ge me^{-at} \mid \underline{x_0} \mid, \\ \mid e^{At}\overline{x_0} \mid &\ge me^{-at} \mid \overline{x_0} \mid \end{aligned}$$

for $t \le 0$. Thus, by Definition 2.1 we have

$$d(e^{At}x_0, \tilde{0}) \le Me^{-ct} d(x_0, \tilde{0})$$

for $t \ge 0$, and

$$d(e^{At}x_0, \tilde{0}) \ge me^{-at} d(x_0, \tilde{0})$$

for $t \le 0$.
$(c \Rightarrow a)$: If this last pair of inequalities is satisfied for all $x_0 \in E^n$, it follows by taking the limit as $t \to \pm\infty$ on each side of the inequalities that $\lim_{t\to\infty} d(e^{At}x_0, \tilde{0}) = 0$ and that $\lim_{t\to-\infty} d(e^{At}x_0, \tilde{0}) = \infty$ for $x_0 \ne 0$.

The next theorem is proved in exactly the same manner as Theorem 3.1 using the Theorem 2.1 and Remark 3.1.

Theorem 3.2 *The following statements are equivalent:*

(a) *For all $x_0 \in E^n, \lim_{t\to-\infty} d(e^{At}x_0, \tilde{0}) = 0$ and for $x_0 \ne \tilde{0}, \lim_{t\to-\infty} d(e^{At}x_0, \tilde{0}) = \infty$.*

(b) *All eigenvalues of A have positive real part.*

(c) *There are positive constants a, c, m and M such that for all $x_0 \in E^n$*

$$d(e^{At}x_0, \tilde{0}) \le Me^{ct} d(x_0, \tilde{0})$$

for $t \leq 0$, *and*

$$d(e^{At}x_0, \tilde{0}) \geq me^{at}d(x_0, \tilde{0})$$

for $t \geq 0$.

4 Fuzzy Lyapunov Stability

In this section, we discuss the fuzzy stability of the equilibrium points of the nonlinear system

$$\dot{x}(t) = f(x(t)), x(0) = x_0 \in E^n. \quad (4.3)$$

In the following definition, we denote the maximal interval of existence (α, β) of the solution $\phi(t, x_0)$ of the initial value problem (4.3) by $I(x_0)$, since the endpoints α and β of the maximal interval generally depend on x_0.

Definition 4.1 *Let F be an open subset of E^n, and let $f \in C^1(F)$. For $x_0 \in F$, let $\phi(t, x_0)$ be the solution of the initial value problem (4.3) defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the set of mappings ϕ_t defined by $\phi_t(x_0) = \phi(t, x_0)$ is called the flow of the differential Eq. (4.3) or the flow defined by the differential Eq. (4.3); ϕ_t is also referred to as the flow of the vector field $f(x)$.*
If we think of the initial point x_0 as being fixed and let $I = I(x_0)$, then the mapping $\phi(., x_0) : I \to F$ defines a solution curve or trajectory of the system (4.3) through the point $x_0 \in F$.

Theorem 4.1 *Let F be an open set of E^n and let $f \in C^1(F)$. Then for all $x_0 \in F$, if $t \in I(x_0)$ and $s \in I(\phi_t(x_0))$, it follows that $s + t \in I(x_0)$ and $\phi_{s+t}(x_0) = \phi_s(\phi_t, x_0)$.*

Suppose that $s > 0, t \in I(x_0)$ and $s \in I(\phi_t(x_0))$. Let the maximal interval $I(x_0) = (\alpha, \beta)$ and define the function $x : (\alpha, s + t) \to F$ by

$$x(r) = \begin{cases} \phi(r, x_0) \;\; if \;\; \alpha < r \leq t, \\ \phi(r - t, \phi_t(x_0)) \;\; if \;\; t \leq r \leq s + t. \end{cases}$$

Then $x(r)$ is a solution of the initial value problem (4.3) on $(\alpha, s + t]$. Hence $s + t \in I(x_0)$ and by uniqueness of solutions

$$\begin{aligned} \phi_{s+t}(x_0) &= x(s + t) = \phi(s, \phi_t(x_0)) \\ &= \phi_s(\phi_t, x_0). \end{aligned}$$

If $s = 0$, the statement of the theorem follows immediately. And if $s < 0$, then we define the function $x : [s + t, \beta) \to F$ by

$$x(t) = \begin{cases} \phi(r, x_0) \;\; if \;\; t \leq r \leq \beta, \\ \phi(r - t, \phi_t(x_0)) \;\; if \;\; s + t \leq r \leq t. \end{cases}$$

Then $x(r)$ is a solution of the initial value problem (4.3) on $[s + t, \beta)$ and the last statement of the theorem follows from the uniqueness of solutions as above.

$x_e \in E^n$ is called an equilibrium point or critical point of (4.3), if $f(x_e) = \tilde{0}$.

Definition 4.2 *Let ϕ_t denote the solution of the fuzzy differential equation (4.3) defined for all $t \in \Re$. An equilibrium point x_e of Eq. (4.3) is said to be fuzzy stable in the sense of Lyapunov, if for every real number $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x that $d(x, x_e) < \delta$ and $t \geq 0$ we have $d(\phi_t(x), x_e) < \varepsilon$. The equilibrium point x_e is fuzzy unstable, if it is not fuzzy stable, and x_e is fuzzy exponentially stable if there exists a $\delta > 0$ such that for all x that $d(x, x_e) < \delta$ we have $\lim_{t\to\infty} \phi_t(x) = x_e$.*

It is an immediate consequence of the Definition 4.2 that if an equilibrium point x_e of Eq. (4.3) is fuzzy exponentially stable, then x_e is fuzzy stable in the sense of Lyapunov.

Definition 4.3 *If $F \subset E^n, f \in C^1(F), V \in C^1(F)$ and ϕ_t is the solution of the fuzzy differential equation (4.3), then for $x \in F$ the derivative of the function $V(x)$ along the solution $\phi_t(x)$ is*

$$\dot{V}(x) = \frac{\mathrm{d}}{\mathrm{dt}}V(\phi_t(x)).$$

A function $V : E^n \to E$ satisfying the hypotheses of the next theorem is called a fuzzy Lyapunov function.

Theorem 4.2 *Let F be an open subset of E^n containing x_e. Suppose that $f \in C^1(F)$ and that $f(x_e) = \tilde{\mathbf{0}}$. Suppose further that there exists a function $V \in C^1(F)$ satisfying $V(x_e) = \tilde{0}$ and $V(x) > \tilde{0}$ if $x \neq x_e$, then*

(a) *if $\dot{V}(x) \leq \tilde{0}$ for all $x \in F, x_e$ is fuzzy stable.*

(b) *if $\dot{V}(x) < \tilde{0}$ for all $x \in F - \{x_e\}, x_e$ is fuzzy exponentially stable.*

(c) *if $\dot{V}(x) > \tilde{0}$ for all $x \in F - \{x_e\}, x_e$ is fuzzy unstable.*

Without loss of generality, we shall assume that the equilibrium point $x_e = \tilde{\mathbf{0}}$. (a) Choose $\varepsilon > 0$ sufficiently small that $\overline{N_\varepsilon(\tilde{\mathbf{0}})} \subset F$ and let m_ε be the minimum of the fuzzy continuous function $V(x)$ on the compact set

$$S_\varepsilon = \{x \in E^n | d(x, \tilde{\mathbf{0}}) = \varepsilon\}.$$

Then since $V(x) > \tilde{0}$ for $x_e \neq \tilde{\mathbf{0}}$, it follows that $m_\varepsilon > \tilde{0}$. Since $V(x)$ is continuous and $V(\tilde{\mathbf{0}}) = \tilde{0}$, it follows that there exists a $\delta > 0$ such that $d(x, \tilde{\mathbf{0}}) < \delta$ implies that $V(x) < m_\varepsilon$. Since $\dot{V}(x) \leq \tilde{0}$ for $x \in F$, it follows that $V(x)$ is decreasing along trajectories of Eq. (4.3). Thus, if ϕ_t is the solution of the differential equation (4.3), it follows that for all $x_e \in N_\delta(\tilde{\mathbf{0}})$ and $t > 0$ we have

$$V(\phi_t(x_e)) \leq V(x_e) < m_\varepsilon. \tag{4.4}$$

Now suppose that for $d(x_e, \tilde{\mathbf{0}}) < \delta$ there is a $t_1 > 0$ such that $d(\phi_{t_1}(x_e), \tilde{\mathbf{0}}) = \varepsilon$; i.e., such that $\phi_{t_1}(x_e) \in S_\varepsilon$. Then since m_ε is the minimum of $V(x)$ on S_ε, this would imply that

$$m_\varepsilon \leq V(\phi_{t_1}(x_e))$$

which contradicts the inequality (4.4). Thus for $d(x_e, \tilde{\mathbf{0}}) < \delta$ and $t \geq 0$ it follows that $d(\phi_t(x_e), \tilde{\mathbf{0}}) < \varepsilon$; i.e., $\tilde{\mathbf{0}}$ is a stable equilibrium point. (b) Suppose that $\dot{V}(x) < \tilde{0}$ for all $x \in F$. Then $V(x)$ is strictly decreasing along trajectories of Eq. (4.3). Let ϕ_t be the solution of Eq. (4.3), and let $d(x_e, \tilde{\mathbf{0}}) < \delta$, the neighborhood defined in part (a). Then, by part (a), $\phi_t(x_e) \subset N_\varepsilon(\tilde{\mathbf{0}})$ for all $t \geq 0$. Let $\{t_k\}$ be any sequence with $t_k \to \infty$. Then since $\overline{N_\varepsilon(\tilde{\mathbf{0}})}$ is compact, there is a subsequence of $\{\phi_{t_k}(x_e)\}$ that converges to a point in $\overline{N_\varepsilon(\tilde{\mathbf{0}})}$. But for any subsequence $\{t_n\}$ of $\{t_k\}$ such that $\{\phi_{t_n}(x_e)\}$ converges, we show below that the limit is $\tilde{\mathbf{0}}$. It then follows that $\phi_{t_k}(x_e) \to \tilde{\mathbf{0}}$ for any sequence $t_k \to \infty$, and therefore that $\phi_t(x_e) \to \tilde{\mathbf{0}}$ as $t \to \infty$; i.e., that $\tilde{\mathbf{0}}$ is exponentially stable. It remains to show that if $\phi_{t_n}(x_e) \to y_e$, then $y_e = \tilde{\mathbf{0}}$. Since $V(x)$ is strictly decreasing along trajectories of Eq. (4.3), and since $V(\phi_{t_n}(x_e)) \to V(y_e)$ by the continuity of V, it follows that $V(\phi_t(x_e)) > V(y_e)$ for all $t > 0$. But if $y_e \neq \tilde{\mathbf{0}}$, then for $s > 0$ we have $V(\phi_s(y_e)) < V(y_e)$ and, by continuity, it follows that for all y sufficiently close to y_e we have $V(\phi_s(y)) < V(y_e)$ for $s > 0$. But then for $y = \phi_{t_n}(x_e)$ and n sufficiently large, we have $V(\phi_{s+t_n}(x_e)) < V(y_e)$ which contradicts the above inequality. Therefore, $y_e = \tilde{\mathbf{0}}$, and it follows that $\tilde{\mathbf{0}}$ is exponentially stable. (c) Let M be the maximum of the continuous function $V(x)$ on the compact set $\overline{N_\varepsilon(\tilde{\mathbf{0}})}$. Since $\dot{V}(x) > \tilde{0}$, $V(x)$ is strictly increasing along trajectories of Eq. (4.3). Thus, if ϕ_t is the solution of Eq. (4.3), then for any $\delta > 0$ and $x_e \in N_\delta(\tilde{\mathbf{0}}) - \{\tilde{\mathbf{0}}\}$ we have

$$V(\phi_t(x_e)) > V(x_e) > \tilde{0}$$

for all $t > 0$. And since $\dot{V}(x) > \tilde{0}$, this last statement implies that

$$\inf_{t>0} \dot{V}(\phi_t(x_e)) = m > \tilde{0}.$$

Thus,

$$V(\phi_t(x_e)) - V(x_e) \geq mt$$

for all $t > 0$. Therefore,

$$V(\phi_t(x_e)) > mt > M$$

for t sufficiently large; i.e., $\phi_t(x_e)$ lies outside the closed set $\overline{N_\varepsilon(\tilde{\mathbf{0}})}$. Hence, $\tilde{\mathbf{0}}$ is unstable.

5 Examples

In this section, two numerical examples are given to support the theoretical results.

Example 5.1 *Consider the fuzzy homogeneous linear system*

$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = 4x_1 - 2x_2$$

with $x_{01} = (-0.5, 0.1)$ and $x_{02} = (-2, 0.3)$. Eigenvalues of coefficient matrix are $\lambda_1 = -4.5616, \lambda_2 = -0.4384$ which have negative real parts. Based on Definition 3.1 and Theorem 3.1, the system is fuzzy exponentially stable, because all eigenvalues of matrix lie in left half plan.

Example 5.2 *Consider the nonlinear system*

$$\dot{x}_1 = -x_1 - 2x_2^2$$

$$\dot{x}_2 = x_1 x_2 - x_2^3.$$

The equilibrium point is $x_e = \tilde{0}$, and

$$V(x) = a(\alpha)(\left(\frac{x_1^2}{2}\right) + x_2^2)$$

with $a(\alpha) = [2+\alpha, 4+\alpha], \alpha \in [0,1]$ *is a Lyapunov function for this system. Computing* $\dot{V}(x)$, *we find*

$$\begin{aligned}\dot{V}(x) =& [(2+\alpha)(-x_1^2 - 2x_2^4), (4+\alpha)(-x_1^2 - 2x_2^4)] \\ &< [\underline{0}, \overline{0}].\end{aligned}$$

Therefore, based on Theorem 4.2 the equilibrium point is stable.

6 Conclusion

In this paper, exponential stability and Lyapunov stability for a fuzzy homogeneous system of differential equations have been investigated and some related theorems have been proved. Also, some numerical examples are presented.

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