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A Piecewise Approximate Method for Solving Second Order Fuzzy Differential Equations Under Generalized Differentiability

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Abstract

In this paper a numerical method for solving second order fuzzy differential equations under generalized differentiability is proposed. This method is based on the interpolating a solution by piecewise polynomial of degree 4 in the range of solution. Moreover we investigate the existence, uniqueness and convergence of approximate solutions. Finally the accuracy of piecewise approximate method by some examples are shown.

Keywords : Generalized differentiability; Numerical Solution; Fuzzy Differential Equations.

1 Introduction

Fuzzy differential equations (FDE) are a suitable tool to model problem in science and engineering in which uncertainties or vagueness pervade. There are many idea to define a fuzzy derivative and in consequence, to study FDE. The first and most popular approach is using the Hukuhara differentiability for fuzzy valued function. Kaleva in [19] proposed FDE using Hukuhara derivative and it was developed by some other authors [15, 23]. Hukuhara differentiability has the drawback that the solution of FDE need to have increasing length of its support, so in order to overcome this weakness, Bede and Gal [9], introduced the strongly generalized differentiability of fuzzy valued function. This concept allows us to solve the above-mentioned shortcoming, also the strongly generalized derivative is defined for a larger class of fuzzy valued functions than the Hukuhara derivatives.

Many researchers some numerical method for solving FDE under Hukuhara differentiability presented in [1, 2, 5], and under generalized differentiability investigated in [6, 7]. Higher-order fuzzy differential equations with Hukuhura differentiability were presented in [18, 13, 3, 4]. Khastan in [20], proposed a analytic method to solve higher-order fuzzy differential equations based on the selection different type of derivatives, they obtained several solution to fuzzy initial value problem. In this paper a numerical method for second order fuzzy differential equations is proposed. The idea of this method is based on interpolating the solution by polynomial of degree 4 in the range of solution, the step size used is of length $H = 3h$. Also existence, uniqueness and convergency of the approximate solutions are proved.

The paper is organized as follows: In section 2, some basic definitions are brought. A proposed method for solving second order fuzzy differential equations is introduced also the existence,

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uniqueness and convergency are proved in section 3. A numerical example are presented in section 4 and finally conclusion is drawn.

2 Notation and definitions

First notations which shall be used in this paper are introduced.
We denote by $\mathbb{R}_\mathcal{F}$, the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which are defined over the real line.
For $0 < r \leq 1$, set $[u]^r = \left\{t \in \mathbb{R} \middle| u(t) \geq r\right\}$, and $[u]^0 = cl\left\{t \in \mathbb{R} \middle| u(t) > 0\right\}$. We represent $[u]^r = [u^-(r), u^+(r)]$, so if $u \in \mathbb{R}_\mathcal{F}$, the r-level set $[u]^r$ is a closed interval for all $r \in [0,\ 1]$. For arbitrary $u,\ v \in \mathbb{R}_\mathcal{F}$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $[u+v]^r = [u]^r + [v]^r$, $[ku]^r = k[u]^r$ respectively.

A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_\mathcal{F}$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \leq b \leq c$ such that $u^-(r) = a + (b-a)r$ and $u^+(r) = c - (c-b)r$ are the endpoints of r-level sets for all $r \in [0, 1]$.

Definition 2.1 *[16]The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as*

$$D(u,\ v) = \sup_{r\in[0,\ 1]} \max\left\{|u^-(r) - v^-(r)|,\ |u^+(r) - v^+(r)|\right\}. \tag{2.1}$$

Consider $u, v, w, z \in \mathbb{R}_\mathcal{F}$ and $\lambda \in \mathbb{R}$, then the following properties are well-known for metric D,

1. *$D(u \oplus w,\ v \oplus w) = D(u,\ v)$, for all $u, v, w \in \mathbb{R}_\mathcal{F}$,*
2. *$D(\lambda u,\ \lambda v) = |\lambda| D(u,\ v)$, for all $u, v \in \mathbb{R}_\mathcal{F}$, $\lambda \in \mathbb{R}$*
3. *$D(u \oplus v,\ w \oplus z) \leq D(u,\ w) + D(v,\ z)$, for all $u, v, w, z \in \mathbb{R}_\mathcal{F}$,*
4. *$D(u \ominus v,\ w \ominus z) \leq D(u,\ w) + D(v,\ z)$, as long as $u \ominus v$ and $w \ominus z$ exist, where $u,\ v,\ w,\ z \in \mathbb{R}_\mathcal{F}$.*

where, $\ominus$ is the Hukuhara difference(H-difference), it means that $w \ominus v = u$ if and only if $u \oplus v = w$.

Definition 2.2 *[9] Let $u, v \in \mathbb{R}_\mathcal{F}$. If there exists $w \in \mathbb{R}_\mathcal{F}$ such that*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) & u = v + w, \\ & \text{or} \\ (ii) & v = u + (-1)w, \end{cases}$$

Then w is called the generalized Hukuhara difference of u and v.

Remark 2.1 *[9] Throughout the rest of this paper, we assume that $u \ominus_{gH} v \in \mathbb{R}_\mathcal{F}$.*

Note that a function $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}_\mathcal{F}$ is called fuzzy-valued function. The r-level representation of this function is given by $f(t; r) = [f^-(t; r)\ ,\ f^+(t; r)]$, for all $t \in [a, b]$ and $r \in [0, 1]$.

Definition 2.3 *([16]) A fuzzy valued function $f : [a, b] \rightarrow \mathbb{R}_\mathcal{F}$ is said to be continuous at $t_0 \in [a, b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $D(f(t), f(t_0)) < \epsilon$, whenever $t \in [a, b]$ and $|t - t_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $t_0 \in [a,\ b]$.*

Definition 2.4 *([12]) The generalized Hukuhara derivative of the fuzzy-valued function $f : (a, b) \rightarrow \mathbb{R}_\mathcal{F}$ at $t_0 \in (a, b)$ is defined as*

$$f'_{gH}(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h}. \tag{2.2}$$

If $f'_{gH}(t_0) \in \mathbb{R}_\mathcal{F}$ satisfying (2.2) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at t_0.

Definition 2.5 *([12]) Let $f : [a, b] \rightarrow \mathbb{R}_\mathcal{F}$ and $t_0 \in (a, b)$, with $f^-(t; r)$ and $f^+(t; r)$ both differentiable at t_0 for all $r \in [0, 1]$. We say that*

- *f is $[(i) - gH]$-differentiable at t_0 if*

$$f'_{i.gH}(t_0; r) = [(f^-)'(t_0; r)\ ,\ (f^+)'(t_0; r)], \tag{2.3}$$

- *f is $[(ii) - gH]$-differentiable at t_0 if*

$$f'_{ii.gH}(t_0; r) = [(f^+)'(t_0; r)\ ,\ (f^-)'(t_0; r)]. \tag{2.4}$$

Definition 2.6 *([12]) We say that a point $t_0 \in (a, b)$, is a switching point for the differentiability of f, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that*

type(I) *at t_1 (2.3) holds while (2.4) does not hold and at t_2 (2.4) holds and (2.3) does not hold, or*

type(II) *at t_1 (2.4) holds while (2.3) does not hold and at t_2 (2.3) holds and (2.4) does not hold.*

Theorem 2.1 *[6] Let $T = [a, a+\beta] \subset \mathbb{R}$, with $\beta > 0$ and $f \in \mathcal{C}^n_{gH}([a,b],\ \mathbb{R}$ $F).For s \in T$*

(i) *If $f^{(i)}$, $i = 0, 1, \ldots, n-1$ are $[(i)-gH]$-differentiable, provided that type of gH-differentiability has no change. Then*

$$f(s) = f(a) \oplus f'_{i.gH}(a) \odot (s-a) \oplus f''_{i.gH}(a) \odot \frac{(s-a)^2}{2!} \oplus \ldots \oplus f^{(n-1)}_{i.gH}(a) \odot \frac{(s-a)^{n-1}}{(n-1)!} \oplus R_n(a,s),$$

where

$$R_n(a,s) := \int_a^s \Big(\int_a^{s_1} \ldots \Big(\int_a^{s_{n-1}} f^{(n)}_{i.gH}(s_n) ds_n \Big) ds_{n-1} \ldots \Big) ds_1.$$

(ii) *If $f^{(i)}$, $i = 0, 1, \ldots, n-1$ is $[(ii)-gH]$-differentiable, provided that type of gH-differentiability has no change. Then*

$$f(s) = f(a) \ominus (-1) f'_{ii.gH}(a) \odot (s-a) \ominus (-1) f''_{ii.gH}(a) \odot \frac{(a-s)^2}{2!} \ominus (-1) \ldots \ominus (-1) f^{(n-1)}_{ii.gH}(a) \odot \frac{(a-s)^{n-1}}{(n-1)!} \ominus (-1) R_n(a,s),$$

where

$$R_n(a,s) := \int_a^s \Big(\int_a^{s_1} \ldots \Big(\int_a^{s_{n-1}} f^{(n)}_{ii.gH}(s_n) ds_n \Big) ds_{n-1} \ldots \Big) ds_1.$$

(iii) *If $f^{(i)}$ are $[(i)-gH]$-differentiable for $i = 2k-1$, $k \in \mathbb{N}$, and $f^{(i)}$ are $[(ii)-gH]$-differentiable for $i = 2k$, $k \in \mathbb{N} \cup \{0\}$. Then*

$$f(s) = f(a) \ominus (-1) f'_{ii.gH}(a) \odot (s-a) \oplus f''_{i.gH}(a) \odot \frac{(a-s)^2}{2!} \ominus (-1) \ldots \ominus (-1) f^{(\frac{i-1}{2})}_{ii.gH}(a) \odot \frac{(a-s)^{\frac{i}{2}-1}}{(\frac{i}{2}-1)!} \oplus f^{(\frac{i}{2})}_{i.gH}(a) \odot \frac{(a-s)^{\frac{i}{2}}}{(\frac{i}{2})!} \ominus (-1) \ldots \ominus (-1) R_n(a,s),$$

where

$$R_n(a,s) := \int_a^s \Big(\int_a^{s_1} \ldots \Big(\int_a^{s_{n-1}} f^{(n)}_{i.gH}(s_n) ds_n \Big) ds_{n-1} \ldots \Big) ds_1.$$

(iv) *Suppose that $f \in \mathcal{C}^n_{gH}([a,b],\ \mathbb{R}$*

$F)$,$n \geq 3$.

Furthermore let f in $[a,\xi]$ is $[(i)-gH]$-differentiable and in $[\xi, b]$ is $[(ii)-gH]$-differentiable, in fact ξ is switching point type I for first order derivative of f and $t_0 \in [a,\xi]$ in a neighborhood of ξ. Moreover suppose that second order derivative of f in ζ_1 of $[t_0,\xi]$ have switching point type II. Moreover type of differentiability for $f^{(i)}$, $i \leq n$ on $[\xi,\ b]$ don't change. So

$$f(s) = f(t_0) \oplus f'_{i.gH}(t_0) \odot (\xi - t_0) \ominus f''_{ii.gH}(t_0) \odot (t_0 - \zeta_1) \odot (\xi - t_0) \oplus f''_{i.gH}(\zeta_1) \Big(\frac{(\xi-\zeta_1)^2}{2} - \frac{(t_0-\zeta_1)^2}{2} \Big) \odot \ominus (-1) f'_{ii.gH}(\xi) \odot (s-\xi) \ominus (-1) f''_{ii.gH}(\xi) \odot \frac{(s-\xi)^2}{2!} \ominus (-1) \int_{t_0}^{\xi} \Big(\int_{t_0}^{\zeta_1} \Big(\int_{t_0}^{s_2} f'''_{ii.gH}(s_4) ds_4 \Big) ds_2 \Big) ds_1 \oplus \int_{t_0}^{\xi} \Big(\int_{\zeta_1}^{s_1} \Big(\int_{\zeta_1}^{s_3} f'''_{i.gH}(s_5) ds_5 \Big) ds_3 \Big) ds_1 \ominus (-1) \int_{\xi}^{s} \Big(\int_{\xi}^{t_1} \Big(\int_{t_0}^{t_2} f'''_{ii.gH}(t_3) dt_3 \Big) dt_2 \Big) dt_1.$$

3 Piecewise Approximate Method (PWA Method)

Consider the following second order fuzzy differential equation

$$\begin{cases} y''(t) = f(t, y(t)), & t \in I = [0, T], \\ y(0) = y_0, y'(0) = y'_0, \end{cases} \tag{3.5}$$

where the derivative $y^{(i)}$, $i = 1, 2$, is considered in the sense of gH-differentiable, where at the end points of I we consider only the one-sided derivatives, and the fuzzy function $f : I \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is sufficiently smooth function. The initial data y_0, y'_0 are assumed in $\mathbb{R}_{\mathcal{F}}$. The interval I may be $[0, T]$ for some $T > 0$ or $I = [0, \infty)$. We assume that $f : I \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy function, such that there exists $k > 0$ such that

$$D(f(t, x), f(t, z)) \leq kD(x, z), \quad \forall t \in I, \ x, z \in \mathbb{R}_{\mathcal{F}}. \tag{3.6}$$

Our construction of the fuzzy approximate solution $s(t)$ is as follows:
let $y(t)$ be the fuzzy solution of (3.5) determined by the fuzzy initial value problem y_0 and y'_0 . We divided the range of solution $[0, T]$ into subintervals of equal length $H = 3h = \frac{T}{n}$, and let $I_k = [kH, (k + 1)H]$, where $k = 0, \cdots, n - 1$. Let $s(t)$, $0 \leq t \leq T$ be a fuzzy approximate function of degree m.
In this paper we assume that $m = 4$, and we approximate fuzzy solution of (3.5) by fuzzy piecewise polynomial of order 4. Piecewise approximate solution $s(t)$ on $I_k = [kH, (k + 1)H]$, is construct step by step as follows:

Step 1: We define the first component of $s(t)$ by $s_0(t)$, in three cases:

Case(i): Let us suppose that the unique solution of problem (3.5), $y(t)$ is $[(i) - gH]$-differentiable, therefore

$$s_0(t) = y(0) \oplus t \odot y'_{i.gH}(0) \oplus \sum_{i=2}^{4} \alpha_{i,0} \odot \frac{t^i}{i!}, \tag{3.7}$$

for $0 \leq t \leq H$,

Case(ii): Now, consider $y(t)$ is $[(ii) - gH]$-differentiable, then $s_0(t)$ is obtained as follows:

$$s_0(t) = y(0) \ominus (-1)t \odot y'_{ii.gH}(0) \oplus \sum_{i=2}^{4} \alpha_{i,0} \odot \frac{t^i}{i!}, \tag{3.8}$$

for $\quad 0 \leq t \leq H$,

In Eqs (3.7) and (3.8), the coefficients $\alpha_{i,0}$ for $i = 2, 3, 4$ as yet undetermined and to be obtained where $s_0(t)$ satisfy the relations:

$$s''_0(jh) = f(jh, s_0(jh)), \tag{3.9}$$

for $j = 1, 2, 3$. By using Hausdorff distance(2.1), for $j = 1, 2, 3$ we obtain:

$$(s_0^-)''(jh, r) = f^-(jh, s_0(jh, r)), \tag{3.10}$$

$$(s_0^+)''(jh, r) = f^+(jh, s_0(jh, r)), \tag{3.11}$$

by solving (3.10) and (3.11), the value of $\alpha_{i,0}$ for $i = 2, 3, 4$ are obtained and $s_0(t)$ is constructed.

Step 2: The approximate solution $s(t)$ in interval $[H, 2H]$ is obtained as follows:

$$s(t) = \sum_{i=0}^{1} s_0^{(i)}(t) \odot \frac{(t-H)^i}{i!} \oplus \sum_{i=2}^{4} \alpha_{i,k} \odot \frac{(t-H)^i}{i!}, \tag{3.12}$$

where $s_0(t)$ is obtained by step 1. The value of $\alpha_{i,k}$ are to be determined where $s(t)$ satisfy the following relations:

$$s''(jh) = f(jh, s(jh)). \tag{3.13}$$

This means for $j = 4, 5, 6$,

$$(s^-)''(jh, r) = f^-(jh, s(jh, r)), \tag{3.14}$$

$$(s^+)''(jh, r) = f^+(jh, s(jh, r)), \tag{3.15}$$

by solving (3.14) and (3.15), the values of $\alpha_{i,k}$ are obtained.

Step 3: The approximate solution $s(t)$ in interval $[kH, (k + 1)H]$ for $k = 2, \cdots, n - 1$ is obtained as follows:

$$s(t) = \sum_{i=0}^{1} s_{3k}^{(i)}(t) \odot \frac{(t-kH)^i}{i!} \oplus \sum_{i=2}^{4} \alpha_{i,k} \odot \frac{(t-kH)^i}{i!}, \tag{3.16}$$

The value of $\alpha_{i,k}$ are to be determined where $s(t)$ satisfy the following relations:

$$s''(jh) = f(jh, s(jh)). \tag{3.17}$$

This means for $j = 3k+1, 3k+2, 3k+3$; $k = 2, \cdots, n-1$,

$$(s^-)''(jh, r) = f^-(jh, s(jh, r)), \tag{3.18}$$

$$(s^+)''(jh, r) = f^+(jh, s(jh, r)), \tag{3.19}$$

by solving (3.18) and (3.19), the values of $\alpha_{i,k}$ are obtained.

Finally the PWA method is obtained as follows

$$s(t) = \sum_{i=0}^{1} s_{3k}^{(i)}(t) \odot \frac{(t-kH)^i}{i!} \oplus \sum_{i=2}^{4} \alpha_{i,k} \odot \frac{(t-kH)^i}{i!}, \tag{3.20}$$

where

$$s_0(t) = y(0) \oplus t \odot y'_{i.gH}(0) \oplus \sum_{i=2}^{4} \alpha_{i,0} \odot \frac{t^i}{i!}, \tag{3.21}$$

if $y(t)$ is $[(i)-gH]-differentiable$.

$$s_0(t) = y(0) \ominus (-1)t \odot y'_{ii.gH}(0) \oplus \sum_{i=2}^{4} \alpha_{i,0} \odot \frac{t^i}{i!}, \tag{3.22}$$

if $y(t)$ is $[(ii)-gH]-differentiable$.

3.1 Existence and uniqueness

In this section we prove that there exist a unique fuzzy function $s(t)$ where approximate the solution of second order fuzzy differential equation (3.5), provided that the size of the subinterval h satisfies some constraints.

Theorem 3.1 *If $h = \min\{h_1, h_2, h_3\}$, where*

$$h_1 < \sqrt{\frac{2}{L}}, h_2 < \sqrt{\frac{6}{L}}, h_3 < \sqrt{\frac{24}{L}} \tag{3.23}$$

then the approximate solution defined by (3.20), exists and unique.

Proof: Let $t = jh$ and $j = 3k+\eta$ for $\eta = 1, 2, 3$, therefore

$$s''((3k+\eta)h) = s''_{3k+\eta} = \sum_{i=2}^{4} \alpha_{i,k} \frac{(\eta h)^{i-2}}{(i-2)!} \tag{3.24}$$

By solving system (3.24) we obtain:

$$\alpha^+_{2,k} = 3(s^+_{3k+1})'' - 3(s^+_{3k+2})'' + (s^+_{3k+3})'', \tag{3.25}$$

$$\alpha^+_{3,k} = \frac{1}{h}[-\frac{5}{2}(s^+_{3k+1})'' + 4(s^+_{3k+2})'' - \frac{3}{2}(s^+_{3k+3})''], \tag{3.26}$$

$$\alpha^+_{4,k} = \frac{1}{h^2}[(s^+_{3k+1})'' - 2(s^+_{3k+2})'' + (s^+_{3k+3})''], \tag{3.27}$$

and

$$\alpha^-_{2,k} = 3(s^-_{3k+1})'' - 3(s^-_{3k+2})'' + (s^-_{3k+3})'', \tag{3.28}$$

$$\alpha^-_{3,k} = \frac{1}{h}[-\frac{5}{2}(s^-_{3k+1})'' + 4(s^-_{3k+2})'' - \frac{3}{2}(s^-_{3k+3})''], \tag{3.29}$$

$$\alpha^-_{4,k} = \frac{1}{h^2}[(s^-_{3k+1})'' - 2(s^-_{3k+2})'' + (s^-_{3k+3})''], \tag{3.30}$$

To prove the existence and uniqueness of $s(t)$, let us define the operator $G_\nu : \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ by $\alpha_{j,k} = G_\nu(\alpha_{j,k})$ for $j = 2, 3, 4$ and $v = 1, 2, 3$. According to condition (3.6) and equations (3.25), (3.26), (3.27) and (3.28), (3.29), (3.30) we conclude that

$$D(G_1(\alpha_{2,k}), G_1(\alpha^*_{2,k}) \leq L\frac{h^2}{2} D(\alpha_{2,k}, \alpha^*_{2,k})|3-3+1|, \tag{3.31}$$

$$D(G_2(\alpha_{3,k}), G_2(\alpha^*_{3,k}) \leq L\frac{h^3}{6} D(\alpha_{3,k}, \alpha^*_{3,k})|\frac{1}{h}(-\frac{5}{2}+8-\frac{9}{2})|, \tag{3.32}$$

$$D(G_3(\alpha_{4,k}), G_3(\alpha^*_{4,k}) \leq L\frac{h^4}{24} D(\alpha_{4,k}, \alpha^*_{4,k})|\frac{1}{h^2}(\frac{1}{2}-4+\frac{9}{2})|, \tag{3.33}$$

From Equations (3.31), (3.32), (3.33), and

$$h_1 < \sqrt{\frac{2}{L}}, \quad h_2 < \sqrt{\frac{6}{L}}, \quad h_3 < \sqrt{\frac{24}{L}}$$

it follows that $G_\nu, \quad \nu = 1, 2, 3$ are contraction operators. This implies the existence and uniqueness of approximate solution under the stated conditions of theorem.

3.2 Consistency relations and convergence

It is well-known that a linear method will be convergent if and only if, It is both consistent and stable.

Theorem 3.2 *The piecewise approximate functions (3.20), are consistent.*

proof: In the case of [(i)-gH]-differentiability, $s(t)$ is defined on I_k as:

$$s(t) = \sum_{i=0}^{1} s_{3k}^{(i)}(t) \odot \frac{(t-3kh)^i}{i!} \oplus \sum_{i=2}^{4} \alpha_{i,k} \odot \frac{(t-3kh)^i}{i!}, \quad (3.34)$$

and the parametric form of $s(t) = (s^-(t,r), s^+(t,r))$ is as following:

$$s^-(t,r) = \sum_{i=0}^{1} \frac{(s_{3k}^-)^{(i)}(t)}{i!}(t-3kh)^i + \sum_{i=2}^{4} \frac{\alpha_{i,k}^-}{i!}(t-3kh)^i, \quad (3.35)$$

$$s^+(t,r) = \sum_{i=0}^{1} \frac{(s_{3k}^+)^{(i)}(t)}{i!}(t-3kh)^i + \sum_{i=2}^{4} \frac{\alpha_{i,k}^+}{i!}(t-3kh)^i, \quad (3.36)$$

without lose generality, we just proof consistency for s^+, and for s^- is similar.

On differentiating equation (3.36) and setting $t = jh$ with $j = 3k+1,\ 3k+2,\ 3k+3$, we obtain

$$(s^+)''((3k+\eta)h) = (s^+)''_{3k+\eta} = \sum_{i=2}^{4} \alpha_{i,k}^+ \frac{(\eta h)^{i-2}}{(i-2)!}, for \quad \eta = 1(1)3, \quad (3.37)$$

on eliminating $\alpha_{i,k}^+$, we obtain:

$$s_{3(k-1)}^+ - 2s_{3k}^+ + s_{3(k+1)}^+ = h^2\{\frac{405}{12}(s_{3k+1}^+)'' - \frac{486}{12}(s_{3k+2}^+)'' + \frac{189}{12}(s_{3k+3}^+)''\} \quad (3.38)$$

Hence, the associative polynomials $\rho(\xi)$ and $\sigma(\xi)$ are

$$\rho(\xi) = \xi^6 - 2\xi^3 + 1, \qquad \sigma(\xi) = \frac{405}{12}\xi^4 - \frac{486}{12}\xi^5 + \frac{189}{12}\xi^6, \quad (3.39)$$

clearly $\rho(1) = 0, \rho'(1) = 0$ and $\rho''(1) = 2\sigma(1)$, and the method is consistent. Also the condition of stability is fulfilled since the zeros of $\rho(\xi)$ do not exceed unity in modulus, multiple zeros of multiplicity 2 and thus the method is convergent.

Table 1: Error of PWA method by Hausdorff distance in example 4.1

t	Error of PWA method Case (i)	Case(ii)
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0

Table 2: Error of PWA method by Hausdorff distance in example 4.2

t	Case (i)	Case(ii)
0	0	0
0.1	0.000003073	0.0000030737
0.2	0.000007067	0.0000070678
0.3	0.000010994	0.0000109946
0.4	0.000018675	0.0000186745
0.5	0.000027282	0.0000272813
0.6	0.000035617	0.0000356173
0.7	0.000047701	0.0000477022
0.8	0.000060486	0.00006048718
0.9	0.000072667	0.0000726680

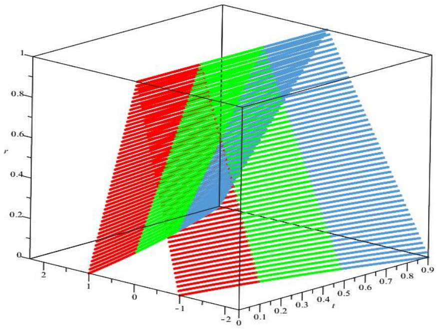

Figure 1: Approximate solution for case(i) in example 4.1. Red points: $s_0(t)$; Green points: $s_3(t)$; Blue. points: $s_6(t)$

4 Numerical Example

Example 4.1 *[20] Let us consider the following second-order fuzzy initial value problem*

$$y''(t) = \sigma_0, \quad y_0 = \gamma_0, \quad y'(0) = \gamma_1, \qquad (4.40)$$

where $\sigma_0 = \gamma_0 = \gamma_1$ are the triangular fuzzy number having r-level sets $[r-1, 1-r]$.

Case(i) If $y(t)$ is $[(i)-gH]$-differentiable, the real solution is:

$$y^-(t,r) = (r-1)\{\frac{t^2}{2} + t + 1\},$$
$$y^+(t,r) = (1-r)\{\frac{t^2}{2} + t + 1\},$$

Now we use PWA method to obtain piecewise approximate solution $s(t)$. Let $I_k = [kH, (k+1)H]$, for $k = 0, 1, 2$, $H = 3h$ and $h = 0.1$. $s_0(t)$, $s_3(t)$ and $s_6(t)$ are obtained as follows:

$$\begin{aligned}
s_0^-(t) &= (r-1) + t(r-1) + \frac{t^2}{2}(r-1),\\
s_0^+(t) &= (1-r) + t(1-r)t + \frac{t^2}{2}(1-r),\\
s_3^-(t) &= 1.345r - 1.345\\
&+ (t-0.3)(1.3r-1.3)\\
&+ \frac{(t-0.3)^2}{2}(r-1),\\
s_3^+(t) &= 1.345 - 1.345r\\
&+ (t-0.3)(1.3-1.3r)\\
&+ \frac{(t-0.3)^2}{2}(1-r),\\
s_6^-(t) &= 1.78r - 1.78\\
&+ (t-0.6)(1.6r-1.6)\\
&+ \frac{(t-0.6)^2}{2}(r-1),\\
s_6^+(t) &= 1.78 - 1.78r\\
&+ (t-0.6)(1.6-1.6r))\\
&+ \frac{(t-0.6)^2}{2}(1-r),
\end{aligned}$$

The approximate solution $s_i(t)$ in Case(i), for $i = 0, 1, 2$, is plotted in Fig 1.

Case(ii) If $y(t)$ is $[(ii) - gH]$-differentiable, the real solution is:

$$y^-(t,r) = (r-1)\{\frac{t^2}{2} - t + 1\},$$
$$y^+(t,r) = (1-r)\{\frac{t^2}{2} - t + 1\},$$

in this case $s_0(t)$, $s_3(t)$ and $s_6(t)$ are obtained as follows:

$$\begin{aligned}
s_0^-(t) &= (r-1)+t(1-r)+\frac{t^2}{2}(r-1),\\
s_0^+(t) &= (1-r)+t(r-1)t+\frac{t^2}{2}(1-r),\\
s_3^-(t) &= .745r-.745+(t-0.3)(0.7-.7r)\\
&+ \frac{(t-0.3)^2}{2}(r-1),\\
s_3^+(t) &= .745-.745r+(t-0.3)(0.7r-.7)\\
&+ \frac{(t-0.3)^2}{2}(1-r),\\
s_6^-(t) &= .58r-.58+(t-0.6)(.4-.4r)\\
&+ \frac{(t-0.6)^2}{2}(r-1),\\
s_6^+(t) &= .58-.58r+(t-0.6)(.4r-.4r)\\
&+ \frac{(t-0.6)^2}{2}(1-r),
\end{aligned}$$

The approximate solution $s_i(t)$ in Case(ii), for $i = 0,1,2$, is plotted in Fig 2.

Example 4.2 *[20] Consider the fuzzy initial value problem*

$$y''(t)+y(t)=\sigma_0, \quad y(0)=\gamma_0, \quad y'(0)=\gamma_1,$$

where σ_0 is the fuzzy number having r-level sets $[r,2-r]$. $[\gamma_0]^r=[\gamma_1]^r=[r-1,1-r]$.

Case(i) If $y(t)$ is $[(i)-gH]$-differentiable, the real solution is:

$$\begin{aligned}
y^-(t,r) &= r(1+\sin(t))-\sin(t)-\cos(t),\\
y^+(t,r) &= (2-r)(1+\sin(t))\\
&- \sin(t)-\cos(t),
\end{aligned}$$

Let $I_k=[kH,(k+1)H]$, for $k=0,1,2$, $H=3h$ and $h=0.1$. $s_0(t)$, $s_3(t)$ and $s_6(t)$ are obtained as follows:

$$\begin{aligned}
s_0^-(t) &= (r-1)+t(r-1)\\
&+ \frac{t^2}{2}(.9992+0.00099r)\\
&+ \frac{t^3}{3!}(1.016-1.01817r)\\
&+ \frac{t^4}{4!}(-1.1778+.1986r),\\
s_0^+(t) &= (1-r)+t(1-r)\\
&+ \frac{t^2}{2}(1.001-0.00099r)\\
&+ \frac{t^3}{3!}(-1.021+1.0182r)\\
&+ \frac{t^4}{4!}(-.7807-.1985r),\\
s_3^-(t) &= (1.295r-1.2509)\\
&+ (t-0.3)(.9554r-.6599)\\
&+ \frac{(t-0.3)^2}{2}(1.251-.2947r)\\
&+ \frac{(t-0.3)^3}{3!}(.6688-.972r)\\
&+ \frac{(t-0.3)^4}{4!}(-1.356+.4791),\\
s_3^+(t) &= (1.3402-1.296r)\\
&+ (t-0.3)(1.2509-.9554r)\\
&+ \frac{(t-0.3)^2}{2}(.6612+.2946r)\\
&+ \frac{(t-0.3)^3}{3!}(-1.275+.972r)\\
&+ \frac{(t-0.3)^4}{4!}(-.3978-.4791r),\\
s_6^-(t) &= (1.565r-1.39)\\
&+ (t-0.6)(.8254r-.2608)\\
&+ \frac{(t-0.6)^2}{2}(1.39-.564r)\\
&+ \frac{(t-0.6)^3}{3!}(.26201-.839r)\\
&+ \frac{(t-0.3)^4}{4!}(-1.413+.7169r),\\
s_6^+(t) &= (1.74-1.565r)\\
&+ (t-0.6)(1.3901-.8254r)\\
&+ \frac{(t-0.6)^2}{2}(.26208+.56394r)\\
&+ \frac{(t-0.6)^3}{3!}(-1.416+.839r)\\
&+ \frac{(t-0.3)^4}{4!}(0.0206-.7168r),
\end{aligned}$$

The approximate solution $s_i(t)$ in Case(i), for $i = 0, 1, 2$, is plotted in Fig 3.

Case(ii) If $y(t)$ is $[(ii) - gH]$-differentiable, the real solution is:

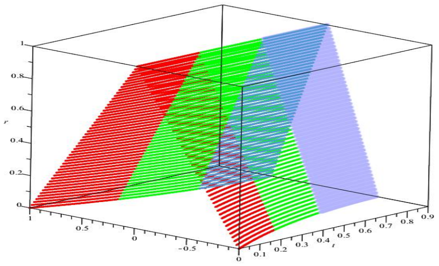

Figure 2: Approximate solution for case(ii) in example 4.1. Red points: $s_0(t)$; Green points: $s_3(t)$; Blue points: $s_6(t)$

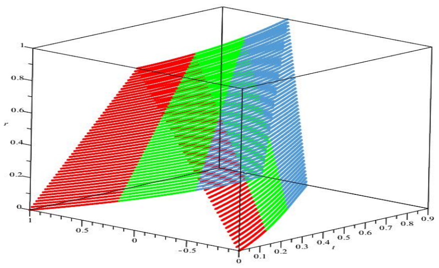

Figure 3: Approximate solution for case(ii) in example 4.2. Red points: $s_0(t)$; Green points:$s_3(t)$; Blue points: $s_6(t)$

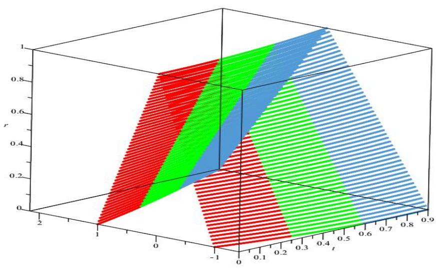

Figure 4: Approximate solution for case(i) in example 4.2. Red points: $s_0(t)$; Green points: $s_3(t)$; Blue points: $s_6(t)$

$$y^-(t, r) = r(1 - \sin(t)) + \sin(t) - \cos(t),$$

$$y^+(t, r) = (2 - r)(1 - \sin(t)) + \sin(t) - \cos(t),$$

$s_0(t)$, $s_3(t)$ and $s_6(t)$ are obtained as follows:

$$\begin{aligned}
s_0^-(t) &= (r-1) + t(1-r) \\
&+ \frac{t^2}{2}(1.0011 - 0.00099r) \\
&+ \frac{t^3}{3!}(-1.021 + 1.0182r) \\
&+ \frac{t^4}{4!}(-.78074 - .19851r), \\
s_0^+(t) &= (1-r) + t(r-1) \\
&+ \frac{t^2}{2}(.9992 + 0.00099r) \\
&+ \frac{t^3}{3!}(1.0157 - 1.01818r) \\
&+ \frac{t^4}{4!}(-1.1778 + .1985r), \\
s_3^-(t) &= (.7045r - .6599) \\
&+ (t-0.3)(1.251 - .95537r) \\
&+ \frac{(t-0.3)^2}{2}(.66114 + .2946r) \\
&+ \frac{(t-0.3)^3}{3!}(-1.7527 + .97199r) \\
&+ \frac{(t-0.3)^4}{4!}(-.3978 - .4791r), \\
s_3^+(t) &= (.7492 - .70447r) \\
&+ (t-0.3)(.9554r - .65985) \\
&+ \frac{(t-0.3)^2}{2}(1.2504 - .29463r) \\
&+ \frac{(t-0.3)^3}{3!}(.66872 - .97199r) \\
&+ \frac{(t-0.3)^4}{4!}(-1.3559 + .47905r), \\
s_6^-(t) &= (.43533r - .26066) \\
&+ (t-0.6)(1.3901 - .825399r) \\
&+ \frac{(t-0.6)^2}{2}(.26207 + .56394r) \\
&+ \frac{(t-0.6)^3}{3!}(-1.41597 + .83899r) \\
&+ \frac{(t-0.3)^4}{4!}(0.0207 - .71680r), \\
s_6^+(t) &= (.610 - .43533r) \\
&+ (t-0.6)(.8254r - .26074) \\
&+ \frac{(t-0.6)^2}{2}(1.3899 - .56394r) \\
&+ \frac{(t-0.6)^3}{3!}(.26201 - .838989r) \\
&+ \frac{(t-0.3)^4}{4!}(-1.413 + .7168r),
\end{aligned}$$

The approximate solution $s_i(t)$ in Case(ii), for $i = 0, 1, 2$, is plotted in Fig 4.

5 Conclusion

In this paper a new approach for solving second order fuzzy differential equations under generalized differentiability was proposed. We used piecewise fuzzy polynomial of degree 4 based on the Taylor expansion for approximating solutions of second order fuzzy differential equations. Also, we can develop this method for Nth-order fuzzy differential equations under generalized derivatives.

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A Piecewise Approximate Method for Solving Second Order Fuzzy Differential Equations Under Generalized Differentiability

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روش تقریب قطعه ای برای حل معادلات دیفرانسیل فازی از مرتبه دوم تحت مشتق تعمیم یافته

چکیده:

در این مقاله یک روش عددی برای حل معادلات دیفرانسیل فازی از مرتبه دوم ارائه شده است. این روش بر اساس درونیابی جواب توسط یک قطعه ای چند جمله ای از درجه چهار در بازه جواب می باشد. وجود، کتایی و همگرایی جواب تقریبی نیز مورد بررسی قرار گرفته است. در آخر دقت روش تقریب قطعه ای با چند مثال نشان داده شده است.