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Fuzzy number-valued fuzzy relation

M. Adabitabar Firozja *†, S. Firouzian ‡

Abstract

It is well known fact that binary relations are generalized mathematical functions. Contrary to functions from domain to range, binary relations may assign to each element of domain two or more elements of range. Some basic operations on functions such as the inverse and composition are applicable to binary relations as well. Depending on the domain or range or both are fuzzy value fuzzy set, interval fuzzy value fuzzy set or fuzzy number value fuzzy set, define of the fuzzy relation is different. Given a fuzzy relation, its domain and range are fuzzy number value fuzzy sets. In this paper, initially we define fuzzy number value fuzzy sets and then propose fuzzy number-valued fuzzy relation (FN-VFR). We also introduce property of reflexive, symmetric, transitive and equivalence relation of FN-VFR. As follow, we prove some theorems for FN-VFR with property of reflexive, symmetric and transitive. Also, we show examples for FN-VFR.

Keywords : Fuzzy numbers; Relation; Fuzzy relation; Reflexive; Symmetric and transitive.

1 Introduction

One of the most fundamental notions in pure and applied sciences is the concept of a relation. Science has been described as the discovery of relations between objects, states and events. In 1965, L. A. Zadeh introduced the concept of fuzzy set theory[10]. Fuzzy set theory is an extension of classical set theory. Fuzzy relations generalize the concept of relations in the same manner as fuzzy sets generalize the fundamental idea of sets. The fuzzy relation theory as a generalization of Euler's graph theory was first introduced by Rosenfeld [8] in 1975. Some researchers work in fuzzy graph [6], bipolar fuzzy graph [9, 1, 2] and fuzzy interval graph [3, 7]. In this paper, we propose fuzzy number-valued fuzzy relation (FN-VFR) and also introduce property of reflexive, symmetric, transitive and equivalence relation of FN-VFR.
The paper has been organized as follows: a background of fuzzy concepts is presented in Section 2. In Section 3, we will introduce the fuzzy number-valued fuzzy relation and is given example. Subsequently, in Section 4, equivalence relation of FN-VFR is presented. Finally, conclusions are presented in Section 5.

2 Preliminaries

A fuzzy subset of X is a mapping $\mu : X \to [0,1]$ where μ as assigning to each element $x \in X$ a degree of membership, $0 \leq \mu(x) \leq 1$.
Let S be a set and μ and ν be fuzzy subsets of S:
1. A fuzzy subset $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in S$.
2. $(\mu \cup \nu)(x) = \mu(x) \bigvee \nu(x)$ for all $x \in S$.
3. $(\mu \cap \nu)(x) = \mu(x) \bigwedge \nu(x)$ for all $x \in S$.
Where, max and min are show with $\bigvee$ and $\bigwedge$

*Corresponding author. mohamadsadega@yahoo.com
†Department of mathematics, Qaemshar Branch, Islamic Azad University, Qaemshahr, Iran.
‡Department of Mathematics, Payame Noor University (PNU), Tehran, Iran.

respectively.

Definition 2.1 *(Rosenfeld, [8]) Let S and T be two sets and μ and ν be fuzzy subsets of S and T, respectively. A fuzzy relation ρ from the fuzzy subset μ into the fuzzy subset ν is a fuzzy subset ρ of $S \times T$ such that $\rho(x, y) \leq \mu(x) \bigwedge \nu(y), \forall x \in S$ and $\forall y \in T$.*

Definition 2.2 *(Rosenfeld, [8]) Let $\rho : S \times T \to [0, 1]$ be a fuzzy relation from a fuzzy subset μ of S into a fuzzy subset ν of T and $\omega : T \times U \to [0, 1]$ be a fuzzy relation from a fuzzy subset ν of T into a fuzzy subset ξ of U.*

Define the composition ρ of ω and denote by $\rho o \omega : S \times U \to [0, 1]$ where for all $x \in S$ and $z \in U$

$$\rho o \omega(x, z) = \bigvee \{\rho(x, y) \bigwedge \omega(y, z) | y \in T\} \quad (2.1)$$

Notation ρ^2 to denote the composition $\rho o \rho$, ρ^k to denote the composition $\rho^{k-1} o \rho; k > 1$.
Define $\rho^{\infty}(x, y) = \bigvee \{\rho^k(x, y) | k = 1, 2, ...\}$ for all $x, y \in S$.

Definition 2.3 *(Coroianu, [4]) The set of all fuzzy numbers is denoted by FN and for fuzzy number $A \in FN$, we show the membership function by $A(x)$ which is given by*

$$A(x) = \begin{cases} 0 & x \leq a_1, \\ l_A(x) & a_1 \leq x \leq a_2, \\ 1 & a_2 \leq x \leq a_3, \\ r_A(x) & a_3 \leq x \leq a_4, \\ 0 & a_4 \leq x \end{cases} \quad (2.2)$$

Where $a_1, a_2, a_3, a_4 \in R$ and $l_A(.)$ is nondecreasing and $r_A(.)$ is non-increasing and $l_A(a_1) = 0$, $l_A(a_2) = 1$, $r_A(a_3) = 1$ and $r_A(a_4) = 0$. We can show every fuzzy number A as family of crisp intervals as follows

$$A = \{x \in R : A(x) \geq r\} = [A_l(r), A_u(r)] \quad (2.3)$$

Let A be a fuzzy subset of X; the support of A, denoted $supp(A)$ whose

$$supp(A) = \{x \in X | A(x) > 0\} \quad (2.4)$$

Sometimes we use this parametric crisp interval instead of corespondent fuzzy number.

Definition 2.4 *[5] let $[A]^r = [A_l(r), A_u(r)]$ and $[B]^r = [B_l(r), B_u(r)]$ be two fuzzy numbers. We get*

$$A \bigvee B = [A_l(r) \bigvee B_l(r), A_u(r) \bigvee B_u(r)]$$

and

$$A \bigwedge B = [A_l(r) \bigwedge B_l(r), A_u(r) \bigwedge B_u(r)].$$

And we used the following ordering method

$$A \preceq B \Leftrightarrow \begin{cases} A_l(r) \leq B_l(r), \\ A_u(r) \leq B_u(r) \end{cases} \forall\ r \in (0, 1]. \quad (2.5)$$

Proposition 2.1 *If $A, B \in FN$ then:*

1. *According to the above ordering we have $A \bigwedge B \preceq A$ and $A \preceq A \bigvee B$.*

2. *$A \bigwedge B \in FN$ and $A \bigvee B \in FN$.*

3 Fuzzy number-valued fuzzy relation (FN-VFR)

Definition 3.1 *Let S is arbitrary set then μ is a fuzzy number value fuzzy set (FN-VFS) on S if $\mu : S \to FN$ and for all $x \in S$, $\mu(x) \in FN$.*

We show the set of all fuzzy numbers with support in $[0, 1]$ by FIN.

Definition 3.2 *Let S and T be two sets and $\mu : S \to FIN$ and $\nu : T \to FIN$ be two FN-VFSs of S and T, respectively. A FN-VFR ρ from μ into ν is a fuzzy number valued subset $\rho : S \times T \to FIN$ such that $\rho(x, y) \preceq \mu(x) \bigwedge \nu(y), \forall x \in S$ and $\forall y \in T$.*

With definition (2.4), we define $[\rho]^r : S \times T \to IN$ for every $r \in (0, 1]$ and $\rho : S \times T \to FIN$ such that $[\rho]^r(x, y) = [\rho(x, y)]^r = [\rho_l(x, y)(r), \rho_u(x, y)(r)], \forall x \in S$ and $\forall y \in T$ where IN shows the set of all interval numbers on $[0, 1]$.

Example 3.1 *Let $S = \{x, y\}$, $T = \{a, b\}$, $\mu : S \to FIN$, $\nu : T \to FIN$ with $\mu = \{(x, \mu(x)), (y, \mu(y))\}$ and $\nu = \{(a, \nu(a)), (b, \nu(b))\}$*

where

$$\mu_{(x)}(t) = \begin{cases} \frac{t-0.2}{0.2} & 0.2 \leq t \leq 0.4, \\ 1 & 0.4 \leq t \leq 0.6, \\ \frac{0.8-t}{0.2} & 0.6 \leq t \leq 0.8, \\ 0 & Otherwise \end{cases}$$

$$\mu_{(y)}(t) = \begin{cases} \frac{t-0.3}{0.1} & 0.3 \leq t \leq 0.4, \\ \frac{0.6-t}{0.2} & 0.4 \leq t \leq 0.6, \\ 0 & Otherwise \end{cases}$$

$$\nu_{(a)}(t) = \begin{cases} \frac{t-0.4}{0.1} & 0.4 \leq t \leq 0.5, \\ \frac{0.6-t}{0.1} & 0.5 \leq t \leq 0.6, \\ 0 & Otherwise \end{cases}$$

$$\nu_{(b)}(t) = \begin{cases} \frac{t-0.3}{0.4} & 0.3 \leq t \leq 0.7, \\ \frac{0.8-t}{0.1} & 0.7 \leq t \leq 0.8, \\ 0 & Otherwise \end{cases}$$

Then, a function $\rho : S \times T \rightarrow FIN$ such that $\rho = \{\rho(x,a), \rho(x,b), \rho(y,a), \rho(y,b)\}$ with

$$\rho_{(x,a)}(t) = \begin{cases} \frac{t-0.2}{0.2} & 0.2 \leq t \leq 0.4, \\ \frac{0.5-t}{0.1} & 0.4 \leq t \leq 0.5, \\ 0 & Otherwise \end{cases}$$

$$\rho_{(x,b)}(t) = \begin{cases} \frac{t-0.2}{0.1} & 0.2 \leq t \leq 0.3, \\ \frac{0.7-t}{0.4} & 0.3 \leq t \leq 0.7, \\ 0 & Otherwise \end{cases}$$

$$\rho_{(y,a)}(t) = \begin{cases} \frac{t-0.2}{0.1} & 0.2 \leq t \leq 0.3, \\ 1 & 0.3 \leq t \leq 0.4, \\ \frac{0.5-t}{0.1} & 0.4 \leq t \leq 0.5, \\ 0 & Otherwise \end{cases}$$

$$\rho_{(y,b)}(t) = \begin{cases} \frac{0.6-t}{0.3} & 0.3 \leq t \leq 0.6, \\ 0 & Otherwise \end{cases}$$

is a FN-VFR.

For every $r \in [0,1]$, $[\rho]^r = \{[\rho(x,a)]^r, [\rho(x,b)]^r, [\rho(y,a)]^r, [\rho(y,b)]^r\}$

where

$[\rho(x,a)]^r = [0.2r + 0.2, 0.5 - 0.1r], [\rho(x,b)]^r = [0.2r + 0.1, 0.7 - 0.4r]$

$[\rho(y,a)]^r = [0.1r + 0.2, 0.5 - 0.1r], [\rho(y,b)]^r = [0.3, 0.6 - 0.3r]$.

Proposition 3.1 *Two fuzzy number $[\rho]^r(x,y)$ and $\mu(x)]^r \bigwedge [\nu(y)]^r$ are compatible, in other word $[\rho]^r(x,y) \preceq [\mu(x)]^r \bigwedge [\nu(y)]^r, \forall x \in S$ and $\forall y \in T$.*

Proof. It directly follows from definition.

Definition 3.3 *Let $\rho : S \times T \rightarrow FIN$ be a FN-VFR from μ into ν and $\omega : T \times U \rightarrow FIN$ be a FN-VFR from ν into ξ. Define the composition ρ of ω and denote by $\rho o \omega : S \times U \rightarrow FIN$*

where for all $x \in S$ and $z \in U$

$$\rho o \omega(x,z) = \bigvee \{\rho(x,y) \bigwedge \omega(y,z) | y \in T\} \quad (3.6)$$

Example 3.2 *Let $\rho : S \times T \rightarrow FIN$ and $\omega : T \times U \rightarrow FIN$ be two FN-VFRs such that $\rho = \{\rho(x,a), \rho(x,b), \rho(y,a), \rho(y,b)\}$ and $\omega = \{\omega(a,e), \omega(a,f), \omega(b,e), \omega(b,f)\}$*

with

$$\rho_{(x,a)}(t) = \begin{cases} \frac{t-0.2}{0.2} & 0.2 \leq t \leq 0.4, \\ \frac{0.5-t}{0.1} & 0.4 \leq t \leq 0.5, \\ 0 & Otherwise \end{cases}$$

$$\rho_{(x,b)}(t) = \begin{cases} \frac{t-0.2}{0.1} & 0.2 \leq t \leq 0.3, \\ \frac{0.7-t}{0.4} & 0.3 \leq t \leq 0.7, \\ 0 & Otherwise \end{cases}$$

$$\rho_{(y,a)}(t) = \begin{cases} \frac{t-0.2}{0.1} & 0.2 \leq t \leq 0.3, \\ 1 & 0.3 \leq t \leq 0.4, \\ \frac{0.5-t}{0.1} & 0.4 \leq t \leq 0.5, \\ 0 & Otherwise \end{cases}$$

$$\rho_{(y,b)}(t) = \begin{cases} \frac{t-0.2}{0.1} & 0.2 \leq t \leq 0.3, \\ \frac{0.6-t}{0.3} & 0.3 \leq t \leq 0.6, \\ 0 & Otherwise \end{cases}$$

$$\omega_{(a,e)}(t) = \begin{cases} \frac{t-0.1}{0.4} & 0.1 \leq t \leq 0.5, \\ \frac{0.7-t}{0.2} & 0.5 \leq t \leq 0.7, \\ 0 & Otherwise \end{cases}$$

$$\omega_{(a,f)}(t) = \begin{cases} \frac{t-0.2}{0.1} & 0.2 \leq t \leq 0.3, \\ 1 & 0.3 \leq t \leq 0.4, \\ \frac{0.7-t}{0.3} & 0.4 \leq t \leq 0.7, \\ 0 & Otherwise \end{cases}$$

$$\omega_{(b,e)}(t) = \begin{cases} \frac{t-0.2}{0.2} & 0.2 \leq t \leq 0.4, \\ \frac{0.5-t}{0.1} & 0.4 \leq t \leq 0.5, \\ 0 & Otherwise \end{cases}$$

$$\omega_{(b,f)}(t) = \begin{cases} \frac{t-0.2}{0.2} & 0.2 \leq t \leq 0.4, \\ \frac{0.8-t}{0.4} & 0.4 \leq t \leq 0.8, \\ 0 & Otherwise \end{cases}$$

then composition $\rho o \omega : S \times U \rightarrow FIN$ is a FN-VFR where

$\rho o \omega = \{\rho o \omega(x,e), \rho o \omega(x,f), \rho o \omega(y,e), \rho o \omega(y,f)\}$ and

$$\rho o \omega_{(x,e)}(t) = \begin{cases} \frac{t-0.2}{0.1} & 0.2 \leq t \leq 0.233, \\ \frac{t-0.1}{0.4} & 0.233 \leq t \leq 0.3, \\ \frac{t-0.2}{0.2} & 0.3 \leq t \leq 0.4, \\ \frac{0.5-t}{0.1} & 0.4 \leq t \leq 0.5, \\ 0 & Otherwise \end{cases}$$

$$\rho o\omega_{(x,f)}(t)=\begin{cases}\frac{t-0.2}{0.1} & 0.2\leq t\leq 0.3,\\ 1 & 0.3\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.433,\\ \frac{0.7-t}{0.4} & 0.433\leq t\leq 0.7,\\ 0 & Otherwise\end{cases}$$

$$\rho o\omega_{(y,e)}(t)=\begin{cases}\frac{t-0.2}{0.1} & 0.2\leq t\leq 0.3,\\ 1 & 0.3\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

$$\rho o\omega_{(y,f)}(t)=\begin{cases}\frac{t-0.2}{0.1} & 0.2\leq t\leq 0.3,\\ 1 & 0.3\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.45,\\ \frac{0.6-t}{0.3} & 0.45\leq t\leq 0.6,\\ 0 & Otherwise\end{cases}$$

Proposition 3.2 *Let ρ,μ,ω and ν as defined in Definition 7. Then $\rho o\omega$ is a FN-VFR from μ into ξ.*

Proof. Let, $x\in S, y\in T$ and $z\in U$. Then $\rho(x,y)\preceq\mu(x)\bigwedge\nu(y)$ and $\omega(y,z)\preceq\nu(y)\bigwedge\xi(z)$. Hence, $\rho(x,y)\bigwedge\omega(y,z)\preceq\mu(x)\bigwedge\nu(y)\bigwedge\xi(z)$. Thus

$\rho o\omega(x,z)=\bigvee\{\rho(x,y)\bigwedge\omega(y,z)|y\in T\}\preceq\mu(x)\bigwedge\xi(z)$.

Notation ρ^2 to denote the composition $\rho o\rho$, ρ^k to denote the composition $\rho^{k-1}o\rho; k>1$.
Define $\rho^\infty(x,y)=\bigvee\{\rho^k(x,y)|k=1,2,...\}$ and $\rho^0(x,y)=0$ if $x\neq y$ $\rho^0(x,y)=\mu(x)$ for all $x,y\in S$.

Definition 3.4 *Let $\rho: S\times T\rightarrow FIN$ be a FN-VFR from μ into ν. Define the FN-VFR $\rho^{-1}: T\times S\rightarrow FIN$ of ν into μ by $\rho^{-1}(y,x)=\rho(x,y)$ for all $(y,x)\in T\times S$.*

Definition 3.5 *Let $\rho: S\times T\rightarrow FIN$ and $\rho^{'}: S\times T\rightarrow FIN$ be two FN-VFRs from μ into ν.*

1. $\rho\subseteq\rho^{'}$ if and only if $\rho(x,y)\preceq\rho^{'}(x,y)$
2. $(\rho\cup\rho^{'})(x,y)=\rho(x,y)\vee\rho^{'}(x,y)$
3. $(\rho\cap\rho^{'})(x,y)=\rho(x,y)\wedge\rho^{'}(x,y)$
for all $(x,y)\in S\times T$.

Example 3.3 *Let $\rho: S\times T\rightarrow FIN$ and $\rho^{'}: S\times T\rightarrow FIN$ be two FN-VFRs such that $\rho=\{\rho(x,a),\rho(x,b),\rho(y,a),\rho(y,b)\}$ and $\rho^{'}=\{\rho^{'}(x,a),\rho^{'}(x,b),\rho^{'}(y,a),\rho^{'}(y,b)\}$*

with

$$\rho_{(x,a)}(t)=\begin{cases}\frac{t-0.2}{0.2} & 0.2\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

$$\rho_{(x,b)}(t)=\begin{cases}\frac{t-0.2}{0.1} & 0.2\leq t\leq 0.3,\\ 1 & 0.3\leq t\leq 0.4,\\ \frac{0.7-t}{0.3} & 0.4\leq t\leq 0.7,\\ 0 & Otherwise\end{cases}$$

$$\rho_{(y,a)}(t)=\begin{cases}\frac{t-0.1}{0.1} & 0.1\leq t\leq 0.2,\\ 1 & 0.2\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

$$\rho_{(y,b)}(t)=\begin{cases}\frac{t-0.2}{0.2} & 0.2\leq t\leq 0.4,\\ \frac{0.6-t}{0.2} & 0.4\leq t\leq 0.6,\\ 0 & Otherwise\end{cases}$$

$$\rho^{'}_{(x,a)}(t)=\begin{cases}\frac{t-0.3}{0.1} & 0.3\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

$$\rho^{'}_{(x,b)}(t)=\begin{cases}\frac{t-0.2}{0.1} & 0.2\leq t\leq 0.3,\\ \frac{0.7-t}{0.4} & 0.3\leq t\leq 0.7,\\ 0 & Otherwise\end{cases}$$

$$\rho^{'}_{(y,a)}(t)=\begin{cases}\frac{t-0.2}{0.1} & 0.2\leq t\leq 0.3,\\ 1 & 0.3\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

$$\rho^{'}_{(y,b)}(t)=\begin{cases}\frac{t-0.3}{0.1} & 0.3\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

Then,

$$(\rho\cup\rho^{'})_{(x,a)}(t)=\begin{cases}\frac{t-0.3}{0.1} & 0.3\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

$$(\rho\cup\rho^{'})_{(x,b)}(t)=\begin{cases}\frac{t-0.2}{0.1} & 0.2\leq t\leq 0.3,\\ 1 & 0.3\leq t\leq 0.4,\\ \frac{0.7-t}{0.3} & 0.4\leq t\leq 0.7,\\ 0 & Otherwise\end{cases}$$

$$(\rho\cup\rho^{'})_{(y,a)}(t)=\begin{cases}\frac{t-0.2}{0.1} & 0.2\leq t\leq 0.3,\\ 1 & 0.3\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

$$(\rho\cup\rho^{'})_{(y,b)}(t)=\begin{cases}\frac{t-0.3}{0.1} & 0.3\leq t\leq 0.4,\\ \frac{0.6-t}{0.2} & 0.4\leq t\leq 0.6,\\ 0 & Otherwise\end{cases}$$

$$(\rho\cap\rho^{'})_{(x,a)}(t)=\begin{cases}\frac{t-0.2}{0.2} & 0.2\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

$$(\rho\cap\rho^{'})_{(x,b)}(t)=\begin{cases}\frac{t-0.2}{0.1} & 0.2\leq t\leq 0.3,\\ \frac{0.7-t}{0.4} & 0.3\leq t\leq 0.7,\\ 0 & Otherwise\end{cases}$$

$$(\rho\cap\rho^{'})_{(y,a)}(t)=\begin{cases}\frac{t-0.1}{0.1} & 0.1\leq t\leq 0.2,\\ 1 & 0.2\leq t\leq 0.4,\\ \frac{0.5-t}{0.1} & 0.4\leq t\leq 0.5,\\ 0 & Otherwise\end{cases}$$

$$(\rho \cap \rho')_{(y,b)}(t) = \begin{cases} \frac{t-0.2}{0.2} & 0.2 \le t \le 0.4, \\ \frac{0.5-t}{0.1} & 0.4 \le t \le 0.5, \\ 0 & Otherwise \end{cases}$$

Proposition 3.3 *Let τ, π, ρ and ω be FN-VFRs on a fuzzy number subset μ of a set S. Then the following properties hold.*

1. $\rho \cup \omega = \omega \cup \rho$
2. $\rho \cap \omega = \omega \cap \rho$
3. $\pi \cup (\rho \cup \omega) = (\pi \cup \rho) \cup \omega$
4. $\pi \cap (\rho \cap \omega) = (\pi \cap \rho) \cap \omega$
5. $\pi o(\rho o\omega) = (\pi o\rho)o\omega$
6. $\pi \cup (\rho \cap \omega) = (\pi \cup \rho) \cap (\pi \cup \omega)$
7. $\pi \cap (\rho \cup \omega) = (\pi \cap \rho) \cup (\pi \cap \omega)$
8. If $\tau \subseteq \rho$ and $\pi \subseteq \omega$ then $\tau \cup \pi \subseteq \rho \cup \omega$
9. If $\tau \subseteq \rho$ and $\pi \subseteq \omega$ then $\tau \cap \pi \subseteq \rho \cap \omega$
10. If $\tau \subseteq \rho$ and $\pi \subseteq \omega$ then $\tau o\pi \subseteq \rho o\omega$
11. For all $r \in [0,1]$, $[(\rho \cup \omega)]^r = [\rho]^r \cup [\omega]^r$
12. For all $r \in [0,1]$, $[(\rho \cap \omega)]^r = [\rho]^r \cap [\omega]^r$

Proof. We provide the proofs for 9 and 10.

Let $x, y, z \in S$.

9. $(\tau \cap \pi)(x,y) = \tau(x,y) \bigwedge \pi(x,y) \preceq \rho(x,y) \bigwedge \omega(x,y) = (\rho \cap \omega)(x,z)$.

10. $\tau o\pi(x,z) = \bigvee\{\tau(x,y) \bigwedge \pi(y,z) | y \in S\} \preceq \bigvee\{\rho(x,y) \bigwedge \omega(y,z) | y \in S\} = \rho o\omega(x,z)$.

4 Equivalence relation on FN-VFR

Let ρ, ω be FN-VFRs on a fuzzy number subset $\mu : S \to FIN$. It is quite natural to represent a FN-VFR in the form of a matrix. We use the matrix representation of a FN-VFR to explain the properties of a FN-VFR.

Definition 4.1 *We call FN-VFR, ρ is reflexive on μ if $\rho(x,x) = \mu(x)$ for all $x \in S$. If ρ reflexive on μ, then $\rho(x,y) \preceq \mu(x) \wedge \mu(y) \preceq \mu(x) = \rho(x,x)$ for all $x, y \in S$ and it follows that any diagonal element is larger than or equal to any element in its column and row.*

Theorem 4.1 *Let ρ, ω be FN-VFRs on a fuzzy number subset μ of a set S. Then the following properties hold.*

1. If ρ is reflexive then $\omega \subseteq \rho o\omega$ and $\omega \subseteq \omega o\rho$.
2. If ρ is reflexive then $\rho \subseteq \rho^2 \subseteq ... \subseteq \rho^\infty$.
3. If ρ is reflexive then $\rho(x,x) = \rho^2(x,x) = ... = \rho^\infty(x,x) = \mu(x)$.
4. If ρ and ω are reflexive, so is $\rho o\omega$ and $\omega o\rho$.
5. If ρ is reflexive then $[\rho]^r$ is reflexive for $r \in [0,1]$.

Proof. Let $x, y, z \in S$.
1. $\rho o\omega(x,y) = \bigvee\{\rho(x,z) \bigwedge \omega(z,y) | z \in S\} \succeq \rho(x,x) \bigwedge \omega(x,y) = \mu(x) \wedge \omega(x,y)$.
Since $\omega(x,y) \preceq \mu(x) \wedge \mu(y)$, then $\mu(x) \wedge \omega(x,y) = \omega(x,y)$. Thus $\omega \subseteq \rho o\omega$. Similarly, $\omega \subseteq \omega o\rho$.
2. Choose ω as ρ in 1.
3. Note that $\rho(x,x) = \mu(x)$ and $\rho(x,y) \preceq \rho(x,x), \forall x, y \in S$, then
$\mu(x) = \rho(x,x) \wedge \rho(x,x) \preceq \bigvee\{\rho(x,y) \bigwedge \rho(y,x) | y \in S\} = \rho^2(x,x)$
$\mu(x) = \bigvee\{\rho(x,x) \bigwedge \rho(x,x)\} \succeq \bigvee\{\rho(x,y) \bigwedge \rho(y,x) | y \in S\} = \rho^2(x,x)$.
Therefore, $\rho(x,x) = \rho^2(x,x)$ and similarly $\rho(x,x) = \rho^2(x,x) = ... = \rho^\infty(x,x)$.
4. $\rho o\omega(x,x) = \bigvee\{\rho(x,y) \bigwedge \omega(y,x) | y \in S\} \preceq \bigvee\{\rho(x,x) \bigwedge \omega(x,x)\} = \mu(x)$
and $\rho o\omega(x,x) = \bigvee\{\rho(x,y) \bigwedge \omega(y,x) | y \in S\} \succeq \{\rho(x,x) \bigwedge \omega(x,x)\} = \mu(x)$.
Therefore $\rho o\omega(x,x) = \mu(x)$. Similarly, $\omega o\rho(x,x) = \mu(x)$.
5. Proof is evident.

Definition 4.2 *We call FN-VFR, ρ is symmetric on μ if $\rho(x,y) = \rho(y,x)$ for all $x, y \in S$. In the matrix representation, ρ is symmetric FN-VFR if matrix representation of ρ is symmetric.*

Theorem 4.2 *Let ρ, ω be FN-VFRs on a fuzzy number subset $\mu : S \to FIN$. Then the following properties hold.*

1. If ρ is symmetric, then so is every power of ρ.
2. If ρ and ω are symmetric, then $\rho o\omega$ is symmetric if and only if $\rho o\omega = \omega o\rho$.
5. If ρ is symmetric then $[\rho]^r$ is symmetric for $r \in [0,1]$.

Proof. Let $x, y, z \in S$.

1. Assume that ρ^n is symmetric for $n \in N$. Then, $\rho^{n+1}(x,y) = \bigvee\{\rho^n(x,z) \bigwedge \rho(z,y) | z \in S\} = \bigvee\{\rho^n(z,x) \bigwedge \rho(y,z) | z \in S\} = \bigvee\{\rho(y,z) \bigwedge \rho^n(z,x) | z \in S\} = \rho^{n+1}(y,x)$.
2. $\rho o\omega(x,y) = \rho o\omega(y,x) \leftrightarrow$

$\bigvee\{\rho(x,z)\bigwedge\omega(z,y)|z \in S\} = \bigvee\{\rho(y,z)\bigwedge\omega(z,x)|z \in S\} \leftrightarrow$
$\bigvee\{\rho(x,z)\bigwedge\omega(z,y)|z \in S\} = \bigvee\{\omega(z,x)\bigwedge\rho(y,z)|z \in S\} \leftrightarrow$
$\bigvee\{\rho(x,z)\bigwedge\omega(z,y)|z \in S\} = \bigvee\{\omega(x,z)\bigwedge\rho(z,y)|z \in S\} \leftrightarrow \rho o\omega = \omega o\rho$
3. Proof is evident.

Definition 4.3 *We call FN-VFR, ρ is transitive on μ if $\rho^2 \subseteq \rho$.*

Theorem 4.3 *Let π, ρ and ω be FN-VFRs on a fuzzy number subset $\mu : S \to FIN$. Then the following properties hold.*

1. If ρ is transitive and $\pi \subseteq \rho$ and $\omega \subseteq \rho$ then $\pi o\omega \subseteq \rho$.
2. If ρ is transitive, then so is every power of ρ.
3. If ρ is transitive and ω is reflexive and $\omega \subseteq \rho$ then $\rho o\omega = \omega o\rho = \rho$.
4. If ρ is reflexive and transitive then $\rho^0 \subseteq \rho = \rho^2 = ... = \rho^\infty$.
5. If ρ and ω are transitive and $\rho o\omega = \omega o\rho$, then $\rho o\omega$ is transitive.
6. If ρ is symmetric and transitive, then $\rho(x,y) \subseteq \rho(x,x)$ and $\rho(y,x) \subseteq \rho(x,y)$ for all $x, y \in S$.
7. If ρ is transitive then $[\rho]^r$ is transitive for $r \in [0,1]$.

Proof. Let $x, y, z \in S$.

1. $\pi o\omega(x,y) = \bigvee\{\pi(x,z)\bigwedge\omega(z,y)|z \in S\} \subseteq \bigvee\{\rho(x,z)\bigwedge\rho(z,y)|z \in S\} = \rho^2(x,y) \subseteq \rho(x,y)$
2. Assume that $\rho^n o\rho^n \subseteq \rho^n$. Then $\rho^{n+1}o\rho^{n+1} = \rho^n o\rho^n o\rho^2 \subseteq \rho^n o\rho = \rho^{n+1}$.
3. By 1. taking π to be ρ, $\rho o\omega \subseteq \rho$. $\rho o\omega(x,y) = \bigvee\{\rho(x,z)\bigwedge\omega(z,y)|z \in S\} \succeq \bigvee\{\rho(x,y)\bigwedge\omega(y,y)\} = \rho(x,y)\bigwedge\mu(y) = \rho(x,y)$. Hence $\rho o\omega = \rho$.
4. By 3. taking ω to be ρ, $\rho^2 = \rho$. $\rho^3 = \rho^2 o\rho = \rho o\rho = \rho$ and similarly $\rho^n = \rho$ for all $n \in N$.
5. $(\rho o\omega)o(\rho o\omega) = \rho o(\omega o\rho)o\omega) = (\rho o\rho)o(\omega o\omega) = \rho^2 o\omega^2 \subseteq \rho o\omega$.
6. Since ρ is symmetric and transitive, $\rho o\rho \subseteq \rho$. Hence $\rho o\rho(x,x) \subseteq \rho(x,x) \Leftrightarrow$
$\bigvee\{\rho(x,y)\bigwedge\rho(y,x)|y \in S\} \subseteq \rho(x,x) \Leftrightarrow \bigvee\{\rho(x,y)\bigwedge\rho(x,y)|y \in S\} \subseteq \rho(x,x) \Leftrightarrow \rho(x,y) \subseteq \rho(x,x)$.
Since ρ is symmetric then $\rho(y,x) \subseteq \rho(x,x)$.
7. Is evident.

Definition 4.4 *A FN-VFR, ρ on S which reflexive, symmetric and transitive is called a fuzzy number-valued equivalence relation on S.*

5 Conclusions

It is well known that fuzzy number valued fuzzy sets constitute a generalization of the notion of fuzzy sets. The fuzzy number-valued fuzzy models give more precision, flexibility and compatibility to the system as compared to the classical and fuzzy models. So we have introduced fuzzy number-valued fuzzy relation and have presented several properties for this relation. The further study of fuzzy number-valued fuzzy relation may also be extended with fuzzy number-valued fuzzy graph an application in database theory, an expert system, neural networks and geographical information system.

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Mohammad Adabitabar Firozja, he has bornin the mazandaran, Babol in 1975. Got B.Sc and M.Sc degrees in appliedmathematics, operation research field from Teacher Training University, Tehran Iran andPHD degreein applied mathematics, fuzzy numerical analysis field from Science and Research Branch, Islamic Azad University. Main research interest includefuzzy logic, ranking, fuzzy metric distance, ordinary differential equation, partial differential equation, fuzzy system and similarity measure.

Siyamak firouzian, he has born in the mazandaran, babol in 1957. Got B.Sc degrees in mathematics, from Teacher Training University, Tehran Iran and M.Sc degrees in mathematics, finite group theory field from Iran University of Science and Technology University, Tehran Iran and PHD degree in mathematics, arithmetic geometric field from University of Regensburg in Germany. Main research fuzzy logic, fuzzy rough set, fuzzy graph.

Fuzzy number-valued fuzzy relation

M. Adabitabar Firozja, S. Firouzian

رابطه فازی با مقدار عدد فازی

چکیده:

این حقیقت که روابط دوتایی تعمیمی از توابع ریاضی هستند شناخته شده است. بر خلاف توابع از دامنه به برد روابط دوتایی ممکن است هر عضو از دامنه به یک یا بیشتر عضو از برد اختصاص دهد. برخی از عملیات اصلی در توابع مانند معکوس و ترکیب قابل اجرا به روابط باینری هستند. بسته به دامنه یا برد و یا هر دو مجموعه فازی با مقدار فازی، مجموعه فازی با مقدار بازه فازی و یا مجموعه فازی با مقدار عدد فازی، تعریف رابطه فازی متفاوت است. در نظر بگیرید رابطه فازی که دامنه و برد آن مجموعه فازی با مقدار عدد فازی است. در این مقاله، ابتدا مجموعه فازی با مقدار عدد فازی را معرفی و سپس رابطه فازی با مقدار عدد فازی را معرفی می کنیم. ما همچنین خواص انعکاسی، تقارنی، تعدی و رابطه هم ارزی از رابطه با مقدار عدد فازی را معرفی می کنیم. در ادامه، بعضی از قضایا در رابطه فازی با مقدار عدد فازی از خواص انعکاسی، تقارنی، تعدی را اثبات می کنیم. همچنین مثالهایی را برای رابطه فازی با مقدار عدد فازی نشان می دهیم.