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Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 8, No. 1, 2016 Article ID IJIM-00484, 8 pages
Research Article

Science and Research Branch (IAU)

Fuzzy efficiency: Multiplier and enveloping CCR models

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Received Date: 2014-4-20 Revised Date: 2014-11-18 Accepted Date: 2015-07-01

Abstract

Comparing the performance of a set of activities or organizations under uncertainty environment has been performed by means of Fuzzy Data Envelopment Analysis (FDEA) since the traditional DEA models require accurate and precise performance data. As regards a method for dealing with uncertainty environment, many researchers have introduced DEA models in fuzzy environment. Some of these models are solved by transforming fuzzy models into their crisp counterparts. In this paper applying a fuzzy metric and a ranking function, obtained from it, the multiplier fuzzy CCR model converts to its crisp counterpart. Solving this model yields the optimal solution of fuzzy multiplier model. Moreover, in the following some properties and theorems about mentioned enveloping and multiplier models have been proved.

Keywords : Fuzzy number; Fuzzy DEA; Ranking.

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1 Introduction

Data Envelopment Analysis (DEA) is a very effective method to evaluate the relative efficiency of decision making units (DMUs). As a result of its comprehensive practical use, DEA has been adapted to many fields to deal with problems that have occurred in practice. Since, in some cases, the data of production processes cannot be measured in a precise manner the uncertain theory has played an significant role in DEA. For this reason, the possibility of having available a methodology that permits the analyst to focus on imprecise data becomes a subject of great attention in these situations. To deal with imprecise data, the notion of fuzziness has been introduced. Considering fuzzy inputs-outputs, the efficiency evaluation of DMUs are done by fuzzy data envelopment analysis (FDEA). FDEA is a tool for comparing the performance of a set of activities or organizations under uncertainty environment. By extending to fuzzy environment, the DEA approach is made more powerful for applications. There exist many papers carry out some researches to DEA under fuzzy environment. Kao and Liu [2] developed a procedure to measure the efficiencies of DMUs with fuzzy observations. They formulated a pair of parametric programs to describe that family of crisp DEA models, via which the membership functions of the efficiency measures are derived. Since the efficiency measures are expressed by membership functions rather than by crisp values, more information is provided for management. Guo and Tanaka [3], based on the fundamental CCR model, proposed a fuzzy DEA model to deal with

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the efficiency evaluation problem with the given fuzzy input and output data. Furthermore, they proposed an extension of the fuzzy DEA model to a more general form with considering the relationship between DEA and RA (regression analysis). Using the proposed fuzzy DEA models, the crisp efficiency in CCR model is extended to be a fuzzy number to reflect the inherent uncertainty in real evaluation problems. Saati et al. [4] presented a fuzzy version of CCR model with asymmetrical triangular fuzzy number. They also suggested a procedure for its solution and proposed a ranking method for fuzzy DMUs using presented fuzzy DEA approach. Kao and Liu [5] devised a method to rank the fuzzy efficiency scores without knowing the exact form of the membership functions. Via a skillful modeling technique, the requirement of the membership functions is avoided. The efficiency rankings are consequently determined by solving a pair of nonlinear programs for each DMU. Leon et al. [6] by using some ranking methods based on the comparison of α-cuts developed some fuzzy versions of the the BCC model. The obtained crisp problems can be solved by the usual DEA software. Their approaches can be seen as an extension of the DEA methodology that provides users and practitioners with models which represent some real life processes more appropriately. Lertworasirikul et al. [7] studied the FDEA model of the BCC type (FBCC). They also provided possibility and credibility approaches and compared with an α-level based approach for solving the FDEA models. Using the possibility approach, the relationship between the primal and dual models of FBCC models is revealed and fuzzy efficiency can be constructed. Using the credibility approach, an efficiency value for each DMU is obtained as a representative of its possible range. Wang et al. [8] proposed two new fuzzy DEA models constructed from the perspective of fuzzy arithmetic to deal with fuzziness in input and output data in DEA. These fuzzy DEA models are formulated as linear programming models and can be solved to determine fuzzy efficiencies of a group of DMUs. Wena and Li [9] attempted to extend the traditional DEA models to a fuzzy framework, thus producing a fuzzy DEA model based on credibility measure. They also provided a method of ranking all the DMUs. In the case When the inputs and outputs are all trapezoidal or triangular fuzzy variables, the model can be transformed to linear programming. Zerafat Angiz et al. [12] developed a non-radial model to evaluate DMUs under uncertainty using Fuzzy DEA and to include α-level to the model under fuzzy environment. Wena et al. [10] defined a fuzzy comparison of fuzzy variables and extended the CCR model to be a fuzzy DEA model based on credibility measure. They also proposed a full ranking method in order to rank all the DMUs. In their paper a fuzzy simulation is designed and embedded into the genetic algorithm to establish a hybrid intelligent algorithm since the ranking method involves a fuzzy function. Tlig and Rebai [11] developed DEA models using imprecise data represented by LR fuzzy numbers with different shapes. The resulting FDEA models take the form of fuzzy linear programming and can be solved by the use of some approaches to rank fuzzy numbers.

This paper has focused on FDEA models of the CCR type. We emphasize that when some observations are fuzzy, the efficiencies become fuzzy as well. Thus, the obtained efficiencies are also fuzzy numbers which reflect the inherent ambiguity in evaluation problems under uncertainty. While considering a fuzzy metric and a ranking function, obtained from it, the multiplier fuzzy CCR model converts to its crisp counterpart which can be easily solved. Moreover, some properties and theorems about mentioned envelopment and multiplier models will be proved.

The current article proceeds as follows: In the next section, Preliminaries of fuzzy set are briefly reviewed. Then, in Section 3, Metric for fuzzy numbers will be discussed. In Section 4, fuzzy DEA and fuzzy efficiency score will be introduced. Finally, some conclusions are drawn based on preceding discussion.

2 Preliminaries

In this section give a brief review of essential notions of fuzzy set theory which will be used throughout this paper. Below, we give definitions and notations taken from Bezdek [1], Goetschel and Voxman [15], Zimmermann [16], Dubois and Prade [17] and Zadeh [18].

Definition 2.1 *Let X be the universal set. $\tilde{A}$ is called a fuzzy set in X if $\tilde{A}$ is a set of ordered pairs*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\},$$

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where $\mu_{\tilde{A}}(x)$ *is the membership value of* x *in* $\tilde{A}$.

Definition 2.2 *A convex fuzzy set* $\tilde{A}$ *on* $\Re$ *is a fuzzy number if the following conditions hold:*

(a) Its membership function is piecewise continuous.

(b) There exist only one x_0 *that* $\mu_A(x_0) = 1$.

Definition 2.3 *The support of a fuzzy set* $\tilde{A}$ *is a set of elements in* X *for which* $\mu_{\tilde{A}}(x)$ *is positive, that is,*

$$\text{supp}\tilde{A} = \overline{\{x \in X | \mu_{\tilde{A}}(x) > 0\}}.$$

Definition 2.4 *A fuzzy number* $\tilde{A}$ *is called positive , if* $inf\ supp(A) \geq 0$.

Definition 2.5 *(Generakized Left Right fuzzy number) A GLRFN fuzzy number is of L-R type fuzzy number if there exists reference function L (L for left), (R for right) and* $a_1 \leq a_2 \leq a_3 \leq a_4$ *with*

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a_2 - x}{a_2 - a_1}), & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ R(\frac{x - a_3}{a_4 - a_3}), & a_3 \leq x \leq a_4 \\ 0, & Otherwise \end{cases}$$

$\tilde{A}$ *is denoted by* $(a_1, a_2, a_3, a_4)_{LR}$.

Where L and R are strictly decreasing functions defined on $[0, 1]$ *and satisfying the conditions:*

$$\begin{array}{ll} L(x) = R(x) = 1 & if \quad x \leq 0 \\ L(x) = R(x) = 0 & if \quad x \geq 1 \end{array}$$

For $a_2 = a_3$, we have the classical definition of left right fuzzy numbers (LRFN) of Dubois and Prade [17], a LRFN $\tilde{B}$ is denoted as $\tilde{B} = (b_1, b_2, b_3)_{LR}$. Trapezoidal fuzzy numbers (TrFN) are special cases of GLRFN with $L(x) = R(x) = 1 - x$. Triangular fuzzy numbers (TFN) are also special cases of GLRFN with $L(x) = R(x) = 1 - x$ and $a_2 = a_3$. It should be noted that L_A^{-1} and R_A^{-1} are the inverse of L_A and R_A functions.
A GLRFN $\tilde{A}$ is denoted as $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$ and an $\alpha-$level interval of fuzzy number $\tilde{A}$ as:

$$[\tilde{A}]^\alpha = [A_l(\alpha), A_r(\alpha)] =$$

$$[a_2 - (a_2 - a_1)L_A^{-1}(\alpha), a_3 + (a_4 - a_3)R_A^{-1}(\alpha)]$$

Definition 2.6 *Parametric form of a fuzzy number has been introduced and represented by* $\tilde{A} = (\underline{A}(r), \overline{A}(r))$, *where* $\underline{A}(r)$ *and* $\overline{A}(r)$, $0 \leq r \leq 1$, *satisfying the following requirements:*
1. $\underline{A}(r)$ *is monotonically increasing left continuous function.*
2. $\overline{A}(r)$ *is monotonically decreasing right continuous function.*
3. $\underline{A}(r) \leq \overline{A}(r)$, $0 \leq r \leq 1$.

It should be noted that in this paper we consider a singleton fuzzy number as a LR fuzzy number.

2.1 Metric for fuzzy numbers

Definition 2.7 *Let* $f(x) = (a - b)x + b$ *and* $g(x) = (c - d)x + d$. *The distance of two interval* $[a, b]$ *and* $[c, d]$, $(a \leq b, c \leq d)$ *is denoted by* $d_{TMI}^{(p)}([a, b], [c, d])$ *such that:*

$$d_{TMI}^{(p)}([a, b], [c, d]) = (D_{TMI}^{(p)}([a, b], [c, d]))^{\frac{1}{p}} \quad (2.1)$$

and

$$D_{TMI}^{(p)}([a, b], [c, d]) = \| f(x) - g(x) \|_{L_p}^p \quad (2.2)$$

Where $\| \, . \, \|$ *is the usual norm in the* L_p *space on the* $[0, 1]$ $(p > 1)$.

Definition 2.8 *A distance between two GLRFNs* $\tilde{A}$ *and* $\tilde{B}$ *can be defined as:*

$$d_{TMF}^{(p)}(\tilde{A}, \tilde{B}, s) = (D_{TMF}^{(p)}(\tilde{A}, \tilde{B}, s))^{\frac{1}{p}} \quad (2.3)$$

Such that

$$D_{TMF}^{(p)}(\tilde{A}, \tilde{B}, s) = \frac{\int_0^1 s(\alpha) D_{TMI}^{(p)}([\tilde{A}]^\alpha, [\tilde{B}]^\alpha) d\alpha}{\int_0^1 s(\alpha) d\alpha} \quad (2.4)$$

Here s, *is a weight function such that continuous positive function defined on* $[0, 1]$. *It can be proved that* $d_{TMF}^{(p)}(\tilde{A}, \tilde{B}, s)$ *is a metric on GLRFNs. This distance satisfies the following properties:*

1. *If* $\tilde{A} = \tilde{B} \Longleftrightarrow d_{TMF}^{(p)}(\tilde{A}, \tilde{B}, s) = 0$.
2. $d_{TMF}^{(p)}(\tilde{A}, \tilde{B}, s) = d_{TMF}^{(p)}(\tilde{B}, \tilde{A}, s)$.
3. $d_{TMF}^{(p)}(\tilde{A}, \tilde{B}, s) + d_{TMF}^{(p)}(\tilde{B}, \tilde{C}, s)$
$\geq d_{TMF}^{(p)}(\tilde{A}, \tilde{C}, s)$.

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Proof: For more details about the proofs you can see [13].

Proposition 7. If $\tilde{E} = (a_1, a_2, a_3, a_4)$ and $\tilde{E} = (b_1, b_2, b_3, b_4)$ are two fuzzy numbers and p=1 with $s(\alpha) = 1$

$$\begin{aligned} & d(\tilde{A}, \tilde{B}) =: d^{(1)}_{TMF}(\tilde{A}, \tilde{B}, 1) = \\ & \int_0^1 D^{(1)}_{TMI}([A_l{}^{\alpha}, A_u{}^{\alpha}], [B_l{}^{\alpha}, B_u{}^{\alpha}])d\alpha \\ & = \int_0^1 (\int_0^1 |(1-x)A_u{}^{\alpha} + xA_l{}^{\alpha} \\ & -((1-x)B_u{}^{\alpha} + xB_l{}^{\alpha})|dx)d\alpha \end{aligned} \tag{2.5}$$

Definition 2.9 *The ranking method for two positive fuzzy numbers as it discussed in [14] is as follows:*

$$\tilde{A} \precsim \tilde{B} \quad \Leftrightarrow \quad \gamma^{(p)}_d(\tilde{A}, m) \leq \gamma^{(p)}_d(\tilde{B}, m)$$

$$\tilde{A} = \tilde{B} \quad \Leftrightarrow \quad \gamma^{(p)}_d(\tilde{A}, m) = \gamma^{(p)}_d(\tilde{B}, m)$$

$$\tilde{A} \succsim \tilde{B} \quad \Leftrightarrow \quad \gamma^{(p)}_d(\tilde{A}, m) \geq \gamma^{(p)}_d(\tilde{B}, m)$$

where

$$\gamma^p_d(\tilde{A}, m) = \frac{d^{(p)}_{TMF}(\tilde{A}, m, s)}{d^{(p)}_{TMF}(\tilde{A}, m, s) + d^{(p)}_{TMF}(\tilde{A}, M, s)}, \tag{2.6}$$

such that $d^{(p)}_{TMF}(\tilde{A}, m, s)$ and $d^{(p)}_{TMF}(\tilde{A}, M, s)$ are distances between fuzzy number $\tilde{A}$ and crisp numbers max(M) and min(m), respectively. Also, $m \leq min(supp(\tilde{A}) \cup supp(\tilde{B}))$ and $M \geq max(supp(\tilde{A}) \cup supp(\tilde{B}))$.

Since $d(m, M) = d(\tilde{A}, m) + d(\tilde{A}, M)$ thus, the denominator in (2.6) is ineffective in comparing two fuzzy numbers. Therefore, we give the following definition for comparing two fuzzy numbers.

Definition 2.10 *Considering the ranking method for two positive fuzzy numbers as discussed in [14] we define the following ranking method:*

$$\begin{aligned} \tilde{A} \precsim \tilde{B} \quad &\Leftrightarrow \quad d(\tilde{A}, m) \leq d(\tilde{B}, m), \\ \tilde{A} = \tilde{B} \quad &\Leftrightarrow \quad d(\tilde{A}, m) = d(\tilde{B}, m), \\ \tilde{A} \succsim \tilde{B} \quad &\Leftrightarrow \quad d(\tilde{A}, m) \geq d(\tilde{B}, m) \end{aligned}$$

where $m \leq min(supp(\tilde{A}) \cup supp(\tilde{B}))$.

3 Fuzzy efficiency in DEA

This paper has focused on FDEA models of the CCR type. In this section applying a fuzzy metric and a ranking function, obtained from it, the multiplier fuzzy CCR model converts to its crisp counterpart. Moreover, some properties and theorems about mentioned envelopment and multiplier models will be proved.

We assume there are n decision-making units (DMUs) and each DMU_j(j= 1,...,n) transforms inputs, $\tilde{X}_j$, into outputs, $\tilde{Y}_j$ and for each DMU_j $\tilde{X}_{ij}(i = 1, ..., m)$ and $\tilde{Y}_{rj}(r = 1, ..., s)$ are positive LR fuzzy numbers. Further, observed quantities of inputs and outputs are assumed to be positive. The multiplier form of the CCR model [14] in input orientation for assessing DMU_o is as follows:

$$\begin{aligned} \max \tilde{Z} = \quad & \sum_{r=1}^{s} u_r \tilde{y}_{ro} \\ s.t. \quad & \sum_{i=1}^{m} v_i \tilde{x}_{io} = \tilde{1}, \\ & \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \precsim \tilde{0}, \\ & j = 1, ..., n, \\ & u \geq 0, \quad v \geq 0. \end{aligned} \tag{3.7}$$

Theorem 3.1 *Model (3.7) is always feasible.*

Proof: Let us assume $d(\tilde{x}_{ko}, m) = max_{1\leq i\leq k} \; d(\tilde{x}_{ko}, m)$ thus $\tilde{x}_{ko} = max_{1\leq i\leq k} \; \tilde{x}_{ko}$. Also, assume $\tilde{1} = (\tilde{x}^{(-1)}_{ko}) \otimes (\tilde{x}_{ko})$ be a fuzzy number. This multiplication is defined on basis of the extension principle. Now, let $u^t = (0, ..., 0)$ and $v^t = (0, ..., \tilde{x}^{(-1)}_{ko}, ..., 0)$. Therefore,a feasible solution for model (3.7) is at hand.

Definition 3.1 *$(u^*, v^*)^t$ is an optimal solution of model (3.7) if for every feasible solution such as $(u, v)^t$ for this model we have $d(u^t \tilde{y}_o, m) \leq d(u^{t*} \tilde{y}_o, m)$.*

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Taking into account the proposed ranking method in [13] and the definition (2.10) model (3.7) will be converted into the following one:

$$\begin{aligned} \max \tilde{Z} = & \sum_{r=1}^{s} d(u_r\tilde{y}_{ro}, m) \\ s.t. \quad & \sum_{i=1}^{m} d(v_i\tilde{x}_{io}, m) = d(\tilde{1}, m), \\ & d(\sum_{r=1}^{s} u_r\tilde{y}_{rj} - \sum_{i=1}^{m} v_i\tilde{x}_{ij}, m) \leq d(\tilde{0}, m), \\ & j = 1, ..., n, \\ & u \geq 0, \quad v \geq 0. \end{aligned} \tag{3.8}$$

For simplification of notation hereafter consider $\sum_{r=1}^{s} u_r\tilde{y}_{rj} - \sum_{i=1}^{m} v_i\tilde{x}_{ij} = \tilde{k}_j$ for all j. Note that $\tilde{1} = (\alpha, 1, \alpha)$ and $\tilde{0} = (-\varepsilon, 1, \varepsilon)$.

Theorem 3.2 *Model (3.8) is always feasible.*

Proof: Let us assume $d(\tilde{x}_{ko}, m) = max_{1\leq i\leq k} \; d(\tilde{x}_{ko}, m)$ thus $\tilde{x}_{ko} = max_{1\leq i\leq k} \; \tilde{x}_{ko}$. Also, assume $\tilde{1} = (\tilde{x}_{ko}^{(-1)}) \otimes (\tilde{x}_{ko})$ be a fuzzy number. This multiplication is defined on basis of the extension principle. Now, let $u^t = (0, ..., 0)$ and $v^t = (0, ..., \tilde{x}_{ko}^{(-1)}, ..., 0)$. Therefore, a feasible solution for model (3.8) is at hand.

Theorem 3.3 *An optimal solution of model (3.7) is the optimal solution of model (3.8) as well and vice versa.*

Proof: Let us assume (u^{t*}, v^{t*}) is an optimal solution of model (3.8) thus $(u^{t*}, v^{t*}) \in S_2$ hence for all $(u^t, v^t) \in S_2$, $d(u^t\tilde{y}_o, m) \leq d(u^{t*}\tilde{y}_o, m)$. Since $(u^{t*}, v^{t*}) \in S_2$, according to the definition (2.10) $(u^{t*}, v^{t*}) \in S_1$. Now, for all $(u^t, v^t) \in S_2$ because $(u^t, v^t) \in S_1$ therefore $d(u^t\tilde{y}_o, m) \leq d(u^{t*}\tilde{y}_o, m)$ thus, (u^{t*}, v^{t*}) is optimal for model (3.7). The proof of the other part is identical.
Considering proposition (*2.1*) model (3.8) will be converted into the following model:

$$\begin{aligned} \max \quad & t \\ s.t. \quad & \sum_{r=1}^{s} u_r\bar{y}_{ro} = t, \\ & \sum_{i=1}^{m} v_i\bar{x}_{io} = \bar{1}, \\ & \bar{k}_j \leq \bar{q}, \quad j = 1, ..., n, \\ & u \geq 0, \quad v \geq 0. \end{aligned} \tag{3.9}$$

in which

$$\bar{y}_{ro} = \int_0^1 \int_0^1 |y_{ro}^l x + y_{ro}^u(1-x) - m| dx d\alpha,$$

$r = 1, ..., s,$

$$\bar{x}_{ro} = \int_0^1 \int_0^1 |x_{ro}^l x + x_{ro}^u(1-x) - m| dx d\alpha,$$

$i = 1, ..., m,$

$$\bar{k}_j = \int_0^1 \int_0^1 |k_j^l x + k_j^u(1-x) - m| dx d\alpha,$$

$j = 1, ..., n.$

$$\bar{q} = \int_0^1 \int_0^1 |0^l x + 0^u(1-x) - m| dx d\alpha,$$

$j = 1, ..., n.$

$$\bar{1} = \int_0^1 \int_0^1 |1^l x + 1^u(1-x) - m| dx d\alpha.$$

Theorem 3.4 *Model (3.9) is always feasible.*

Proof: let $u^t = (0, ..., 0)$ and $v^t = (0, ..., x_{ko}, ..., 0)$. Therefore, a feasible solution for model (3.7) is at hand.
Regarding to what has been mentioned above the optimal solution of model (3.9) is the optimal solution of model (3.7); therefore, by solving this linear and feasible model the optimal solution of model (3.7) is at hand.

Definition 3.2 *Let $(u, v)^t$ be an optimal solution of model (3.9) by substituting this solution in the objective function of model (3.7), $\tilde{Z}_o^* = u_1^*\tilde{y}_{1o} \oplus ... \oplus u_1^*\tilde{y}_{1o}$ will be acquired. In regard of the extension principle, $\tilde{Z}_o^*$ is a fuzzy number which is equal to the **fuzzy efficiency** of DMU_o.*

As a result, in data envelopment analysis with fuzzy inputs and outputs the efficiency measure of a DMU has been obtained a fuzzy number. It should be noted that this is the significant feature of the proposed method.

Definition 3.3 *DMU_o is called Pareto-efficient if there exists a solution for the multiplier model for which $d(u^{t*}\tilde{y}_o) = 1$ and $(u^{t*}, v^{t*}) > 0$.*

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Now, consider model (3.7) we consider its corresponding dual as follows:

$$
\begin{aligned}
\min \quad & \tilde{\theta} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \precsim \tilde{\theta}\tilde{x}_{io}, \quad i = 1, ..., m, \\
& \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \succsim \tilde{y}_{ro}, \quad r = 1, ..., s, \\
& \lambda_j \geq 0, \quad j = 1, ..., n.
\end{aligned} \tag{3.10}
$$

Theorem 3.5 *Model (3.10) is always feasible and its optimal value is bounded.*

Proof: Let $\tilde{\theta} = \tilde{1}, \lambda_j = 0$ for all j except o and $\lambda_o = 1$ where $\tilde{1} = (\alpha, 1, \alpha)_{LR}$. According to the bundles of input-output constraints of this model;

$$\tilde{x}_{io} \precsim \tilde{1}.\tilde{x}_{io} \Leftrightarrow d(\tilde{x}_{io}, m) \leq d(\tilde{1}.\tilde{x}_{io}, m)$$

$$\tilde{y}_{ro} \succsim \tilde{y}_{ro} \Leftrightarrow d(\tilde{y}_{ro}, m) \geq d(\tilde{y}_{ro}, m)$$

therefore, a feasible solution of the model is at hand. On the other had due to the input constraints

$$\tilde{0} \precsim \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \precsim \tilde{\theta}\tilde{x}_{io}, \quad i = 1, ..., m.$$

Since the output vector is always assumed to be semi-positive then we will come to the conclusion that;

$$\tilde{0} \prec \tilde{\theta}^* \precsim \tilde{1},$$

Where $\tilde{0} = (0, 0, \varepsilon)_{LR}$. Thus, this model has bounded optimal value.

Theorem 3.6 *For all $(\tilde{\theta}, \lambda)^t \in S_1$ and $(u, v)^t \in S_2$, by assuming $\tilde{1} = (\alpha, 1, \alpha)$, $u^t\tilde{Y}_o \precsim \tilde{\theta}$. Where S_1 and S_2 are feasible regions of model (3.7) and (3.10), respectively.*

Proof: Let $(\tilde{\theta}, \lambda)^t \in S_1$ and $(u, v)^t \in S_2$ be two feasible solutions of models (3.7) and (3.10), respectively. Considering model (3.10) $v^t\tilde{x}_o = \tilde{1}$ thus $\tilde{\theta}v^t\tilde{x}_o = \tilde{\theta}\tilde{1}$. As a result, since

$$d(\tilde{\theta}, m) = d(\tilde{\theta}\tilde{1}, m) \Leftrightarrow \tilde{\theta} = \tilde{\theta}\tilde{1}, \tag{3.11}$$

thus $v^t\tilde{\theta}\tilde{x}_o = \tilde{\theta}$. Moreover, $u^t\tilde{y} - v^t\tilde{x} \precsim \tilde{0}$ therefore;

$$d(u^t\tilde{y}\lambda - v^t\tilde{x}\lambda, m) \leq d(\tilde{0}, m)$$

$$d(u^t\tilde{y}\lambda - v^t\tilde{x}\lambda, m) - d(\tilde{0}, m) \leq 0, \tag{3.12}$$

on the other hand

$$d(u^t\tilde{y}\lambda - v^t\tilde{x}\lambda, m) - d(\tilde{0}, m)$$

$$\geq d(u^t\tilde{y}\lambda, m) - d(v^t\tilde{x}\lambda, m)$$

hence;

$$d(u^t\tilde{y}\lambda, m) - d(v^t\tilde{x}\lambda, m) \leq 0. \tag{3.13}$$

Also, by considering model (3.7) $\lambda\tilde{y} \succsim \tilde{y}_o$ thus $\lambda u^t\tilde{y} \succsim u^t\tilde{y}_o$ therefore;

$$d(u^t\tilde{y}\lambda, m) \geq d(u^t\tilde{y}_o, m). \tag{3.14}$$

Moreover, $\tilde{x}\lambda \precsim \tilde{\theta}\tilde{x}_o$ thus $v^t\tilde{x}\lambda \precsim v^t\tilde{\theta}\tilde{x}_o$ therefore;

$$d(v^t\tilde{x}\lambda, m) \geq d(v^t\tilde{\theta}\tilde{x}_o, m),$$

which results:

$$d(v^t\tilde{x}\lambda, m) - d(v^t\tilde{\theta}\tilde{x}_o, m) \geq 0. \tag{3.15}$$

In regard of expression (3.14) and (3.15) we have:

$$d(v^t\tilde{\theta}\tilde{x}_o, m) - d(v^t\tilde{x}\lambda, m) + d(u^t\tilde{y}\lambda, m) \geq d(u^t\tilde{y}_o, m). \tag{3.16}$$

Furthermore, considering expression (3.11) and (3.13):

$$d(\tilde{\theta}) \geq d(v^t\tilde{\theta}\tilde{x}_o) + d(u^t\tilde{y}\lambda) - d(v^t\tilde{x}\lambda).$$

Consequently;

$$d(\tilde{\theta}) \geq d(u^t\tilde{y}_o, m).$$

Definition 3.4 *$(\tilde{\theta}^*, \lambda^*)^t$ is an optimal solution of model (3.10) if for every feasible solution such as $(\theta, \lambda)^t$, from this model we have: $d(\tilde{\theta}, m) \leq d(\tilde{\theta}^*\tilde{y}_o, m)$.*

Theorem 3.7 *Let $(\tilde{\theta}, \lambda)^t \in S_1$ and $(u, v)^t \in S_2$ be two feasible solutions of models (3.7) and (3.10) respectively and $u^tY_o = \tilde{\theta}$, then these two feasible solutions are optimal for their corresponding models.*

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Proof: Let $(\tilde{\theta}, \lambda)^t \in S_1$, $(v^*, u^*)^t \in S_2$ and $u^{t*}\tilde{Y}_o = \tilde{\theta}$ thus $d(u^{t*}Y_o, m) = d(\tilde{\theta}, m)$. According to Theorem (3.4) for all $(u, v) \in S_2$, $u^t\tilde{y}_o \leq \tilde{\theta}$ and $d(u^t\tilde{y}_o, m) \leq d(\tilde{\theta}, m)$ therefore $d(u^t\tilde{y}_o, m) \leq d(u^{t*}\tilde{Y}_o, m)$ thus (v^{t*}, u^{t*}) is an optimal solution for model (3.7). Now, let $(\tilde{\theta}^*, \lambda^*)^t \in S_1$, $(v, u)^t \in S_2$ and $u^t\tilde{Y}_o = \tilde{\theta}^*$ thus $d(u^t\tilde{Y}_o, m) = d(\tilde{\theta}^*, m)$. According to Theorem (3.4) for all $(\tilde{\theta}, \lambda)^t \in S_1$, $\tilde{\theta} \geq u^t\tilde{y}_o$ and $d(\theta, m) \geq d(u^t\tilde{y}_o, m)$ therefore $d(\tilde{\theta}^*, m) \leq d(\tilde{\theta}, m)$ thus $(\theta^*, \lambda^*)^t$ is an optimal solution for model (3.10).

In model (3.7) consider the bundles of input and output constraints. By introducing input excess and output shortfall those constraints convert to the following equalities:

$$\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} + \tilde{s}_i^- = \tilde{\theta}\tilde{x}_{io}, \quad i = 1, ..., m,$$

$$\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \tilde{s}_r^+ = \tilde{y}_{ro}, \quad r = 1, ..., s.$$

in which

$$\lambda_j \geq 0, \quad j = 1, ..., n, \quad \tilde{s}_i^- \geq \tilde{0},$$

$$i = 1, ..., m, \quad \tilde{s}_r^+ \geq \tilde{0}, \quad r = 1, ..., s.$$

Definition 3.5 *DMU_o is called Pareto-efficient if $d(\tilde{\theta}^*, m) = 1$ and for each solution $\tilde{s}^- = \tilde{0}$ and $\tilde{s}^+ = \tilde{0}$ where $\tilde{0} = (0, 0, \varepsilon)$.*

4 Conclusion

Data Envelopment Analysis (DEA) is recognized as a modern approach to the assessment of performance of a set of homogeneous DMUs that use identical sources to produce identical outputs. Recently several approaches are introduced for evaluating DMUs with uncertain data since DEA commonly is used with precise data. Fuzzy Data Envelopment Analysis (FDEA) is an mathematical approach which compares the performance of a set of activities under uncertainty environment. The purpose of this paper is to develop a new model to evaluate DMUs under uncertainty using Fuzzy DEA. In this paper the frequently used DEA model, the CCR model, is used to obtain fuzzy efficiency. Since, We emphasize that when some observations are fuzzy, the efficiencies become fuzzy as well. Considering a fuzzy metric and a ranking function, obtained from it, the multiplier fuzzy CCR model converts to its crisp counterpart. Solving this model yields the optimal solution of fuzzy multiplier model. The significant feature of this model is that it can compute fuzzy efficiency through solving a crisp model. Moreover, some properties and theorems about mentioned enveloping and multiplier models have been proved. Although the fuzzy efficiency has been obtained while the FCCR model has been converted into its crisp counterpart, the lack of fuzzy perception in this method is felt. Also, it should be noted that utilizing a ranking with large equivalence classes is a weak point that can be considered for further investigations.

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Int. J. of Industrial Mathematics **Vol. 8, No. 1** (2016)

Fuzzy efficiency: Multiplier and enveloping CCR models

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کارایی فازی: مدلهای پوششی ومضربی CCR

چکیده:

قیاس عملکرد یک مجموعه ازفعالیتها وسازمانهاکه درشرایط غیرقطعی قراردارند بوسیله تحلیل پوششی دادههای فازی انجام میشود. زیرامدلهای کلاسیک تحلیل پوششی داده ها به داده های دقیق وقطعی برای اینکارنیازدارند. برای معرفی یک روش برای انجام ارزیابی درشرایط نادقیق بسیاری ازمحققان مدلهای تحلیلپوششی دادهها را درمحیط فازی مطرح نموده اند. بسیاری ازاین مدلهای فازی باتبدیل به معادل قطعی حل میشوند. دراین مقاله با بکاربردن مترفازی وتابع رتبهبندی مدل مضربی CCR به معادل قطعی آن تبدیل شده است. باحل این مدل جواب بهین مدل مضربی فازی به دست می آید. درادامه برخی خواص درمورد مدلهای پوششی ومضربی معرفی شده ذکرمی شود. طول بلوک های مینیمال و ماکسیمال مساوی فاصله روی لگاریتم – لگاریتم مقیاس درمقابل تابع نوسانی، بر واریانس و اریبی تجزیه بلوکی آنالیز نوسانی روند زدایی تاثیر می گذارد. استفاده ازشبیه سازی شبه مونت کارلو وتجزیه چولکسی، برای حداقل وحداکثربلوک ها می باشد که حداقل مجموع میانگین مربع خطا در توان (هورست) را درآن بکاربرده ایم.

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