



Bessel multipliers on the tensor product of Hilbert C^* -modules

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Abstract

In this paper, we first show that the tensor product of a finite number of standard g-frames (resp. fusion frames, frames) is a standard g-frame (resp. fusion frame, frame) for the tensor product of Hilbert C^* -modules and vice versa, then we consider tensor products of g-Bessel multipliers, Bessel multipliers and Bessel fusion multipliers in Hilbert C^* -modules. Moreover, we obtain some results for the tensor product of duals using Bessel multipliers.

Keywords : G-frames; Bessel multipliers; tensor products; Hilbert C^* -modules.

1 Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [7] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [6]. Frames are very useful in characterization of function spaces and other fields of applications such as filter bank theory, sigma-delta quantization, signal and image processing and wireless communications. Fusion frames [5] and g-frames [23] are important generalizations of frames.

Hilbert C^* -modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^* -algebra rather than in the field of complex numbers. Hilbert C^* -modules are used in the study of locally compact quantum groups, completely positive maps between C^* -algebras, non-commutative geometry and KK-theory.

Frank and Larson presented a general approach to the frame theory in Hilbert C^* -modules (see [8]).

Also A. Khosravi and B. Khosravi introduced fusion frames and g-frames in Hilbert C^* -modules (see [12]).

Bessel multipliers in Hilbert spaces were introduced by Balazs in [3]. Bessel fusion multipliers and g-Bessel multipliers in Hilbert spaces were introduced in [17] and [21], respectively. Also multipliers were introduced for p -Bessel sequences in Banach spaces (see [22]). Recently the present author and A. Khosravi generalized Bessel multipliers, g-Bessel multipliers and Bessel fusion multipliers to Hilbert C^* -modules (see [15]).

Tensor products of frames, fusion frames and g-frames in Hilbert spaces have been studied by some authors recently, see [4, 11, 13]. Also tensor products of g-frames were considered in Hilbert C^* -modules, see [11, 12, 10, 20]. Tensor products have important applications, for example tensor products are useful in the approximation of multi-variate functions of combinations of uni-variate ones. In this paper, we investigate tensor products of g-frames, fusion frames and frames in Hilbert C^* -modules and we consider their multipliers.

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2 Frames, fusion frames and g-frames in Hilbert C^* -modules

Suppose that \mathfrak{A} is a C^* -algebra and E is a left \mathfrak{A} -module such that the linear structures of \mathfrak{A} and E are compatible. E is a pre-Hilbert \mathfrak{A} -module if E is equipped with an \mathfrak{A} -valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathfrak{A}$, such that

- (i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;
- (ii) $\langle ax, y \rangle = a \langle x, y \rangle$, for each $a \in \mathfrak{A}$ and $x, y \in E$;
- (iii) $\langle x, y \rangle = \langle y, x \rangle^*$, for each $x, y \in E$;
- (iv) $\langle x, x \rangle \geq 0$, for each $x \in E$ and if $\langle x, x \rangle = 0$, then $x = 0$.

For each $x \in E$, we define $|x| = \langle x, x \rangle^{\frac{1}{2}}$ and $\|x\| = \|\langle x, x \rangle\|$

12. If E is complete with $\|\cdot\|$, it is called a *Hilbert \mathfrak{A} -module* or a *Hilbert C^* -module* over \mathfrak{A} . We call $\mathcal{Z}(\mathfrak{A}) = \{a \in \mathfrak{A} : ab = ba, \forall b \in \mathfrak{A}\}$, the *center* of \mathfrak{A} . Let E_1 and E_2 be Hilbert \mathfrak{A} -modules. The operator $T : E_1 \rightarrow E_2$ is called *adjointable* if there exists an operator $T^* : E_2 \rightarrow E_1$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$, for each $x \in E_1$ and $y \in E_2$. Every adjointable operator $T : E_1 \rightarrow E_2$ is bounded and \mathfrak{A} -linear (that is, $T(ax) = aT(x)$ for each $x \in E_1$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from E_1 into E_2 by $\mathfrak{L}_{\mathfrak{A}}(E_1, E_2)$. Note that $\mathfrak{L}_{\mathfrak{A}}(E_1, E_1)$ is a C^* -algebra which is denoted by $\mathfrak{L}_{\mathfrak{A}}(E_1)$, for more details see [16].

A Hilbert \mathfrak{A} -module E is *finitely generated* if there exists a finite set $\{x_1, \dots, x_n\} \subseteq E$ such that every element $x \in E$ can be expressed as an \mathfrak{A} -linear combination $x = \sum_{i=1}^n a_i x_i, a_i \in \mathfrak{A}$. A Hilbert \mathfrak{A} -module E is *countably generated* if there exists a countable set $\{x_i\}_{i \in I} \subseteq E$ such that E equals the norm-closure of \mathfrak{A} -linear hull of $\{x_i\}_{i \in I}$.

Let E be a Hilbert \mathfrak{A} -module. A family $\mathcal{F} = \{f_i\}_{i \in I} \subseteq E$ is a *frame* for E , if there exist real constants $0 < A \leq B < \infty$, such that for each $x \in E$,

$$A \cdot \langle x, x \rangle \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq B \cdot \langle x, x \rangle, \quad (2.1)$$

i.e., there exist real constants $0 < A \leq B < \infty$, such that the series $\sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle$ converges

in the ultraweak operator topology to some element in the universal enveloping Von Neumann algebra of \mathfrak{A} such that the inequality (2.1) holds, for each $x \in E$. The numbers A and B are called the lower and upper bound of the frame, respectively. In this case we call it an (A, B) *frame*. If only the second inequality is required, we call it a *Bessel sequence*. If the sum in (2.1) converges in norm, the frame is called *standard*. If $\mathcal{F} = \{f_i\}_{i \in I}$ is a standard Bessel sequence, then the operator $S_{\mathcal{F}}$ is defined on E by $S_{\mathcal{F}}x = \sum_{i \in I} \langle x, f_i \rangle f_i$. $S_{\mathcal{F}}$ is an adjointable and positive operator and if \mathcal{F} is a standard frame, then $S_{\mathcal{F}}$ is invertible. For more results about frames in Hilbert C^* -modules, see [8, 1].

A closed submodule M of E is *orthogonally complemented* if $E = M \oplus M^{\perp}$. In this case $\pi_M \in \mathfrak{L}_{\mathfrak{A}}(E, M)$, where $\pi_M : E \rightarrow M$ is the projection onto M .

Suppose that $\{\omega_i : i \in I\} \subseteq \mathfrak{A}$ is a family of weights, i.e., each ω_i is a positive, invertible element from the center of \mathfrak{A} , and $\{W_i : i \in I\}$ is a family of orthogonally complemented submodules of E . Then $\{(W_i, \omega_i)\}_{i \in I}$ is a *fusion frame* if there exist positive numbers A and B such that

$$A \cdot \langle x, x \rangle \leq \sum_{i \in I} \omega_i^2 \langle \pi_{W_i}(x), \pi_{W_i}(x) \rangle \leq B \cdot \langle x, x \rangle,$$

for each $x \in E$. If we only require to have the upper bound, then $\{(W_i, \omega_i)\}_{i \in I}$ is called a *Bessel fusion sequence* with upper bound B .

Let $\{E_i\}_{i \in I}$ be a sequence of Hilbert \mathfrak{A} -modules. A sequence $\Lambda = \{\Lambda_i \in \mathfrak{L}_{\mathfrak{A}}(E, E_i) : i \in I\}$ is called a *g-frame* for E with respect to $\{E_i : i \in I\}$ if there exist real constants $A, B > 0$ such that for each $x \in E$,

$$A \cdot \langle x, x \rangle \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \leq B \cdot \langle x, x \rangle.$$

If only the second-hand inequality is required, then Λ is called a *g-Bessel sequence*. Standard g-frames and fusion frames are defined similar to frames.

If $W = \{(W_i, \omega_i)\}_{i \in I}$ is a standard Bessel fusion sequence, then the operator $S_W : E \rightarrow E$ which is defined by $S_W x = \sum_{i \in I} \omega_i^2 \pi_{W_i} x$ is adjointable and called the *operator* of W . For a standard g-Bessel sequence Λ , the operator $S_{\Lambda} : E \rightarrow E$ which is defined by $S_{\Lambda}(x) = \sum_{i \in I} \Lambda_i^* \Lambda_i(x)$ is adjointable and it is called the *operator* of Λ . If Λ is a standard (A, B) g-frame, then $A \cdot Id_E \leq S_{\Lambda} \leq$

$B.Id_E$. For more results about fusion frames and g-frames in Hilbert C^* -modules, see [12, 24]. Also note that fusion frames have been introduced in Hilbert modules over pro- C^* -algebras (see [2]).

In this paper all C^* -algebras are unital and Hilbert C^* -modules are finitely or countably generated. All frames, fusion frames, g-frames and Bessel sequences are standard.

Throughout this paper I and I_k , for each $1 \leq k \leq n$, are subsets of \mathbb{N} . \mathfrak{A}_k is a unital C^* -algebra, E , E_k and $E_{i(k)}$ are finitely or countably generated Hilbert C^* -modules, for each $k \in \{1, \dots, n\}$ and $i(k) \in I_k$.

3 Tensor products of Bessel multipliers

First we recall the definitions of Bessel multipliers, g-Bessel multipliers and Bessel fusion multipliers from [15].

As usual $\ell^\infty(I, \mathfrak{A})$ is the set $\left\{ \{a_i\}_{i \in I} \subseteq \mathfrak{A} : \sup\{\|a_i\| : i \in I\} < \infty \right\}$, and in this note m is always a sequence $\{m_i\}_{i \in I} \in \ell^\infty(I, \mathfrak{A})$ with $m_i \in \mathcal{Z}(\mathfrak{A})$, for each $i \in I$. Each sequence with these properties is called a *symbol*.

Definition 3.1 Let E_1 and E_2 be Hilbert \mathfrak{A} -modules, and let $\mathcal{F} = \{f_i\}_{i \in I} \subseteq E_1$ and $\mathcal{G} = \{g_i\}_{i \in I} \subseteq E_2$ be standard Bessel sequences. The operator $S_{m\mathcal{G}\mathcal{F}} : E_1 \rightarrow E_2$ defined by $S_{m\mathcal{G}\mathcal{F}}(x) = \sum_{i \in I} m_i \langle x, f_i \rangle g_i$, is adjointable and it is called the Bessel multiplier for the Bessel sequences \mathcal{F} and \mathcal{G} .

Recall from Example 3.1 in [12] that if $W = \{(W_i, \omega_i)\}_{i \in I}$ is a standard Bessel fusion sequence (resp. standard fusion frame) for E , then $\Lambda_W = \{\omega_i \pi_{W_i}\}_{i \in I}$ is a standard g-Bessel sequence (resp. standard g-frame) for E with respect to $\{W_i\}_{i \in I}$.

Definition 3.2 Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ be standard g-Bessel sequences for E with respect to $\{E_i\}_{i \in I}$. Then the operator $S_{m\Gamma\Lambda} : E \rightarrow E$ which is defined by $S_{m\Gamma\Lambda}(x) = \sum_{i \in I} m_i \Gamma_i^* \Lambda_i(x)$ is adjointable and it is called the g-Bessel multiplier for the g-Bessel sequences Λ and Γ . Also if $W = \{(W_i, \omega_i)\}_{i \in I}$ and $V = \{(V_i, v_i)\}_{i \in I}$ are standard Bessel fusion sequences for E , we call the operator $S_{mVW}(x) =$

$S_{m\Lambda_V \Lambda_W}(x) = \sum_{i \in I} m_i v_i \omega_i \pi_{V_i} \pi_{W_i}(x)$, the Bessel fusion multiplier for W and V .

Recall that if \mathfrak{A}_k is a C^* -algebra, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n \mathfrak{A}_k$ is a C^* -algebra with the spatial norm and for each $a_k \in \mathfrak{A}_k$, we have $\|a_1 \otimes \dots \otimes a_n\| = \prod_{k=1}^n \|a_k\|$. The multiplication and involution on simple tensors are defined by $(\otimes_{k=1}^n a_k)(\otimes_{k=1}^n b_k) = \otimes_{k=1}^n (a_k b_k)$ and $(\otimes_{k=1}^n a_k)^* = \otimes_{k=1}^n a_k^*$, respectively. As we know if $a_k \geq 0$, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n a_k \geq 0$.

Now if E_k is a Hilbert \mathfrak{A}_k -module, for each $1 \leq k \leq n$, then the (Hilbert C^* -module) tensor product $\otimes_{k=1}^n E_k = E_1 \otimes \dots \otimes E_n$ is a Hilbert $(\otimes_{k=1}^n \mathfrak{A}_k)$ -module. The module action and inner product for simple tensors are defined by

$$\begin{aligned} (\otimes_{k=1}^n a_k)(\otimes_{k=1}^n x_k) &= (a_1 x_1) \otimes \dots \otimes (a_n x_n) \\ &= \otimes_{k=1}^n (a_k x_k), \end{aligned}$$

and

$$\begin{aligned} \langle \otimes_{k=1}^n x_k, \otimes_{k=1}^n y_k \rangle &= \langle x_1, y_1 \rangle \otimes \dots \otimes \langle x_n, y_n \rangle \\ &= \otimes_{k=1}^n \langle x_k, y_k \rangle, \end{aligned}$$

respectively, where $a_k \in \mathfrak{A}_k$ and $x_k, y_k \in E_k$. If U_k is an adjointable operator on E_k , then the tensor product $\otimes_{k=1}^n U_k$ is an adjointable operator on $\otimes_{k=1}^n E_k$. Also $(\otimes_{k=1}^n U_k)^* = \otimes_{k=1}^n U_k^*$ and $\|\otimes_{k=1}^n U_k\| = \prod_{k=1}^n \|U_k\|$. Note that if M_k is an orthogonally complemented submodule of E_k , for each $1 \leq k \leq n$, then it is easy to see that $\otimes_{k=1}^n M_k$ is an orthogonally complemented submodule of $\otimes_{k=1}^n E_k$ and $\pi_{\otimes_{k=1}^n M_k} = \otimes_{k=1}^n \pi_{M_k}$. For more results, see [19, 16].

In this paper $\mathcal{F}^{(k)} = \{f_{i(k)}\}_{i(k) \in I_k}$ and $\mathcal{G}^{(k)} = \{g_{i(k)}\}_{i(k) \in I_k}$ are sequences in E_k and $\otimes_{k=1}^n \mathcal{F}^{(k)}$ is defined by $\{f_{i(1)} \otimes \dots \otimes f_{i(n)}\}_{(i(1), \dots, i(n)) \in (I_1 \times \dots \times I_n)}$. $\Phi^{(k)} = \{\Lambda_{i(k)} \in \mathfrak{L}_{\mathfrak{A}_k}(E_k, E_{i(k)})\}_{i(k) \in I_k}$, $\Psi^{(k)} = \{\Gamma_{i(k)} \in \mathfrak{L}_{\mathfrak{A}_k}(E_k, E_{i(k)}) : i(k) \in I_k\}$, $\mathcal{W}^{(k)} = \{(W_{i(k)}, \omega_{i(k)})\}_{i(k) \in I_k}$, $\mathcal{V}^{(k)} = \{(V_{i(k)}, v_{i(k)}) : i(k) \in I_k\}$, where $W_{i(k)}$ and $V_{i(k)}$ are orthogonally complemented submodules of E_k and $\omega_{i(k)}$ and $v_{i(k)}$ are weights in \mathfrak{A}_k , for each $1 \leq k \leq n$. $\otimes_{k=1}^n \Phi^{(k)}$ and $\otimes_{k=1}^n \mathcal{W}^{(k)}$ are

$$\begin{aligned} &\{\Lambda_{i(1)} \otimes \dots \otimes \Lambda_{i(n)} \in \\ &\mathfrak{L}_{(\mathfrak{A}_1 \otimes \dots \otimes \mathfrak{A}_n)}(\otimes_{k=1}^n E_k, E_{i(1)} \otimes \dots \otimes E_{i(n)}) \\ &, (i(1), \dots, i(n)) \in (I_1 \times \dots \times I_n)\}, \end{aligned}$$

$$\begin{aligned} &\{(W_{i(1)} \otimes \dots \otimes W_{i(n)}, \omega_{i(1)} \otimes \dots \otimes \omega_{i(n)}) \\ &: (i(1), \dots, i(n)) \in (I_1 \times \dots \times I_n)\}, \end{aligned}$$

respectively. Also $m^{(k)} = \{m_{i(k)}\}_{i(k) \in I_k}$ is a symbol in $\ell^\infty(I_k, \mathfrak{A}_k)$ and $\otimes_{k=1}^n m^{(k)}$ is the set $\{m_{i(1)} \otimes \dots \otimes m_{i(n)}\}_{(i(1), \dots, i(n)) \in (I_1 \times \dots \times I_n)}$.

The following theorem is a generalization of [13, Theorem 2.1 (i)] to Hilbert C^* -modules and also generalizes the results obtained for tensor products of g -frames in [12], [20] and [10].

Theorem 3.1 (i) *If $\Phi^{(k)}$ is a g -Bessel sequence, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n \Phi^{(k)}$ is a g -Bessel sequence. Moreover, $\Phi^{(k)}$ is a g -frame, for each $1 \leq k \leq n$ if and only if $\otimes_{k=1}^n \Phi^{(k)}$ is a g -frame.*

(ii) *If $\Phi^{(k)}$'s and $\Psi^{(k)}$'s are g -Bessel sequences, then the operator $S_{(\otimes_{k=1}^n m^{(k)}) (\otimes_{k=1}^n \Psi^{(k)}) (\otimes_{k=1}^n \Phi^{(k)})}$ is well-defined and is equal to $\otimes_{k=1}^n S_{m^{(k)} \Psi^{(k)} \Phi^{(k)}}$.*

Proof. (i) *It is enough to prove the theorem for $n = 2$. Let B_1 and B_2 be upper bounds of $\Phi^{(1)}$ and $\Phi^{(2)}$, respectively, $I_1 = \{i_{11}, \dots, i_{1p}, \dots\}$ and $I_2 = \{i_{21}, \dots, i_{2q}, \dots\}$. Then define $S_{1p}x = \sum_{r=1}^p \Lambda_{i_{1r}}^* \Lambda_{i_{1r}} x$ and $S_{2q}y = \sum_{t=1}^q \Lambda_{i_{2t}}^* \Lambda_{i_{2t}} y$, for each $x \in E_1$ and $y \in E_2$. Now $\|S_{1p}\| \leq \|S_{\Phi^{(1)}}\|$ and $\|S_{2q}\| \leq \|S_{\Phi^{(2)}}\|$, for each $p, q \in \mathbb{N}$ and since $\Phi^{(1)}$ and $\Phi^{(2)}$ are standard g -Bessel sequences, then $0 \leq S_{\Phi^{(k)}} \leq B_k Id_{E_k}$, for each $k \in \{1, 2\}$ and consequently $0 \leq S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}} \leq B_1 B_2 Id_{(E_1 \otimes E_2)}$. Therefore by Lemma 4.1 in [16], for each $z \in E_1 \otimes E_2$ and $p, q \in \mathbb{N}$, we have*

$$\langle (S_{1p} \otimes S_{2q})z, z \rangle \leq \langle (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z, z \rangle \leq B_1 B_2 \langle z, z \rangle. \quad (3.2)$$

It is also easy to see that $\lim_{p,q} (S_{1p} \otimes S_{2q})z = (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z$, for each $z = \sum_{l=1}^m x_l \otimes y_l \in E_1 \otimes_{alg} E_2$. Now if $z \in E_1 \otimes E_2$, then by an appropriate choice of $z_0 \in E_1 \otimes_{alg} E_2$, and the inequality

$$\begin{aligned} & \| (S_{1p} \otimes S_{2q})z - (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z \| \\ & \leq \| S_{\Phi^{(1)}} \| \| S_{\Phi^{(2)}} \| \| z - z_0 \| \\ & + \| (S_{1p} \otimes S_{2q})z_0 - (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z_0 \| \\ & + B_1 B_2 \| z - z_0 \|, \end{aligned}$$

we get $\lim_{p,q} (S_{1p} \otimes S_{2q})z = (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z$. This means that the series

$\sum_{(i(1), i(2)) \in I_1 \times I_2} \langle (\Lambda_{i(1)} \otimes \Lambda_{i(2)})z, (\Lambda_{i(1)} \otimes \Lambda_{i(2)})z \rangle$ converges in norm and by (3.2), we have

$$\sum_{(i(1), i(2)) \in I_1 \times I_2} \langle (\Lambda_{i(1)} \otimes \Lambda_{i(2)})z, (\Lambda_{i(1)} \otimes \Lambda_{i(2)})z \rangle$$

$$= \langle (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z, z \rangle \leq B_1 B_2 \langle z, z \rangle. \quad (3.3)$$

This shows that $\Phi^{(1)} \otimes \Phi^{(2)}$ is a standard g -Bessel sequence with upper bound $B_1 B_2$.

Now suppose that $\Phi^{(1)}$ and $\Phi^{(2)}$ are g -frames with lower bounds A_1 and A_2 , respectively. Since

$$\begin{aligned} & A_1 A_2 Id_{E_1 \otimes E_2} \\ & \leq (\|S_{\Phi^{(1)}}^{-1}\|^{-1} \|S_{\Phi^{(2)}}^{-1}\|^{-1}) Id_{E_1 \otimes E_2} \\ & = \|(S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})^{-1}\|^{-1} Id_{E_1 \otimes E_2} \\ & \leq S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}}, \end{aligned}$$

using (3.2) and (3.3), we obtain that $\otimes_{k=1}^2 \Phi^{(k)}$ is a standard g -frame with lower bound $A_1 A_2$.

Conversely let $\otimes_{k=1}^2 \Phi^{(k)}$ be a standard g -frame with upper bound B and $x \in E_1$. Since $\otimes_{k=1}^2 \Phi^{(k)}$ is a standard g -Bessel sequence, it is clear that the series $\sum_{i(1) \in I_1} \langle \Lambda_{i(1)} x, \Lambda_{i(1)} x \rangle$ converges in norm and for each $y \in E_2$,

$$\begin{aligned} & \left\| \sum_{i(1) \in I_1} \langle \Lambda_{i(1)} x, \Lambda_{i(1)} x \rangle \right\| \times \\ & \left\| \sum_{i(2) \in I_2} \langle \Lambda_{i(2)} y, \Lambda_{i(2)} y \rangle \right\| \\ & = \left\| \sum_{(i(1), i(2)) \in I_1 \times I_2} \langle (\Lambda_{i(1)} \otimes \Lambda_{i(2)})(x \otimes y), (\Lambda_{i(1)} \otimes \Lambda_{i(2)})(x \otimes y) \rangle \right\| \\ & \leq B \|x \otimes y\|^2 = B \|x\|^2 \|y\|^2. \end{aligned}$$

Let $y \in E_2$ with $\|y\| = 1$. Since $\otimes_{k=1}^2 \Phi^{(k)}$ is a g -frame,

$C = \left\| \sum_{i(2) \in I_2} \langle \Lambda_{i(2)} y, \Lambda_{i(2)} y \rangle \right\|$ is a positive number, so we have

$$\left\| \sum_{i(1) \in I_1} \langle \Lambda_{i(1)} x, \Lambda_{i(1)} x \rangle \right\| \leq \frac{B}{C} \|x\|^2.$$

Therefore by [24, Theorem 3.1], $\Phi^{(1)}$ is a standard g -Bessel sequence with upper bound $\frac{B}{C}$.

Now let A be a lower bound for $\otimes_{k=1}^2 \Phi^{(k)}$ and $x \in E_1$. If $y \in E_2$ with $\|y\| = 1$ and $C = \left\| \sum_{i(2) \in I_2} \langle \Lambda_{i(2)} y, \Lambda_{i(2)} y \rangle \right\|$, then it is easy to see that

$$\frac{A}{C} \|x\|^2 \leq \left\| \sum_{i(1) \in I_1} \langle \Lambda_{i(1)} x, \Lambda_{i(1)} x \rangle \right\|.$$

Hence $\Phi^{(1)}$ is a standard g -frame and a similar proof shows that $\Phi^{(2)}$ is also a standard g -frame.

(ii) By part (i), $\otimes_{k=1}^n \Phi^{(k)}$ and $\otimes_{k=1}^n \Psi^{(k)}$ are g -Bessel sequences. Now let $\otimes_{k=1}^n a_k$ be a simple tensor in $\otimes_{k=1}^n \mathfrak{A}_k$. Since $m_{i(k)} \in \mathcal{Z}(\mathfrak{A}_k)$, for each $1 \leq k \leq n$, we have

$$\begin{aligned} & (\otimes_{k=1}^n a_k)(\otimes_{k=1}^n m_{i(k)}) = \otimes_{k=1}^n (a_k m_{i(k)}) \\ & = \otimes_{k=1}^n (m_{i(k)} a_k) \\ & = (\otimes_{k=1}^n m_{i(k)})(\otimes_{k=1}^n a_k). \end{aligned}$$

Because the above equality holds for simple tensors, $N(\otimes_{k=1}^n m_{i(k)}) = (\otimes_{k=1}^n m_{i(k)})N$, for each $N \in \otimes_{k=1}^n \mathfrak{A}_k$. Therefore $\otimes_{k=1}^n m_{i(k)} \in \mathcal{Z}(\otimes_{k=1}^n \mathfrak{A}_k)$ and the relation $\|\otimes_{k=1}^n m_{i(k)}\| = \prod_{k=1}^n \|m_{i(k)}\| \leq \prod_{k=1}^n \|m^{(k)}\|$ yields that $\otimes_{k=1}^n m^{(k)}$ is a symbol, so $S_{(\otimes_{k=1}^n m^{(k)})(\otimes_{k=1}^n \Psi^{(k)})(\otimes_{k=1}^n \Phi^{(k)})}$ is well-defined. Now let $n = 2$ and $x \otimes y \in E_1 \otimes E_2$. Then we have

$$\begin{aligned} & S_{(\otimes_{k=1}^2 m^{(k)})(\otimes_{k=1}^2 \Psi^{(k)})(\otimes_{k=1}^2 \Phi^{(k)})}(x \otimes y) = \\ & \sum_{(i(1), i(2)) \in I_1 \times I_2} (m_{i(1)} \otimes m_{i(2)}) \\ & (\Gamma_{i(1)} \otimes \Gamma_{i(2)})^*(\Lambda_{i(1)} \otimes \Lambda_{i(2)})(x \otimes y) \\ & = \left(\sum_{i(1) \in I_1} m_{i(1)} \Gamma_{i(1)}^* \Lambda_{i(1)} x \right) \otimes \\ & \left(\sum_{i(2) \in I_2} m_{i(2)} \Gamma_{i(2)}^* \Lambda_{i(2)} y \right) \\ & = (S_{m^{(1)} \Psi^{(1)} \Phi^{(1)}} \otimes S_{m^{(2)} \Psi^{(2)} \Phi^{(2)}})(x \otimes y), \end{aligned}$$

and since the operators are bounded, we have

$$\begin{aligned} & S_{(m^{(1)} \otimes m^{(2)})(\Psi^{(1)} \otimes \Psi^{(2)})(\Phi^{(1)} \otimes \Phi^{(2)})} \\ & = S_{m^{(1)} \Psi^{(1)} \Phi^{(1)}} \otimes S_{m^{(2)} \Psi^{(2)} \Phi^{(2)}}, \end{aligned}$$

and the result follows.

Now we get the following result which is a generalization of [13, Theorem 2.1 (ii)], [13, Corollary 2.6] and [4, Theorem 4.1] to Hilbert C^* -modules:

Corollary 3.1 (i) If $\mathcal{W}^{(k)}$ is a Bessel fusion sequence, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n \mathcal{W}^{(k)}$ is a Bessel fusion sequence. Moreover, $\mathcal{W}^{(k)}$ is a fusion frame, for each $1 \leq k \leq n$ if and only if $\otimes_{k=1}^n \mathcal{W}^{(k)}$ is a fusion frame. If $\mathcal{W}^{(k)}$'s and $\mathcal{V}^{(k)}$'s are Bessel fusion sequences, then the operator $S_{(\otimes_{k=1}^n m^{(k)})(\otimes_{k=1}^n \mathcal{W}^{(k)})(\otimes_{k=1}^n \mathcal{V}^{(k)})}$ is well-defined and equals $\otimes_{k=1}^n S_{m^{(k)} \mathcal{W}^{(k)} \mathcal{V}^{(k)}}$.

(ii) If $\mathcal{F}^{(k)}$ is a Bessel sequence, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n \mathcal{F}^{(k)}$ is a Bessel

sequence. Moreover, $\mathcal{F}^{(k)}$ is a frame for each $1 \leq k \leq n$ if and only if $\otimes_{k=1}^n \mathcal{F}^{(k)}$ is a frame for $\otimes_{k=1}^n E_k$. If $\mathcal{F}^{(k)}$'s and $\mathcal{G}^{(k)}$'s are Bessel sequences, then $S_{(\otimes_{k=1}^n m^{(k)})(\otimes_{k=1}^n \mathcal{F}^{(k)})(\otimes_{k=1}^n \mathcal{G}^{(k)})}$ is well-defined and is equal to $\otimes_{k=1}^n S_{m^{(k)} \mathcal{F}^{(k)} \mathcal{G}^{(k)}}$.

Proof. (i) We can get the result using the above theorem, part (a) of Example 3.1 in [12] and the fact that $\Phi^{(k)} = \{\omega_{i(k)} \pi_{W_{i(k)}}\}_{i(k) \in I_k}$ is a standard g -frame for each $1 \leq k \leq n$ if and only if

$$\otimes_{k=1}^n \Phi^{(k)} = \{(\omega_{i(1)} \otimes \dots \otimes \omega_{i(n)}) \pi_{(W_{i(1)} \otimes \dots \otimes W_{i(n)})}\}_{(i(1), \dots, i(n)) \in (I_1 \times \dots \times I_n)}$$

is a standard g -frame.

(ii) The result follows from Theorem 3.1 and part (b) of Example 3.1 in [12].

Recall that if $\Lambda = \{\Lambda_i \in \mathfrak{L}_{\mathfrak{A}}(E, E_i)\}_{i \in I}$ and $\Gamma = \{\Gamma_i \in \mathfrak{L}_{\mathfrak{A}}(E, E_i)\}_{i \in I}$ are standard g -Bessel sequences such that $\sum_{i \in I} \Gamma_i^* \Lambda_i x = x$ or equivalently $\sum_{i \in I} \Lambda_i^* \Gamma_i x = x$, for each $x \in E$, then Γ (resp. Λ) is called a g -dual of Λ (resp. Γ). We define the operator $S_{\Gamma \Lambda}$ on E by $S_{\Gamma \Lambda} = S_{m \Gamma \Lambda}$, where $m = \{m_i\}_{i \in I}$ is a symbol with $m_i = 1_{\mathfrak{A}}$, for each $i \in I$. Then Γ is a g -dual of Λ if and only if $S_{\Gamma \Lambda} = Id_E$. The canonical g -dual for an (A, B) standard g -frame $\Lambda = \{\Lambda_i\}_{i \in I}$ is defined by $\tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i \in I}$, where $\tilde{\Lambda}_i = \Lambda_i S_{\Lambda}^{-1}$ which is an $(\frac{1}{B}, \frac{1}{A})$ standard g -frame and for each $x \in E$, we have

$$x = \sum_{i \in I} \Lambda_i^* \tilde{\Lambda}_i x = \sum_{i \in I} \tilde{\Lambda}_i^* \Lambda_i x.$$

If $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{G} = \{g_i\}_{i \in I}$ are standard Bessel sequences in E , then we say that \mathcal{G} (resp. \mathcal{F}) is a dual of \mathcal{F} (resp. \mathcal{G}), if $x = \sum_{i \in I} \langle x, f_i \rangle g_i$ or equivalently $x = \sum_{i \in I} \langle x, g_i \rangle f_i$, for each $x \in E$. If \mathcal{F} is an (A, B) standard frame, then $\tilde{\mathcal{F}} = \{S$

$\hat{\mathcal{F}}^{-1} f_i\}_{i \in I}$ is an $(\frac{1}{B}, \frac{1}{A})$ standard frame with $x = \sum_{i \in I} \langle x, S$

$$\hat{\mathcal{F}}^{-1} f_i \rangle f_i = \sum_{i \in I} \langle x, f_i \rangle S$$

$\hat{\mathcal{F}}^{-1} f_i\}_{i \in I}$, for each $x \in E$. Hence $\tilde{\mathcal{F}} = \{S$

$\hat{\mathcal{F}}^{-1} f_i\}_{i \in I}$ is a dual of \mathcal{F} called the canonical dual of \mathcal{F} .

Let $W = \{(W_i, \omega_i)\}_{i \in I}$ be a standard Bessel fusion sequence with upper bound B and $V = \{(V_i, v_i)\}_{i \in I}$ be a (C, D) standard fusion frame for E . Since $S_V^{-2} \leq \frac{1}{C^2} Id_E$, by Lemma 4.1 in [16] and the fact that $v_i \in \mathcal{Z}(\mathfrak{A})$, for each $i \in I$,

we have

$$\begin{aligned} & \langle m_i v_i S_V^{-1} \pi_{V_i} x, m_i v_i S_V^{-1} \pi_{V_i} x \rangle \\ &= m_i m_i^* v_i^2 \langle S_V^{-2} \pi_{V_i} x, \pi_{V_i} x \rangle \\ &\leq \frac{\|m\|_\infty^2}{C^2} \cdot \langle v_i \pi_{V_i} x, v_i \pi_{V_i} x \rangle. \end{aligned}$$

Now for each finite subset $\Omega \subseteq I$, using the Cauchy-Schwarz inequality for Hilbert C^* -modules, we obtain that

$$\begin{aligned} & \left\| \sum_{i \in \Omega} m_i v_i \omega_i \pi_{W_i} S_V^{-1} \pi_{V_i} x \right\| \\ &= \sup_{\|y\|=1} \left\| \sum_{i \in \Omega} \langle m_i v_i \omega_i \pi_{W_i} S_V^{-1} \pi_{V_i} x, y \rangle \right\| \\ &= \sup_{\|y\|=1} \left\| \sum_{i \in \Omega} \langle m_i v_i S_V^{-1} \pi_{V_i} x, \omega_i \pi_{W_i} y \rangle \right\| \\ &\leq \left(\frac{\|m\|_\infty}{C} \left\| \sum_{i \in \Omega} |v_i \pi_{V_i} x|^2 \right\|^{\frac{1}{2}} \right) \times \\ &\quad \left(\sup_{\|y\|=1} \left\| \sum_{i \in \Omega} |\omega_i \pi_{W_i} y|^2 \right\|^{\frac{1}{2}} \right) \\ &\leq \frac{\sqrt{B} \|m\|_\infty}{C} \left\| \sum_{i \in \Omega} \langle v_i \pi_{V_i} x, v_i \pi_{V_i} x \rangle \right\|^{\frac{1}{2}}. \end{aligned}$$

Since V is standard, the series $\sum_{i \in I} m_i v_i \omega_i \pi_{W_i} S_V^{-1} \pi_{V_i} x$ converges in E and

$$\begin{aligned} & \left\| \sum_{i \in I} m_i v_i \omega_i \pi_{W_i} S_V^{-1} \pi_{V_i} x \right\| \\ &\leq \frac{\sqrt{BD} \|m\|_\infty \|x\|}{C}. \end{aligned}$$

Now it is easy to see that the operator $S_{m\mathcal{V}_W}$ which is defined on E by

$$S_{m\mathcal{V}_W} x = \sum_{i \in I} m_i v_i \omega_i \pi_{W_i} S_V^{-1} \pi_{V_i} x,$$

is adjointable.

Proposition 3.1 Let $\mathcal{W}^{(k)}$ be a Bessel fusion sequence and $\mathcal{V}^{(k)}$ be a fusion frame, for each $1 \leq k \leq n$. Then

$$\begin{aligned} & S_{(\otimes_{k=1}^n m^{(k)}) (\otimes_{k=1}^n \mathcal{V}^{(k)}) (\otimes_{k=1}^n \mathcal{W}^{(k)})} \\ &= \otimes_{k=1}^n S_{m^{(k)} \mathcal{V}^{(k)} \mathcal{W}^{(k)}}. \end{aligned}$$

Proof. It follows from Corollary 3.1 that $\otimes_{k=1}^n \mathcal{V}^{(k)}$ and $\otimes_{k=1}^n \mathcal{W}^{(k)}$ are standard fusion

frame and standard Bessel fusion sequence, respectively. Now it is easy to see that

$$\begin{aligned} & S_{(\otimes_{k=1}^n m^{(k)}) (\otimes_{k=1}^n \mathcal{V}^{(k)}) (\otimes_{k=1}^n \mathcal{W}^{(k)})} \\ &= \sum_{(i(1), \dots, i(n)) \in (I_1 \times \dots \times I_n)} \left[(m_{i(1)} \otimes \dots \otimes m_{i(n)}) (v_{i(1)} \otimes \dots \otimes v_{i(n)}) (\omega_{i(1)} \otimes \dots \otimes \omega_{i(n)}) \pi_{(W_{i(1)} \otimes \dots \otimes W_{i(n)})} \right. \\ &\quad \left. S_{\otimes_{k=1}^n \mathcal{V}^{(k)}}^{-1} \pi_{(V_{i(1)} \otimes \dots \otimes V_{i(n)})} \right] \\ &= \left(\sum_{i(1) \in I_1} m_{i(1)} v_{i(1)} \omega_{i(1)} \pi_{W_{i(1)}} S_{\mathcal{V}^{(1)}}^{-1} \pi_{V_{i(1)}} \right) \\ &\quad \otimes \dots \otimes \sum_{i(n) \in I_n} m_{i(n)} v_{i(n)} \omega_{i(n)} \pi_{W_{i(n)}} S_{\mathcal{V}^{(n)}}^{-1} \pi_{V_{i(n)}} \\ &= \otimes_{k=1}^n S_{m^{(k)} \mathcal{V}^{(k)} \mathcal{W}^{(k)}} \end{aligned}$$

and the result follows.

Now we have the following definition (see also [9]):

Definition 3.3 Let $V = \{(V_i, v_i)\}_{i \in I}$ be a standard fusion frame and $W = \{(W_i, \omega_i)\}_{i \in I}$ be a standard Bessel fusion sequence for E . Then W is called an alternate dual of V if $x = \sum_{i \in I} v_i \omega_i \pi_{W_i} S_V^{-1} \pi_{V_i} x$, for each $x \in E$.

The following proposition is a generalization of [13, Corollary 3.8] and [14, Proposition 2.10] to Hilbert C^* -modules and also generalizes the result obtained in [18, Proposition 3.6].

Proposition 3.2 (i) If $\Psi^{(k)}$ is a g -dual of $\Phi^{(k)}$, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n \Psi^{(k)}$ is a g -dual of $\otimes_{k=1}^n \Phi^{(k)}$. If $\otimes_{k=1}^n \Psi^{(k)}$ is a g -dual of $\otimes_{k=1}^n \Phi^{(k)}$ and $\Psi^{(k)}$ is a g -dual of $\Phi^{(k)}$, for each $k \in \{1, \dots, n-1\}$, then $\Psi^{(n)}$ is also a g -dual of $\Phi^{(n)}$.

(ii) If $\mathcal{W}^{(k)}$ is an alternate dual of $\mathcal{V}^{(k)}$, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n \mathcal{W}^{(k)}$ is an alternate dual of $\otimes_{k=1}^n \mathcal{V}^{(k)}$.

(iii) If $\Phi^{(k)}$'s are g -frames, then $\widetilde{\otimes_{k=1}^n \Phi^{(k)}} = \otimes_{k=1}^n \widetilde{\Phi^{(k)}}$.

Proof. (i) Let $m_{i(k)} = 1$, for each $1 \leq k \leq n$ and $i(k) \in I_k$. Then Theorem 3.1 implies that

$$\begin{aligned} & S_{(\otimes_{k=1}^n \Psi^{(k)}) (\otimes_{k=1}^n \Phi^{(k)})} \\ &= S_{(\otimes_{k=1}^n m^{(k)}) (\otimes_{k=1}^n \Psi^{(k)}) (\otimes_{k=1}^n \Phi^{(k)})} \\ &= \otimes_{k=1}^n S_{m^{(k)} \Psi^{(k)} \Phi^{(k)}} = \otimes_{k=1}^n S_{\Psi^{(k)} \Phi^{(k)}} \\ &= \otimes_{k=1}^n Id_{E_k} = Id_{\otimes_{k=1}^n E_k}. \end{aligned}$$

This shows that $\otimes_{k=1}^n \Psi^{(k)}$ is a g -dual of $\otimes_{k=1}^n \Phi^{(k)}$. For the rest, we have

$$\begin{aligned} Id_{\otimes_{k=1}^n E_k} &= S_{(\otimes_{k=1}^n \Psi^{(k)}) (\otimes_{k=1}^n \Phi^{(k)})} \\ &= \otimes_{k=1}^n S_{\Psi^{(k)} \Phi^{(k)}} \\ &= (\otimes_{k=1}^{n-1} Id_{E_k}) \otimes S_{\Psi^{(n)} \Phi^{(n)}}, \end{aligned}$$

so $\|Id_{E_n} - S_{\Psi^{(n)} \Phi^{(n)}}\| = \|Id_{E_1} \otimes \dots \otimes Id_{E_{n-1}} \otimes (Id_{E_n} - S_{\Psi^{(n)} \Phi^{(n)}})\| = 0$, and this yields that $S_{\Psi^{(n)} \Phi^{(n)}} = Id_{E_n}$.

(ii) Let $m_{i(k)} = 1_{\mathfrak{A}_k}$, for each $1 \leq k \leq n$. Then Proposition 3.1 implies that

$$\begin{aligned} S_{(\otimes_{k=1}^n m^{(k)}) (\otimes_{k=1}^n \mathcal{V}^{(k)}) (\otimes_{k=1}^n \mathcal{W}^{(k)})} \\ = \otimes_{k=1}^n S_{m^{(k)} \mathcal{V}^{(k)} \mathcal{W}^{(k)}} = \otimes_{k=1}^n Id_{E_k}, \end{aligned}$$

and the result follows.

(iii) By Theorem 3.1, $\otimes_{k=1}^n \Phi^{(k)}$ is a g -frame and by considering $m_{i(k)} = 1$, for each $1 \leq k \leq n$ and $i(k) \in I_k$, similar to part (i), we get $S_{(\otimes_{k=1}^n \Phi^{(k)})} = \otimes_{k=1}^n S_{\Phi^{(k)}}$. This implies that $S_{\otimes_{k=1}^n \Phi^{(k)}}^{-1} = \otimes_{k=1}^n S_{\Phi^{(k)}}^{-1}$, then for each $(i(1), \dots, i(n)) \in I_1 \times \dots \times I_n$, we have

$$\begin{aligned} (\Lambda_{i(1)} \otimes \dots \otimes \Lambda_{i(n)}) S_{\otimes_{k=1}^n \Phi^{(k)}}^{-1} \\ = (\Lambda_{i(1)} S_{\Phi^{(1)}}^{-1}) \otimes \dots \otimes (\Lambda_{i(n)} S_{\Phi^{(n)}}^{-1}). \end{aligned}$$

This shows that $\widetilde{\otimes_{k=1}^n \Phi^{(k)}} = \otimes_{k=1}^n \widetilde{\Phi^{(k)}}$.

Now we obtain the following result which is a generalization of Corollary 2.11 in [14] to Hilbert C^* -modules:

Corollary 3.2 (i) Let $\mathcal{F}^{(k)}$ and $\mathcal{G}^{(k)}$ be Bessel sequences for E . If $\mathcal{G}^{(k)}$ is a dual of $\mathcal{F}^{(k)}$, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n \mathcal{G}^{(k)}$ is a dual of $\otimes_{k=1}^n \mathcal{F}^{(k)}$. If $\otimes_{k=1}^n \mathcal{G}^{(k)}$ is a dual of $\otimes_{k=1}^n \mathcal{F}^{(k)}$ and $\mathcal{G}^{(k)}$ is a dual of $\mathcal{F}^{(k)}$, for each $1 \leq k \leq n-1$, then $\mathcal{G}^{(n)}$ is a dual of $\mathcal{F}^{(n)}$.

(ii) If $\mathcal{F}^{(k)}$ is a frame, for each $1 \leq k \leq n$, then $\widetilde{\otimes_{k=1}^n \mathcal{F}^{(k)}} = \otimes_{k=1}^n \widetilde{\mathcal{F}^{(k)}}$.

Proof. The result follows from the above proposition and Corollary 3.1 by considering $\Phi^{(k)} = \{\Lambda_{i(k)}\}_{i(k) \in I_k}$ and $\Psi^{(k)} = \{\Gamma_{i(k)}\}_{i(k) \in I_k}$, where $\Lambda_{i(k)}x = \langle x, f_{i(k)} \rangle$ and $\Gamma_{i(k)}x = \langle x, g_{i(k)} \rangle$, for each $x \in E$.

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Bessel multipliers on the tensor product of Hilbert C^* -modules

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ضرب گرهای بسل روی حاصلضرب تانسوری C^* -تانسوری هیلبرت

چکیده:

در این مقاله ابتدا نشان می دهیم حاصلضرب تانسوری تعداد متناهی از \mathfrak{g} -قالبهای استاندارد یک \mathfrak{g} - قاب استاندارد برای حاصلضرب تانسوری مدولهای هیلبرت است و برعکس. سپس حاصلضرب های تانسوری ضربگرهای بسل، ضربگرهای بسل و ضربگرهای مخلوط بسل را در مدولهای هیلبرت مورد توجه قرار می دهیم. علاوه بر این با استفاده از ضرب گرهای بسل نتایجی را در مورد حاصلضرب تانسوری دوگانها به دست می آوریم.