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Research Article

Generalized H-differentiability for solving second order linear fuzzy differential equations

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Abstract

In this paper, a new approach for solving the second order fuzzy differential equations (FDE) with fuzzy initial value, under strongly generalized H-differentiability is presented. Solving first order fuzzy differential equations by extending 1-cut solution of the original problem and solving fuzzy integro-differential equations has been investigated by some authors (see for example [5, 6]), but these methods have been done for fuzzy problems with triangular fuzzy initial value. Therefore by extending the r-cut solutions of the original problem we will obviate this deficiency. The presented idea is based on: if a second order fuzzy differential equation satisfy the Lipschitz condition then the initial value problem has a unique solution on a specific interval, therefore our main purpose is to present a method to find an interval on which the solution is valid.

Keywords : Fuzzy differential equations (FDE); Strongly generalized H-differentiability; r-cut solutions.

1 Introduction

The topic of fuzzy differential equations (FDE) has been rapidly growing in recent years. Kandel and Byatt [20] applied the concept of fuzzy differential equations (FDE) to analyze the fuzzy dynamic problems. The FDE and the initial value problem (Cauchy problem) were treated by Kaleva [21, 22], Seikkala [25], He and Yi [17], Kloeden [23] and some others (see [10, 11, 12, 15, 18]). The numerical methods for solving fuzzy differential equations are introduced in [1, 2, 3, 4, 7, 9]. Buckley and Feuring [13] introduced two analytical methods for solving n-order linear differential equations with fuzzy initial value conditions.

A new approach for solving first order fuzzy differential equations with extending 1-cut solution of original problem is introduced by Allahviranloo and salahshour [6]. See [5] for a method for fuzzy integro-differential equations with extending o-cut and 1-cut solutions of the original problem, but these methods have been done for fuzzy problems with triangular fuzzy initial value.

In this paper by extending r-cut solutions of the original problem we will obviate this deficiency.

In [8] we see that, if a second order fuzzy differential equation satisfy the Lipschitz condition, then the initial value problem has a unique solution on a specific interval. The presented method in this paper is an analytical method and with a specific method, we try to find a solution, that the solution is valid. According to the definition of generalized derivative, a fuzzy second order differential equation can be transformed to a four-crisp

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differential equations. Since the initial differential equations satisfy the Lipschitz condition, the obtained solution is unique. The solutions are obtained from a fuzzy initial value problem. To show that these are fuzzy solutions, we find the intervals in which each solution is a fuzzy solution.
The structure of this paper is organized as follow. In section 2, some basic definitions and notations will be given. In section 3, second order fuzzy differential equation is introduced and our method is presented in details. In section 4, the proposed method is illustrated by examples. Conclusion is at the end of section 5.

2 Basic Definitions and Notations

In this section, we give some necessary definitions and notations which will be used throughout the paper.

Definition 2.1 *Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \to [0,1]$. Thus $u(x)$ is interpreted as the degree of membership of an element x in the fuzzy set u for each $x \in X$.*

Let denote by E the class of fuzzy subsets of the real axis (i.e. $u : \mathbb{R} \to [0,1]$) satisfying the following properties:

- u is normal, that is, there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$,
- u is convex fuzzy set (i.e. $u(ts + (1-t)r) \geq \min\{u(s), u(r)\}, \forall t \in [0,1],\ s, r \in \mathbb{R}$),
- u is upper semi-continuous on $\mathbb{R}$,
- $cl\{s \in \mathbb{R} | u(s) > 0\}$ is compact, where cl denotes the closure of a subset.

E is called the space of fuzzy numbers with bounded r-level intervals. This means that if $v \in E$ then the r-level set

$$v^{[r]} = \{s | v(s) \geq r\},$$

is a closed bounded interval which is denoted by

$$v^{[r]} = [\underline{v}(r), \overline{v}(r)] \quad \text{for} \quad r \in (0,1],$$

and

$$v^{[0]} = \overline{\bigcup_{r \in (0,1]} v^{[r]}}.$$

Lemma 2.1 *[24] If $u, v \in E$, then for $r \in (0,1]$,*

$$(u+v)^{[r]} = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)],$$

$$(u.v)^{[r]} = [\min k, \max k],$$

where

$$k = \{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}.$$

Definition 2.2 *[19] The Hausdorff distance between fuzzy numbers is given by*

$$D : E \times E \longrightarrow R^+ \bigcup \{0\},$$

$$D(u,v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\},$$

where

$$u^{[r]} = [\underline{u}(r), \overline{u}(r)], v^{[r]} = [\underline{v}(r), \overline{v}(r)] \subset E$$

is utilized.

Then it is easy to see that D is a metric in E and has the following properties (see[10]).

- $D(u \oplus w, v \oplus w) = D(u,v),\ \forall u, v, w \in E$,
- $D(k \odot u, k \odot v) = |k| D(u,v), \quad \forall k \in R, u, v \in E$,
- $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \quad \forall u, v, w, e \in E$,
- (D, E) is a complete metric space.

Definition 2.3 *[10] Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x \ominus y$.*

Definition 2.4 *[10] Let $f : (a,b) \longrightarrow E$ and $t_0 \in (a,b)$. We say that f is strongly generalized H-differentiable at t_0, if there exists an element $f'(t_0) \in E$, such that:*

(1) for all $h > 0$ sufficiently near to 0, $\exists f(t_0 + h) \ominus f(t_0)$, $\exists f(t_0) \ominus f(t_0 - h)$ such that the following limits hold.

$$\lim_{h \longrightarrow 0^+} \frac{f(t_0+h) \ominus f(t_0)}{h}$$

$$= \lim_{h \longrightarrow 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0), \quad (2.1)$$

(2) for all $h < 0$ sufficiently near to 0, $\exists f(t_0) \ominus f(t_0 + h)$, $\exists f(t_0 - h) \ominus f(t_0)$ such that the following limits hold.

$$\lim_{h \longrightarrow 0^+} \frac{f(t_0) \ominus f(t_0 + h)}{h}$$

$$= \lim_{h \longrightarrow 0^+} \frac{f(t_0 - h) \ominus f(t_0)}{h} = f^{'}(t_0). \quad (2.2)$$

If $f(t)$ is (n)-differentiable at t_0, we denote its first derivatives by $D^1_n f(t_0)$, for $n = 1, 2$.
In the special case when f is a fuzzy-valued function, we have the following results.

Theorem 2.1 *[10] Let $f(t)$ be fuzzy-valued functions and denote $f(t)^{[r]} = (\underline{f}(t; r), \overline{f}(t; r))$, for each $r \in [0, 1]$. Then*

- if $f(t)$ is (1)-differentiable, then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ have second order derivative and $f^{'}(t)^{[r]} = [\underline{f}^{'}(t; r), \overline{f}^{'}(t; r)]$.
- if $f(t)$ is (2)-differentiable, then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ have second order derivative and $f^{'}(t)^{[r]} = [\overline{f}^{'}(t; r), \underline{f}^{'}(t; r)]$.

Theorem 2.2 *[10] let $f : (a, b) \longrightarrow \mathbb{R}$ and $g : (a, b) \longrightarrow E$ be two differentiable functions (g is generalized differentiable as in Definition 2.4).*

- *If $f(t).f^{'}(t) > 0$ and g is (1)-differentiable, then $f.g$ is (1)-differentiable and*

$$(f.g)^{'}(t) = f^{'}(t).g(t) + f(t).g^{'}(t). \quad (2.3)$$

- *If $f(t).f^{'}(t) < 0$ and g is (2)-differentiable, then $f.g$ is (2)-differentiable and*

$$(f.g)^{'}(t) = f^{'}(t).g(t) + f(t).g^{'}(t). \quad (2.4)$$

The main properties of the H-derivatives of first part of the above theorem, some of which still hold for the second part, are well known and can be found in [21] and some other properties of the second part can be found in [14].
Notice that we say fuzzy-valued function f is (1)-differentiable if satisfy in the first form (1) in Definition 2.4. and we say f is (2)-differentiable if satisfy in the second form (2) in Definition 2.4.

3 Second Order Fuzzy Differential Equations

In this section, we are going to investigate a solution of fuzzy differential equations (FDE).
Consider the following second order fuzzy differential equation:

$$\begin{cases} y^{''}(t) = f(t, y(t), y^{'}(t)), \\ y(t_0) = u_0, \\ y^{'}(t_0) = v_0, \end{cases} \quad (3.5)$$

where $f : (a, b) \times E \times E \longrightarrow E$ is linear fuzzy-valued function with positive coefficients, $u_0, v_0 \in E$ and the involved derivatives are strongly generalized H-differentiable which is defined in Definition 2.4.

Theorem 3.1 *[19] Let $f(t)$ and $f^{'}(t)$ are two differentiable fuzzy-valued functions and denote $f(t)^{[r]} = [\underline{f}(t; r), \overline{f}(t; r)]$, for each $r \in [0, 1]$. Then*

- *if $f(t)$ and $f^{'}(t)$ are (1)-differentiable, or $f(t)$ and $f^{'}(t)$ are (2)-differentiable, then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ have second order and second order derivatives and $f^{''}(t)^{[r]} = [\underline{f}^{''}(t; r), \overline{f}^{''}(t; r)]$.*
- *if $f(t)$ is (1)-differentiable and $f^{'}(t)$ is (2)-differentiable, or $f(t)$ is (2)-differentiable and $f^{'}(t)$ is (1)-differentiable, then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ have second order and second order derivatives and $f^{''}(t)^{[r]} = [\overline{f}^{''}(t; r), \underline{f}^{''}(t; r)]$.*

Definition 3.1 *[19] Let $f : (a, b) \rightarrow E$ and $n, m = 1, 2$. One says $f$is (n, m)-differentiable at $t_0 \in (a, b)$, if $D^1_n f$ exists on a neighborhood of t_0 as a fuzzy function and it is (m)-differentiable at t_0. The second derivatives of f are denoted by $D^{(2)}_{n,m} f(t_0)$ for $n, m = 1, 2$.*

Definition 3.2 *[19] Let $y : (a, b) \rightarrow E$ be fuzzy function and $n, m \in \{1, 2\}$. One says that y is an (n, m)-solution for problem (3.5) on (a, b), if $D^1_n y$ and $D^{(2)}_{n,m} y$ exist on (a, b) and $D^{(2)}_{n,m} y = f(t, y(t), D^1_n y(t))$, $y(t_0) = u_0$, $D^1_n \widetilde{y}(t_0) = v_0$.*

Theorem 3.2 *[19] let $f : (a, b) \longrightarrow \mathbb{R}$ and $g : (a, b) \longrightarrow E$ be second order differentiable functions (g is generalized differentiable as in Definition 2.4).*

- *If $f(t).f^{'}(t) > 0$, $f^{'}(t).f^{''}(t) > 0$ and g is $(1,1)$-differentiable, then $f.g$ is $(1,1)$-differentiable and*

$$(f.g)^{''}(t) = f^{''}(t).g(t)+2f^{'}(t).g^{'}(t)+f(t).g^{''}(t). \quad (3.6)$$

- *If $f(t).f^{'}(t) < 0$, $f^{'}(t).f^{''}(t) < 0$ and g is $(2,2)$-differentiable, then $f.g$ is $(2,2)$-differentiable and*

$$(f.g)^{''}(t) = f^{''}(t).g(t)+2f^{'}(t).g^{'}(t)+f(t).g^{''}(t). \quad (3.7)$$

Theorem 3.3 *[8] Let $f : (a,b) \times E \times E \longrightarrow E$ be continuous, and suppose that there exist $M_1, M_2 > 0$ such that:*

$$D(f(t,x_1,x_2), f(t,y_1,y_2)) \leq M_1 D(x_1,x_2) + M_2 D(x_1,x_2),$$

for all $t \in (a,b), x_1,x_2,y_1,y_2 \in E$. Then the initial value problem (3.5) has a unique solution on (a,b) for each case.

Now, we describe our method for solving a FDE (3.5). First, we solve a FDE (3.5) in the sense of r-cut as follows:

$$\begin{cases} D^{(2)}_{n,m} y^{[r]}(t) = f(t, y^{[r]}(t), D^r_n y^{[r]}(t)), \\ y^{[r]}(t_0) = u_0^{[r]}, \\ D^r_n y^{[r]}(t_0) = v_0^{[r]}, \\ \qquad 0 \leq r \leq 1, \ t_0 \in (a,b), \ n,m = 1,2. \end{cases} \quad (3.8)$$

Let $y^{[r]}(t)$ be an (n,m)-solution for problem (3.8). To find it, based on type of differentiability we have the following crisp systems, called (n,m)-systems as follows:

$(1,1)$-system

$$\begin{cases} \underline{y}^{''}(t;r) = f(t, \underline{y}(t;r), \underline{y}^{'}(t)), \\ \overline{y}^{''}(t;r) = f(t, \overline{y}(t;r), \overline{y}^{'}(t;r)), \\ \underline{y}(t_0;r) = \underline{u_0}(r), \ \overline{y}(t_0;r) = \overline{u_0}(r), \\ \underline{y}^{'}(t_0;r) = \underline{v_0}(r), \ \overline{y}^{'}(t_0;r) = \overline{v_0}(r), \\ \qquad 0 \leq r \leq 1. \end{cases} \quad (3.9)$$

$(1,2)$-system

$$\begin{cases} \overline{y}^{''}(t;r) = f(t, \underline{y}(t;r), \underline{y}^{'}(t;r)), \\ \underline{y}^{''}(t;r) = f(t, \overline{y}(t;r), \overline{y}^{'}(t;r)), \\ \underline{y}(t_0;r) = \underline{u_0}(r), \ \overline{y}(t_0;r) = \overline{u_0}(r), \\ \underline{y}^{'}(t_0;r) = \underline{v_0}(r), \ \overline{y}^{'}(t_0;r) = \overline{v_0}(r), \\ \qquad 0 \leq r \leq 1. \end{cases} \quad (3.10)$$

$(2,1)$-system

$$\begin{cases} \overline{y}^{''}(t;r) = f(t, \underline{y}(t;r), \overline{y}^{'}(t;r)), \\ \underline{y}^{''}(t;r) = f(t, \overline{y}(t;r), \underline{y}^{'}(t;r)), \\ \underline{y}(t_0;r) = \underline{u_0}(r), \ \overline{y}(t_0;r) = \overline{u_0}(r), \\ \overline{y}^{'}(t_0;r) = \underline{v_0}(r), \ \underline{y}^{'}(t_0;r) = \overline{v_0}(r), \\ \qquad 0 \leq r \leq 1. \end{cases} \quad (3.11)$$

$(2,2)$-system

$$\begin{cases} \underline{y}^{''}(t;r) = f(t, \underline{y}(t;r), \overline{y}^{'}(t;r)), \\ \overline{y}^{''}(t;r) = f(t, \overline{y}(t;r), \underline{y}^{'}(t;r)), \\ \underline{y}(t_0;r) = \underline{u_0}(r), \ \overline{y}(t_0;r) = \overline{u_0}(r), \\ \overline{y}^{'}(t_0;r) = \underline{v_0}(r), \ \underline{y}^{'}(t_0;r) = \overline{v_0}(r), \\ \qquad 0 \leq r \leq 1. \end{cases} \quad (3.12)$$

Theorem 3.4 *If f is satisfy the Lipschitz condition, then there is an interval I that the solution of (n,m)-system is an (n,m)-solution for the problem (3.5) on the interval I.*

Proof: Since f satisfy the Lipschitz condition, the initial value problem (3.5) has a unique (n,m)-solution such as $y^{[r]} = [\underline{y}(r), \overline{y}(r)]$ on the interval I ([8]). From the Definition 3.2. and theorems 2.1. and 3.1. $y^{[r]} = [\underline{y}(r), \overline{y}(r)]$ is a solution of (n,m)-system. On the other hand, since f satisfy the Lipschitz condition, then according to [16], the (n,m)-system has a unique solution such as $y^{*[r]} = [\underline{y^*}(r), \overline{y^*}(r)]$ and it can be shown that $y = y^*$. Then y^* is an (n,m)-solution for the problem (3.5) on the interval I.
We can choose the interval I such that a solution

of (3.5) be valid on it as fallow:
$I = (a, b),\ b = \min\{\ t \mid \exists\ \varepsilon_i > 0\ ;\ \overline{y}^i(t - \varepsilon_i; r) - \underline{y}^i(t - \varepsilon_i; r) > 0\ ,\ \overline{y}^i(t + \varepsilon_i; r) - \underline{y}^i(t + \varepsilon_i; r) < 0\ ,\ i = 0, 1, 2\}.$ ■

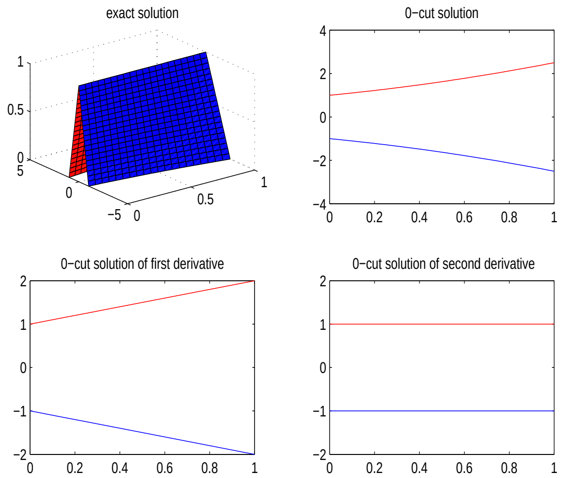

Figure 1.

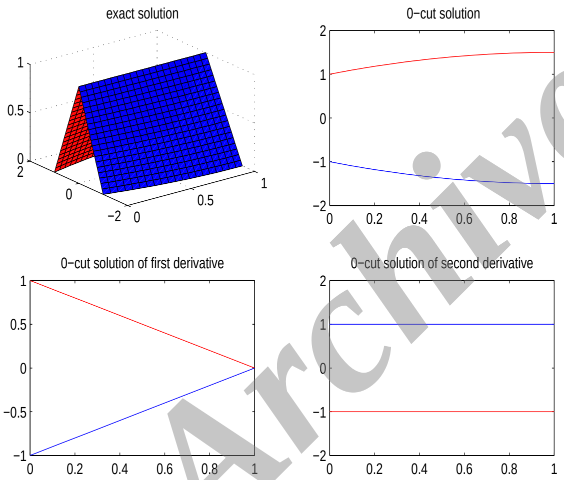

Figure 2.

4 Examples

In this section, some examples are given to illustrate our method and show that our approach is coincide with the exact solutions. Moreover we plot the obtained solutions and derivatives based on the r-cut representation at each case.

Example 4.1 *(see [19]) consider the following second order fuzzy differential equation:*

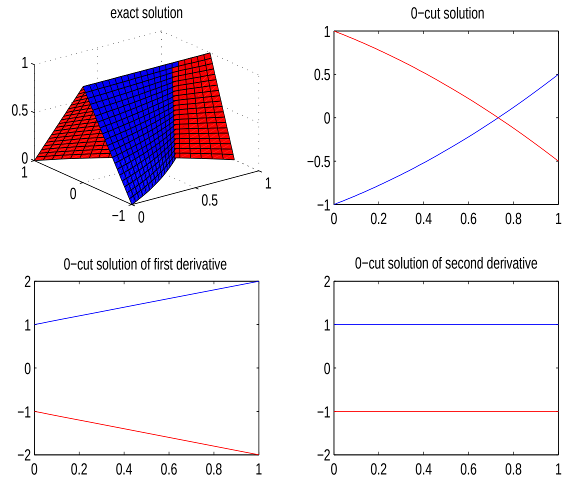

Figure 3.

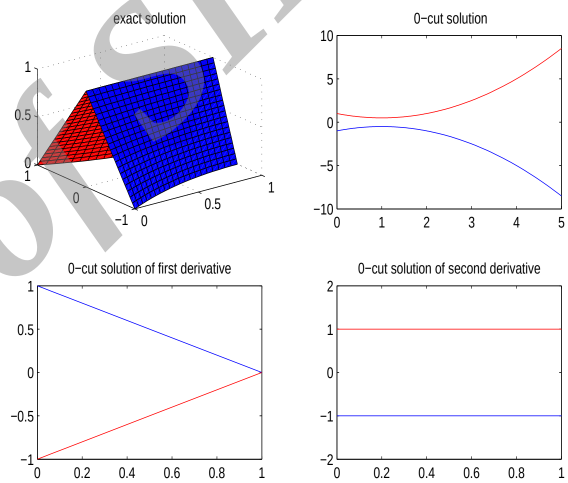

Figure 4.

$$\begin{cases} y''(t) = \sigma_0, \quad {\sigma_0}^{[r]} = [r-1, 1-r], \\ y(0)^{[r]} = [r-1, 1-r], \\ y'(0)^{[r]} = [r-1, 1-r], \quad t \geq 0. \end{cases} \tag{4.13}$$

By our method, 1-cut and 0-cut systems are derived as follows respectively:

Case(I): Suppose that $y(t)$ and $y'(t)$ are (1)-differentiable functions. By solving ODE (3.9) we get:

$$\underline{y}(t; r) = (r-1)(\frac{t^2}{2} + t + 1),$$

$$\overline{y}(t; r) = (1-r)(\frac{t^2}{2} + t + 1),$$

and

$$y(t)^{[r]} = [r-1, 1-r](\frac{t^2}{2} + t + 1),$$

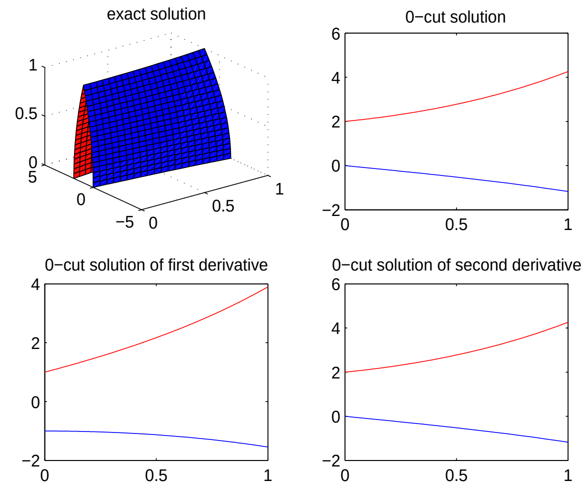

Figure 5.

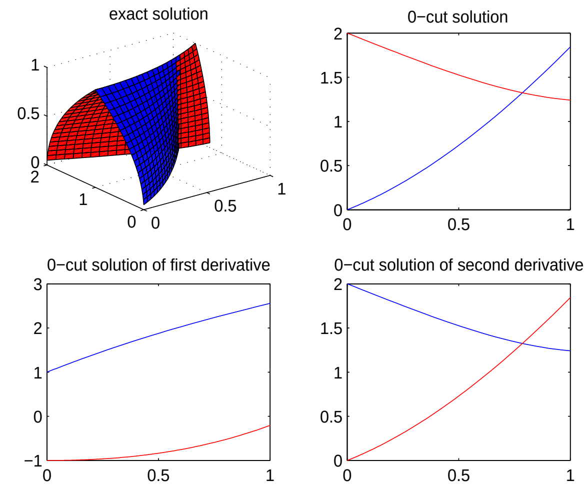

Figure 7.

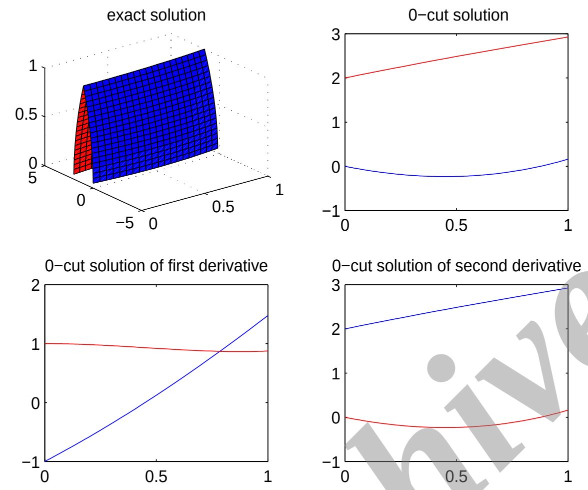

Figure 6.

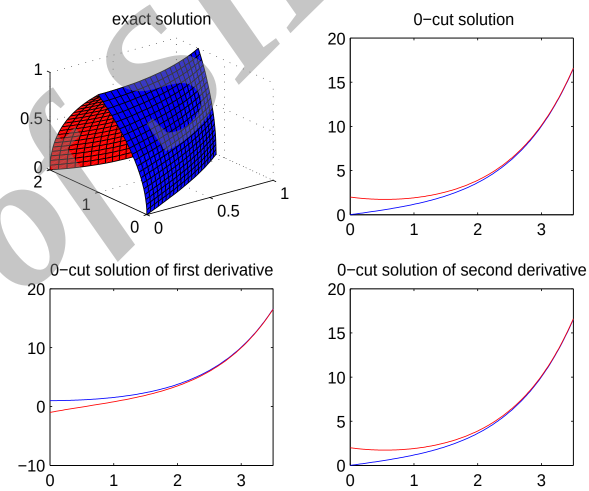

Figure 8.

by drawing the 0-cut solutions of the first and second derivatives we see that $y(t)$ has valid level sets for $t \geq 0$ and $y'(t)$ has valid level sets for $t \geq 0$ and also $y''(t)$ has valid level sets for $t \geq 0$, then by intersection of these valid level sets we get $y(t)$ that is a (1,1)-solution for the original problem on $[0, +\infty)$. (See Figure 1).

Case(II): Let $y(t)$ be a (1)-differentiable function and $y^{'}(t)$ be a (2)-differentiable function. By solving ODE (3.10) , we get:

$$\underline{y}(t; r) = (r - 1)(-\frac{t^2}{2} + t + 1),$$

$$\overline{y}(t; r) = (1 - r)(-\frac{t^2}{2} + t + 1),$$

and

$$y(t)^{[r]} = [r - 1, 1 - r](-\frac{t^2}{2} + t + 1),$$

by drawing the 0-cut solutions of the first and second derivatives we see that $y(t)$ has valid level sets for $t \geq 0$ and $y'(t)$ has valid level sets for $0 \leq t \leq 1$ and also $y''(t)$ has valid level sets for $t \geq 0$, then by intersection of these valid level sets we get $y(t)$ that is a (1,2)-solution for the original problem on $[0, 1]$. (See Figure 2).

Case(III): Let $y(t)$ be a (2)-differentiable function and $y^{'}(t)$ be a (1)-differentiable function. By solving ODE (3.11) we get:

$$\underline{y}(t; r) = (r - 1)(-\frac{t^2}{2} - t + 1),$$

$$\overline{y}(t;r) = (1-r)(-\frac{t^2}{2} - t + 1),$$

and

$$y(t)^{[r]} = [r-1, 1-r](-\frac{t^2}{2} - t + 1),$$

by drawing the 0-cut solutions of the first and second derivatives we see that $y(t)$ has valid level sets for $0 \leq t \leq \sqrt{3} - 1$ and $y'(t)$ has valid level sets for $t \geq 0$ and also $y''(t)$ has valid level sets for $t \geq 0$, then by intersection of these valid level sets we get $y(t)$ that is a (2,1)-solution for the original problem on $[0, \sqrt{3} - 1]$. (See Figure **3**).

Case(IV): Suppose that $y(t)$ and $y^{'}(t)$ are (2)-differentiable functions. By solving ODE (3.12), we get:

$$\underline{y}(t;r) = (r-1)(\frac{t^2}{2} - t + 1),$$

$$\overline{y}(t;r) = (1-r)(\frac{t^2}{2} - t + 1),$$

and

$$y(t)^{[r]} = [r-1, 1-r](\frac{t^2}{2} - t + 1),$$

by drawing the 0-cut solutions of the first and second derivatives we see that $y(t)$ has valid level sets for $t \geq 0$ and $y'(t)$ has valid level sets for $0 \leq t \leq 1$ and also $y''(t)$ has valid level sets for $t \geq 0$, then by intersection of these valid level sets we get $y(t)$ that is a (2,2)-solution for the original problem on $[0, 1]$. (See Figure **4**).

Using 1-cut and 0-cut solutions we show that the discussed method can be applied to solve the fuzzy differential equations.

Example 4.2 *Let us consider the following second order FDE:*

$$\begin{cases} y^{''}(t) = y(t), \\ y(0)^{[r]} = [r^2, 2 - r^2], \\ y^{'}(0)^{[r]} = [r^2 - 1, 1 - r^2] \quad t \geq 0. \end{cases} \tag{4.14}$$

Based on the proposed approach, 1-cut and 0-cut systems are derived as follows respectively:

Case(I):
Suppose that $y(t)$ and $y^{'}(t)$ are (1)-differentiable functions. By solving ODE (3.9), we get:

$$\underline{y}(t;r) = \frac{1}{2}e^{-t} + e^{t}(r^2 - 1/2),$$

$$\overline{y}(t;r) = \frac{1}{2}e^{-t} - e^{t}(r^2 - 3/2),$$

by drawing the 0-cut solutions of the first and second derivatives we see that $y(t)$ has valid level sets for $t \geq 0$ and $y'(t)$ has valid level sets for $t \geq 0$ and also $y''(t)$ has valid level sets for $t \geq 0$, then by intersection of these valid level sets we get $y(t)$ that is a (1,1)-solution for the original problem on $[0, +\infty)$. (See Figure **5**).

Case(II):
Let $y(t)$ be a (1)-differentiable function and $y^{'}(t)$ be a (2)-differentiable function. By solving ODE (3.10) , we get:

$$\underline{y}(t;r) = cosh(t) + 2r.sinh(log(r)).sin(t) + 2r.sinh(log(r)).cos(t),$$

$$\overline{y}(t;r) = cosh(t) - 2r.sinh(log(r)).sin(t) - 2r.sinh(log(r)).cos(t),$$

by drawing the 0-cut solutions of the first and second derivatives we see that $y(t)$ has valid level sets for $t \geq 0$ and $y'(t)$ has valid level sets for $0 \leq t \leq \frac{\pi}{4}$ and also $y''(t)$ has valid level sets for $t \geq 0$, then by intersection of these valid level sets we get $y(t)$ that is a (1,2)-solution for the original problem on $[0, \frac{\pi}{4}]$. (See Figure **6**).

Case(III):
Let $y(t)$ be a (2)-differentiable function and $y^{'}(t)$ be a (1)-differentiable function. By solving ODE (3.11), we get:

$$\underline{y}(t;r) = cosh(t) - 2r.sinh(log(r)).sin(t) + 2r.sinh(log(r)).cos(t),$$

$$\overline{y}(t;r) = r.cosh(t) + 2r.sinh(log(r)).sin(t) - 2r.sinh(log(r)).cos(t),$$

by drawing the 0-cut solutions of the first and second derivatives we see that $y(t)$ has valid level sets for $0 \leq t \leq \frac{\pi}{4}$ and $y'(t)$ has valid level sets for $t \geq 0$ and also $y''(t)$ has valid level sets for $0 \leq t \leq \frac{\pi}{4}$, then by intersection of these valid level sets we get $y(t)$ that is a (2,1)- solution for the original problem on $[0, \frac{\pi}{4}]$. (See Figure 7).

Case(IV):
Suppose that $y(t)$ and $y^{'}(t)$ are (2)-differentiable functions. By solving ODE (3.12) , we get:

$$\underline{y}(t;r) = \frac{1}{2}e^t + (r^2 - 1/2)e^{-t},$$

$$\overline{y}(t;r) = \frac{1}{2}e^t - (r^2 - 3/2)e^{-t},$$

by drawing the 0-cut solutions of the first and second derivatives we see that $y(t)$ has valid level sets for $t \geq 0$ and $y'(t)$ has valid level sets for $t \geq 0$ and also $y''(t)$ has valid level sets for $t \geq 0$, then by intersection of these valid level sets we get $y(t)$ that is a (2,2)-solution for the original problem on $[0, +\infty)$. (See Figure 8).

5 Conclusions

In this paper a new approach for solving second order fuzzy differential equations (FDE) with fuzzy initial value under strongly generalized H-differentiability is considered. The presented idea is based on: if a second order fuzzy differential equation satisfy the Lipschitz condition then the initial value problem has a unique solution on a specific interval. We obtain this solution by transforming the fuzzy initial value of the generalized derivatives to the four-crisp differential equations and then solve them. Using solutions of the first and second derivatives we choose an interval such that the differential equations's solution is valid on it.

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Generalized H-differentiability for solving second order linear fuzzy differential equations

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حل معادلات دیفرانسیل فازی خطی مرتبه دوم به کمک مشتق پذیری تعمیم یافته -H

چکیده:

در این مقاله یک روش جدید براي حل معادلات دیفرانسیل فازي مرتبه دوم با مقدار اولیه فازي، تحت H - مشتق پذیري قوي تعمیم یافته ارایه میگردد. حل معادلات دیفرانسیل فازي مرتبه اول از طریق جواب هاي یک- برشي از مساله اصلي وهمچنین معادلات دیفرانسیل انتگرال فازی، توسط برخي نویسندگان مورد جستجو قرار گرفته است (منابع [6,5]) ، اما این روشها براي مسایل فازي با مقدار اولیه فازي مثلثي انجام شده است. بنابراین با تعمیم جواب های r- برشي از مساله اصلي داده شده، این نقص را برطرف میکنیم. ایده حاضر بر اساس اینست که: اگر یک معادله دیفرانسیل فازي مرتبه دوم که در شرط لیپ شیتس صدق میکند آنگاه مساله با مقدار اولیه در یک بازه مشخص شده داراي جواب منحصربفرد است، بنابراین هدف اصلي ما ارایه روشي براي پیدا کردن بازه ای است که روي آن جواب معتبر باشد.