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Autoconvolution equations and generalized Mittag-Leffler functions

S. Eshaghi *, A. Ansari †‡

Abstract

This article is devoted to study of the autoconvolution equations and generalized Mittag-Leffler functions. These types of equations are given in terms of the Laplace transform convolution of a function with itself. We state new classes of the autoconvolution equations of the first kind and show that the generalized Mittag-Leffler functions are solutions of these types of equations. In view of the inverse Laplace transform, we use the Schouten-Vanderpol theorem to establish an autoconvolution equation for the generalized Mittag-Leffler functions in terms of the Laplace and Mellin transforms. Also, in special cases we reduce the solutions of the introduced autoconvolution equations with respect to the Volterra μ-functions. Moreover, more new autoconvolution equations are shown using the Laplace transforms of generalized Mittag-Leffler functions. Finally, as an application of the autoconvolution equations in thermodynamic systems, we apply the Laplace transform for solving the Boltzmann equation and show its solution in terms of generalized Mittag-Leffler functions.

Keywords : Mittag-leffler function; Volterra function; Autoconvolution equations; Boltzmann equation.

1 Introduction and Preliminaries

1.1 The generalized Mittag-Leffler function

In year 1971, Prabhakar introduced the generalized Mittag-Leffler function on his study on singular integral equations as follows [23]

$$E^{\lambda}_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{\Gamma(\mu k+\nu)} \frac{z^k}{k!}, \qquad (1.1)$$

$$\lambda, \mu, \nu \in \mathbb{C}, \ \Re(\mu) > 0,$$

where $(\lambda)_k$ is the Pochhammer symbol [11]

$$(\lambda)_0 = 1,$$

$$(\lambda)_k = \lambda(\lambda+1)\dots(\lambda+k-1), \quad k = 1, 2, \dots .$$

For $\lambda = 1$, we get the two-parameter Mittag-Leffler function $E_{\mu,\nu}(z)$ defined by

$$E_{\mu,\nu}(z) := E^1_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k+\nu)}, \qquad (1.2)$$

$$\mu, \nu \in \mathbb{C}, \ \Re(\mu) > 0,$$

in addition, for $\lambda = \nu = 1$, this function coincides with the classical Mittag-Leffler function $E_\mu(z)$ [21, 22]

$$E_{\mu}(z) := E^1_{\mu,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k+1)}, \qquad (1.3)$$

where $\mu \in \mathbb{C}, \ \Re(\mu) > 0$.
Recently, many researchers have established

*Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P.O. Box 115, Shahrekord, Iran.
†Corresponding author. alireza_1038@yahoo.com
‡Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P.O. Box 115, Shahrekord, Iran.

many contributions on the generalized Mittag-Leffler function especially in the theory of fractional calculus and detect some of their applications in the physics and engineering. For example, new definitions of generalized fractional derivatives were introduced and solutions of the Cauchy-type initial and boundary value problems were expressed in terms of the generalized Mittag-Leffler function [12, 13, 14, 15, 16, 17, 18, 19, 20], [25, 26, 27].

1.2 The autoconvolution equations

One of the interesting functional equations which their solutions can be led to the generalized Mittag-Leffler functions are autoconvolution equations [30, 31]. These types of equations have been developed much less in the literature and are classified in three kinds. We consider the following form of the first kind

$$\rho(xz * x^{\rho-1}) = \gamma(z * z) + \delta(z * x^{\rho}), \tag{1.4}$$

where * is the Laplace convolution integral given by

$$(f * g)(t) = \int_0^t f(t-\xi)g(\xi)d\xi. \tag{1.5}$$

For more details in the autoconvolution equations and other forms of them see [2, 3, 4, 5, 6], [29].

1.3 Aims

The purpose of this paper is to obtain solutions of the autoconvolution equations in terms of the generalized Mittag-Leffler functions $E^{\lambda}_{\mu,\nu}(z)$. Also, in the particular case we reduce the solution of these equations to the Volterra μ-functions. Finally, as an application of this technique we survey the Boltzmann equation and get its solution in terms of the generalized Mittag-Leffler functions.

2 Autoconvolution equations for generalized Mittag-Leffler function

In this section, we recall the Schouten-Vanderpol theorem for the inverse Laplace transform and present a class of the autoconvolution equations. Next, using this theorem we show that the solution can be expressed as the generalized Mittag-Leffler functions. Also, in special case the solution is reduced to the Volterra μ-function.

Theorem 2.1 *(Schouten-Vanderpol Theorem) Let $F(s)$ and $\phi(s)$ be analytic functions in half plane $\Re(s) > c$, then, the inverse Laplace transform of $F(\phi(s))$ is given by [1]*

$$\mathcal{L}^{-1}\{F(\phi(s)); x\} = \int_0^{\infty} f(t)\mathcal{L}^{-1}\{e^{-t\phi(s)}; x\}dt, \tag{2.6}$$

where $F(s)$ is the Laplace transform $f(x)$

$$F(s) = \int_0^{\infty} f(t)e^{-ts}dt. \tag{2.7}$$

Lemma 2.1 *The Laplace transforms of generalized Mittag-Leffler function (1.1) has the following form [23]*

$$\mathcal{L}[x^{\nu-1}E^{\lambda}_{\mu,\nu}(\omega x^{\mu})](s) = \frac{s^{\lambda\mu-\nu}}{(s^{\mu}-\omega)^{\lambda}}, \tag{2.8}$$

where $\lambda, \mu, \omega \in \mathbb{C}$, $\Re(\nu), \Re(s) > 0$ and $|\frac{\omega}{s^{\mu}}| < 1$.

Theorem 2.2 *The solution of the following autoconvolution equation of the first kind is the generalized Mittag-Leffler function*

$$\rho(xz * x^{\rho-1}) = \gamma(z * z) + \delta(z * x^{\rho}), \tag{2.9}$$

$$x > 0,\ \gamma > 0,\ \rho > -1,\ \delta \in \mathbb{R},$$

where

$$z(x) = \mathcal{M}[\mathcal{L}^{-1}(e^{-\tau\mathcal{L}\{y(x);s\}}); \tau \to \alpha], \tag{2.10}$$

and notation $\mathcal{M}$ is the Mellin transform

$$F(\alpha) = \mathcal{M}\{f(\tau); \alpha\} = \int_0^{\infty} \tau^{\alpha-1}f(\tau)d\tau. \tag{2.11}$$

Proof. First, we evaluate the laplace transform of function $z(x)$ using the Schouten-Vanderpol theorem as

$$\begin{aligned}\mathcal{L}\{z(x); s\} &= \int_0^{\infty} \tau^{\alpha-1}e^{-\tau Y(s)}d\tau \\ &= \mathcal{L}\{\tau^{\alpha-1}; Y(s)\} \\ &= \Gamma(\alpha)Y^{-\alpha}(s).\end{aligned} \tag{2.12}$$

Also, we use the following properties of the Laplace transform

$$\mathcal{L}\{\int_0^t f(t-\xi)g(\xi)d\xi; s\} = \mathcal{L}\{f(t); s\}\mathcal{L}\{g(t); s\}, \tag{2.13}$$

$$\mathcal{L}\{x^n f(x);s\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t);s\}, \quad n\in\mathbb{N}, \tag{2.14}$$

and apply the Laplace transform on (2.9) to get the following Bernoualli differential equation

$$-\rho\frac{d}{ds}[\Gamma(\alpha)Y^{-\alpha}(s)]\frac{\Gamma(\rho)}{s^\rho} = \gamma[\Gamma(\alpha)Y^{-\alpha}(s)]^2 + \delta[\Gamma(\alpha)Y^{-\alpha}(s)]\frac{\Gamma(\rho+1)}{s^{\rho+1}}, \tag{2.15}$$

or equivalently

$$(Y^{-\alpha})'(s) + \frac{\delta}{s}Y^{-\alpha}(s) + \gamma_0 s^\rho (Y^{-\alpha})^2(s) = 0, \tag{2.16}$$

where $\gamma_0 = \frac{\gamma\Gamma(\alpha)}{\Gamma(\rho+1)}$, and Y is the Laplace transform of y. When $\delta \neq 1+\rho$, the general solution of the above equation is

$$Y(s) = \Big(\frac{\rho-\delta+1}{\gamma_0}\Big)^{-\frac{1}{\alpha}} \times \Big(\frac{1}{s^\delta}\Big)^{-\frac{1}{\alpha}} \times \Big(\frac{1}{s^{\rho-\delta+1}-C}\Big)^{-\frac{1}{\alpha}}, \quad C\in\mathbb{R}, \tag{2.17}$$

or equivalently by setting $C=\frac{1}{K}$ for $K\neq 0$ we have

$$Y(s) = \Big(\frac{\delta-1-\rho}{\gamma_0}\Big)^{-\frac{1}{\alpha}} \times \Big(\frac{1}{s^{1+\rho}}\Big)^{-\frac{1}{\alpha}} \times \Big(\frac{K}{s^{\delta-1-\rho}-K}\Big)^{-\frac{1}{\alpha}}, \quad K\in\mathbb{R}. \tag{2.18}$$

When $C=0$, the solution of (2.17) takes the form

$$y(x) = \frac{1}{\Gamma(-\frac{\rho+1}{\alpha})}\Big(\frac{\rho-\delta+1}{\gamma_0}\Big)^{-\frac{1}{\alpha}} x^{-\frac{\rho+1}{\alpha}-1},$$

and for $K=0$,the relation (2.18) gives the trivial solution $y=0$.
Now, by using the relation (2.8) in subcase $\delta < 1+\rho$ of (2.17), we obtain the general solution of (2.9) as follows

$$y(x) = \Big(\frac{1+\rho-\delta}{\gamma_0}\Big)^{-\frac{1}{\alpha}} x^{-\frac{\rho+1}{\alpha}-1} \times E^{-\frac{1}{\alpha}}_{1+\rho-\delta,-\frac{\rho+1}{\alpha}}(Cx^{1+\rho-\delta}), \quad C\in\mathbb{R}, \tag{2.19}$$

and in the subcase $\delta > 1+\rho$, we get

$$y(x) = K^{-\frac{1}{\alpha}}\Big(\frac{\delta-1-\rho}{\gamma_0}\Big)^{-\frac{1}{\alpha}} x^{-\frac{\delta}{\alpha}-1} \times E^{-\frac{1}{\alpha}}_{\delta-1-\rho,-\frac{\delta}{\alpha}}(Kx^{\delta-1-\rho}), \quad K\in\mathbb{R}. \tag{2.20}$$

Corollary 2.1 *In the case $\delta = 1+\rho$, the equation (2.16) gives the solution*

$$Y(s) = \Big(\frac{1}{\gamma_0\ln(s)+K}\Big)^{-\frac{1}{\alpha}}\Big(\frac{1}{s^\delta}\Big)^{-\frac{1}{\alpha}}. \tag{2.21}$$

By putting $K=\gamma_0\ln\alpha_0,\ \alpha_0>0$, we have

$$Y(s) = \Big(\frac{1}{\gamma_0\ln(\alpha_0 s)}\Big)^{-\frac{1}{\alpha}}\Big(\frac{1}{s^\delta}\Big)^{-\frac{1}{\alpha}}, \tag{2.22}$$

$$\delta = 1+\rho,\ \rho>0,$$

which has the general solution

$$y(x) = (\gamma_0)^{\frac{1}{\alpha}}\frac{1}{\alpha_0^{\frac{\rho+1}{\alpha}+1}} \times \mu(\frac{x}{\alpha_0}, -\frac{1}{\alpha}-1, -\frac{\rho+1}{\alpha}-1), \tag{2.23}$$

where μ is the Volterra μ-function [11]

$$\mu(z,\alpha,\beta) = \int_0^\infty \frac{z^{u+\alpha}u^\beta}{\Gamma(\beta+1)\Gamma(u+\alpha+1)}du,$$

$$\Re(\beta) > -1.$$

3 More autoconvolution equations

Now we note another classes of autoconvolution equations.

Theorem 3.1 *For $x>0$, the solution of the following autoconvolution equation*

$$\Big(xy * x^{\mu(\lambda+1)-2}E^{\lambda}_{\mu,\mu(\lambda+1)-1}(-x^\mu)\Big) = (y*y) + \mu\lambda\Big(y * x^{\mu(\lambda+1)-1}E^{\lambda+1}_{\mu,\mu(\lambda+1)}(-x^\mu)\Big), \tag{3.1}$$

is the generalized Mittag-Leffler function, where λ and μ are complex numbers.

Proof. We apply the Laplace transform on the above equation and use the mentioned properties for the Laplace transform (2.13) and (2.14), and Lemma 2.1 to get

$$\frac{1}{s^{\mu-1}(s^\mu+1)^\lambda}Y'(s) + \frac{\mu\lambda}{(s^\mu+1)^{\lambda+1}}Y(s) = -Y^2(s), \tag{3.2}$$

or

$$Y'(s) + \mu\lambda \frac{s^{\mu-1}}{s^{\mu}+1} Y(s) + s^{\mu-1}(s^{\mu}+1)^{\lambda} Y^2(s) = 0, \quad (3.3)$$

that is a Bernoulli differential equation and has the following general solution for $k \neq 0$

$$Y(s) = \mu \frac{1}{(s^{\mu}+1)^{\lambda}} \frac{1}{s^{\mu}+\mu k}. \quad (3.4)$$

Now, by expanding the fraction

$$\frac{1}{s^{\mu}+\mu k}, \quad |\mu k| < |s^{\mu}|,$$

we can easily get

$$Y(s) = \sum_{n=0}^{\infty} (-k)^n \mu^{n+1} \frac{s^{-(n+1)\mu}}{(s^{\mu}+1)^{\lambda}},$$

which by using Lemma 2.1, the relation (3.1) gives the solution

$$y(x) = \sum_{n=0}^{\infty} (-k)^n \mu^{n+1} x^{\mu(\lambda+1+n)-1} \times E^{\lambda}_{\mu,\mu(\lambda+1+n)}(-x^{\mu}). \quad (3.5)$$

Similarly, when $|\mu k| > |s^{\mu}|$ we have

$$Y(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\mu^n k^{n+1}} \frac{s^{\mu n}}{(s^{\mu}+1)^{\lambda}},$$

and therefore

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\mu^n k^{n+1}} x^{\mu(\lambda-n)-1} E^{\lambda}_{\mu,\mu(\lambda-n)}(-x^{\mu}),$$

In the case $s^{\mu} = \mu k$, the relation (3.1) gives the solution

$$y(x) = \frac{1}{2k} x^{\mu\lambda-1} E^{\lambda}_{\mu,\mu\lambda}(-x^{\mu}),$$

or by taking $\mu k = s^{\mu}$

$$y(x) = \frac{1}{2} \mu x^{\mu(\lambda+1)-1} E^{\lambda}_{\mu,\mu(\lambda+1)}(-x^{\mu}).$$

Corollary 3.1 *By setting $k = 0$ and $k = \frac{1}{\mu}$ in relation (3.4), we get the solution of (3.1) as follows*

$$y(x) = \mu x^{\mu(\lambda+1)-1} E^{\lambda}_{\mu,\mu(\lambda+1)}(-x^{\mu}), \quad (3.6)$$

$$y(x) = \mu x^{\mu(\lambda+1)-1} E^{\lambda+1}_{\mu,\mu(\lambda+1)}(-x^{\mu}). \quad (3.7)$$

Here we discuss about two examples of the autoconvolution equation by starting the Mittag-Leffler type function.

Example 3.1 *We consider the generalized Mittag-Leffler function*

$$y = x^{\mu(n+1)} E^{\lambda}_{\mu,\mu(n+1)+1}(x^{\mu}), \quad (3.8)$$

with the Laplace transform

$$Y(s) = \frac{s^{\mu(\lambda-n-1)-1}}{(s^{\mu}-1)^{\lambda}}.$$

We apply the first derivative of $Y(s)$

$$Y'(s) = [\mu(\lambda-n-1)-1] \times \frac{s^{\mu(\lambda-n-1)-2}(s^{\mu}-1)^{\lambda}}{(s^{\mu}-1)^{2\lambda}} - \frac{\lambda\mu(s^{\mu}-1)^{\lambda-1} s^{\mu-1} s^{\mu(\lambda-n-1)-1}}{(s^{\mu}-1)^{2\lambda}}, \quad (3.9)$$

and quadratic product of $Y(s)$

$$Y^2(s) = \frac{s^{2\mu(\lambda-n-1)-2}}{(s^{\mu}-1)^{2\lambda}},$$

to construct the following relation

$$\frac{1}{(s^{\mu}-1)^{\lambda-1}} Y'(s) + \lambda\mu Y^2(s) = \frac{1}{(s^{\mu}-1)^{\lambda-1}} \Big[[\mu(\lambda-n-1)-1]\frac{1}{s} + \lambda\mu s^{\mu(\lambda-n-2)-1}[\frac{1-s^{-\mu(\lambda-n-2)}}{1-s^{-\mu}}]\Big] Y(s),$$

or

$$\frac{1}{(s^{\mu}-1)^{\lambda-1}} Y'(s) + \lambda\mu Y^2(s) = \frac{1}{(s^{\mu}-1)^{\lambda-1}} \Big[[\mu(\lambda-n-1)-1]\frac{1}{s} + \lambda\mu \sum_{k=0}^{\lambda-n-2} s^{\mu(\lambda-n-k-2)-1}\Big] Y(s).$$

At this point, by applying the inverse Laplace transform on the above relation term by term and using Lemma 2.1 and relations (2.13) and (2.13), we deduce that the generalized Mittag-Leffler function (3.8) satisfies the autoconvolution equation

$$(xy * x^{\mu(\lambda-1)-1} E^{\lambda-1}_{\mu,\mu(\lambda-1)}(x^{\mu}))$$

$$= \lambda\mu(y*y)+([1-(\lambda-n-1)\mu]$$
$$(y*x^{\mu(\lambda-1)}E^{\lambda-1}_{\mu,\mu(\lambda-1)+1}(x^{\mu}))$$
$$-\lambda\mu\Big(\sum_{k=0}^{\lambda-n-2} y*x^{\mu(n+k+1)}E^{\lambda-1}_{\mu,\mu(n+k+1)+1}(x^{\mu})\Big). \tag{3.10}$$

Example 3.2 *In the same procedure to previous example, the generalized Mittag-Leffler*

$$y = x^{-n\mu}E^{\lambda}_{\mu,1-n\mu}(x^{\mu}), \tag{3.11}$$

satisfies the autoconvolution equation

$$(xy*x^{\mu(\lambda-1)-1}E^{\lambda-1}_{\mu,\mu(\lambda-1)}(x^{\mu}))$$
$$= \lambda\mu(y*y)+\Big([1-(\lambda+n)\mu]$$
$$(y*x^{\mu(\lambda-1)}E^{\lambda-1}_{\mu,\mu(\lambda-1)+1}(x^{\mu}))\Big)$$
$$-\lambda\mu\Big(\sum_{k=0}^{\lambda+n-2} y*x^{\mu(k-n)}E^{\lambda-1}_{\mu,\mu(k-n)+1}(x^{\mu})\Big). \tag{3.12}$$

4 Application to the Boltzmann equation

In year 1872, Boltzmann introduced a nonlinear evolution equation for description of configuration of particles in a gas. In the modern literature, the Boltzmann equation is often used in a more general sense and refers to any kinetic equation that describes the change of a macroscopic quantity in a thermodynamic system [8, 9, 10, 28]. In this section, we intend to consider an autoconvolution equation for the Boltzmann equation and express its solution with respect to the generalized Mittag-Leffler function. This equation is given in the following form

$$xy(x) = (y*y)(x)+\alpha x^2y'(x). \tag{4.13}$$

In the same manner to previous section, by using the Laplace transform we get

$$Y'(s)+Y^2(s)+\alpha(sY(s))'' = 0, \tag{4.14}$$
$$s\to\infty,\ sY(s)\to 1,$$

or equivalently by setting $Z = sY(s)$ we obtain

$$\alpha s^2Z''+sZ'-Z(1-Z) = 0, \tag{4.15}$$

where $s\to\infty$, $Z(s)\to 1$. At this point, by considering the solution in the form [7]

$$Z = \frac{1}{(s^{-\beta}+1)^2}, \tag{4.16}$$

and applying the following algebraic relations

$$Z' = 2\beta\frac{1}{(s^{-\beta}+1)^3}s^{-\beta-1},$$

$$Z'' = 6\beta^2\frac{1}{(s^{-\beta}+1)^4}s^{-2\beta-2} - 2\beta(\beta+1)\frac{1}{(s^{-\beta}+1)^3}s^{-\beta-2},$$

$$\alpha s^2Z''+sZ' = 6\alpha\beta^2\frac{1}{(s^{-\beta}+1)^4}s^{-2\beta} - 2\beta(\alpha\beta+\alpha-1)\frac{1}{(s^{-\beta}+1)^3}s^{-\beta},$$

$$Z(1-Z) = \frac{1}{(s^{-\beta}+1)^2}-\frac{1}{(s^{-\beta}+1)^4},$$

we rewrite the left hand side of (4.15) as

$$\alpha s^2Z''+sZ'-Z(1-Z) = \frac{6\alpha\beta^2s^{-2\beta}+1}{(s^{-\beta}+1)^4} - \frac{([2\alpha\beta^2+2\beta(\alpha-1)]+1)s^{-\beta}+1}{(s^{-\beta}+1)^3}. \tag{4.17}$$

If $\alpha\beta^2 = -\frac{1}{6}$ and $(\alpha-1)\beta = -\frac{5}{6}$ ($\beta = \frac{1}{2}$, $\alpha = -\frac{2}{3}$ or $\beta = \frac{1}{3}$, $\alpha = -\frac{2}{3}$), then we observe that this equation is identical to zero and the equation is satisfied. This means that there are exactly two solutions for the Boltzmann equation in the form of (4.16). Therefore, the Laplace transform of solution of the Boltzmann equation is

$$Y(s) = \frac{1}{s}\frac{1}{(s^{-\beta}+1)^2}, \qquad \beta = \frac{1}{2},\frac{1}{3}, \tag{4.18}$$

which implies that the solution is

$$y(x) = x^{-2\beta}E^2_{-\beta,1-2\beta}(-x^{-\beta}), \quad \beta = \frac{1}{2},\frac{1}{3}. \tag{4.19}$$

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Shiva Eshaghi has received her B. Sc. and M. Sc. degrees in applied mathematics from Shahrekord University in years 2013 and 2015, respectively. Now, she is teaching mathematics and her research interests are in the areas of applied mathematics including differential equations and fractional calculus.

Alireza Ansari is assistant professor and an academic member at department of applied mathematics, faculty of mathematical sciences, Shahrekord University. He received his Ph.D. degree from University of Guilan in year 2011. Now, he has published more than 30 papers in international journals and conferences. His research areas include fractional differential equation, integral transforms and special functions.

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S. Eshaghi, A. Ansari

معادلات خودپیچش و توابع میتگ-لفلر تعمیم یافته

چکیده:

این مقاله به مطالعه معادلات خود پیچش و توابع میتگ-لفلر تعمیم یافته اختصاص داده شده است. این نوع معادلات شامل جملاتی از تبدیل لاپلاس پیچش یک تابع با خودش است. ما در این مقاله کلاس های جدیدی از معادلات خود پیچش نوع اول را بیان می کنیم و نشان می دهیم که جواب های این نوع معادلات توابع میتگ-لفلر تعمیم یافته هستند. با توجه به تبدیل لاپلاس معکوس، از قضیه سوخوتن واندرپل استفاده می کنیم تا یک معادله خود پیچش را برای توابع میتگ-لفلر تعمیم یافته شامل جملاتی از تبدیلات لاپلاس و ملین تشکیل دهیم. هم چنین در حالت های خاص، جواب هایی از معادلات خود پیچش معرفی شده مربوط به توابع μ- ولترا را نتیجه می گیریم. به علاوه چندین معادله خود پیچش جدید را با اعمال تبدیل لاپلاس از توابع میتگ-لفلر تعمیم یافته بیان می کنیم. سرانجام به عنوان یک کاربرد از معادلات خود پیچش در سیستم های ترمودینامیکی، از تبدیل لاپلاس برای حل کردن معادله بولتزمان استفاده می کنیم و جواب این معادله را درجملاتی از توابع میتگ-لفلر تعمیم یافته به دست می آوریم.