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Planarity of Intersection Graph of submodules of a Module

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Abstract

Let R be a commutative ring with identity and M be an unitary R-module. The intersection graph of an R-module M, denoted by $\Gamma(M)$, is a simple graph whose vertices are all non-trivial submodules of M and two distinct vertices N_1 and N_2 are adjacent if and only if $N_1 \cap N_2 \neq 0$. In this article, we investigate the concept of a planar intersection graph and maximal submodules of an R-module. In particular, we show that if $\Gamma(M)$ is a planar graph, then $M \cong M_1 \oplus M_2$ for a multiplication R-module M with $|Max(M)| \neq 1$.

Keywords: Interval methods; Multiplication modules; Planar Graph; Module Theory; Torsion Graphs.

1 Introduction

 \mathbf{I}^{T} is well known that graph is a very useful tool to model problems originated in all most all areas of our life. In this article, we concentrate our discussion on intersection graphs. Let $S = \{S_i : i \in I\}$ be an arbitrary family of sets. The intersection graph $\Gamma(S)$ of S is the graph whose vertices are S_i , $i \in I$ and there is an edge between two distinct vertices S_i and S_j if and only if $S_i \cap S_j \neq \emptyset$. It is more interesting to study the intersection graphs $\Gamma(S)$ when the elements of S have an algebraic structure. These studies allow us to obtain characterization and representation of the classes of algebraic structure in terms of graphs and vice versa.

Let R be a commutative ring with identity and M be a unitary R-module. The idea of the intersection graph of semigroups was introduced by Bosak in [5]. Inspired by his work, $Cs\acute{a}k\acute{a}ny$ and

Pollák in [8], studied the graph of subgroups of a finite group. The intersection graph of ideals of a ring, was considered by Chakrabarty, Ghosh, Mukherjee and Sen in [7]. Recently, Akbari, Tavallaee and Khaiashi in [1], introduced and investigated the intersection graph of submodules of a module.

In this paper, we investigate the concept of intersection graph of a module. The intersection graph of an *R*-module M, denoted by $\Gamma(M)$, is defined to be the undirected simple graph whose vertices are all non-trivial submodules of M and two distinct vertices are adjacent if and only if the corresponding submodules of M have nonzero intersection. This study helps to illuminate the structure of M, for example, if $\Gamma(M)$ is a planar graph, then M is both Noetherian and Artinian.

Recall that a simple graph is finite if its vertices set is finite, and we use the symbol $|\Gamma(M)|$ to denote the number of vertices in graph $\Gamma(M)$. Also, a graph G is connected if there is a path between any two distinct vertices. The distance, d(x, y)between connected vertices x, y is the length of the shortest path from x to y, $(d(x, y) = \infty$ if there is no such path). An isolated vertex is a

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vertex that has no edges incident to it. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n for the complete graph with n vertices. The complement G of Gis the graph with vertex set $V(\overline{G}) = V(G)$, and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. The complement of a complete graph is the null graph. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A remarkably simple characterization of planar graphs was given by Kuratowski in [5], p.153. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

An R-module M is a multiplication module if for every R-submodule K of M there is an ideal I of R such that K = IM. Note that $I \subseteq [N:M]$, hence $N = IM \subseteq [N:M]M \subseteq N$. So N = [N : M]M. An *R*-module *M* is called a cancellation module if IM = JM for any ideals I and J of R implies that I = J. Also, an R-module M is a weak-cancellation module if IM = JM for any ideals I and J of R implies that I + Ann(M) = J + Ann(M). Finitely generated multiplication modules are weak cancellation, Theorem 3 [2]. Let P be a maximal ideal of R. An R-module M is called P-torsion if for each $m \in M$ there exists $p \in P$ such that (1-p)m = 0. On the other hand, M is called *P*-cyclic if there exists $x \in M$ and $q \in P$ such that $(1-q)M \subseteq Rx$. Theorem 1.2 [6] showed that an R-module M is multiplication if and only if for every maximal ideal P of R either M is *P*-torsion or *P*-cyclic.

In this paper, we study the number of maximal and minimal prime submodule of multiplication modules. It is shown that if $\Gamma(M)$ is a planar graph, then $|Max(M)| \leq 4$ and $|Min(M)| \leq 4$. Also, we show that, if M is a multiplication Rmodule with $|Max(M)| \neq 1$ and $\Gamma(M)$ is a planar graph, then $M \cong M_1 \oplus M_2$.

Throughout the paper, Max(M) is a set of the maximal submodules H of M, we use symbol |Max(M)| to denote the number of maximal submodule of M. As a consequence of Theorem 2.5 [6], for any non-zero multiplication R-module $Max(M) \neq \emptyset$. Also, Min(M) is a set of the minimal prime submodules N of M. let J(R) be the Jacobson radical of R and

$$J(M) := \cap_{H \in Max(M)} H.$$

We follow standard notation and terminology from graph theory [5] and module theory [3].

2 Planar intersection graph

This section is concerned with some basic and important results in the theory of planar torsion graphs over a module.

Lemma 2.1 Let M be an R-module. If $\Gamma(M)$ is a planar graph, then M is both Noetherian and Artinian.

Proof. Let $N_1 \subset N_2 \subset N_3 \subset N_4 \subset N_5 \subset \ldots$ be a chine of nontrivial proper submodule of M. Then vertices N_i , $1 \leq i \leq 5$ form K_5 as an induced subgraph, which is a contradiction. So every chain of nontrivial proper submodule of M is stationary. Therefore M is both Noetherian and Artinian.

Lemma 2.2 Let M be a multiplication R-module and N be a prime submodule of M. If $\bigcap_{i=1}^{n} N_i \subseteq$ N, where N_i be a submodule of M, then there is $1 \leq i \leq n$ such that $N_i \subseteq N$.

Proof. Let $\bigcap_{i=1}^{n} N_i \subseteq N$, where N_i be a submodule of M. Then $[N_1 : M][N_2 : M] \dots [N_n : M]M \subseteq N$. Since N is a prime submodule of M, there is $1 \leq i \leq n$ such that $[N_i : M] \subseteq [N : M]$. Therefore $N_i \subseteq N$.

Lemma 2.3 Let M be a Q-cyclic R-module for all maximal ideal Q of R. Then [N : M] is a prime ideal of R for any proper submodule N of M if and only if [N : M]M is a prime submodule of M.

Proof. Let [N:M] be a prime ideal of R. Clearly [N:M]M is a proper submodule of M. Suppose $ax \in [N:M]M$ such that $a \notin [N:M]$, for some $a \in R$ and $x \in M$. Let $k = \{r \in R | rx \in [N:M]M\}$. If $k \neq R$, then there is a maximal ideal Q of R such that $k \subseteq Q$. Since M is a Q-cyclic R-module, $(1-q)M \subseteq Rm$ for some $q \in Q$ and $m \in M$. Hence $(1-q)ax \in (1-q)[N:M]M \subseteq [N:M]m$. So (1-q)x = sm and $(1-q)ax = \alpha m$ for some $s \in R$ and $\alpha \in [N:M]$. Thus $(\alpha - sa)m = 0$. It is clear that (1 - q)

 $q)ann(m) \subseteq Ann(M)$. Therefore $(1-q)(\alpha-sa) \in Ann(M) \subseteq [N:M]$. Then $(1-q)sa \in [N:M]$. Hence $(1-q) \in k \subseteq Q$, which is a contradiction. This contradiction implies that k = R and so $x \in [N:M]M$. Therefore [N:M]M is a prime submodule of M.

Conversely, let N be a prime submodule of M. Thus [N : M] is a proper ideal of R. Suppose $st \in [N : M]$. So $sM \subseteq N$ or $tM \subseteq N$. Therefore [N : M] is a prime ideal of R.

Theorem 2.1 Let M be a Q-cyclic R-module for all maximal ideal Q of R. If $\Gamma(M)$ is a planar graph, then $|Min(M)| \leq 3$.

Proof. Let $\Gamma(M)$ be a planar graph. Suppose $|Min(M)| \ge 4$ and $N_1, N_2 \dots N_4$ be distinct minimal submodules of M, such that $N_1 \cap N_2 \cap N_3 = 0$. Then $[N_1 : M][N_2 : M][N_3 : M]M \subseteq N_4$. Hence $[N_1 : M][N_2 : M][N_3 : M] \subseteq [N_4 : M]$. It is clear that $[N_4 : M]$ is a prime ideal of R. So $[N_i : M]M \subseteq [N_4 : M]M \subseteq N_4$, for some $1 \le i \le 3$. By Lemma 2.3, [N : M]M is a prime submodule of M. Also, since N is a minimal prime submodule of M, [N : M]M = N. Therefore $N_i = N_4$ for some $1 \le i \le 3$, which is a contradiction. Hence $N_1 \cap N_2 \cap N_3 \ne 0$. Therefore, vertices $N_1 \cap N_2, N_1 \cap N_3, N_2 \cap N_3, N_1, N_2$ and N_3 form K_6 as an induced subgraph, which is a contradiction. Consequently $|Min(M)| \le 3$.

Corollary 2.1 Let M be a multiplication Rmodule. If $\Gamma(M)$ is a planar graph, then $\bigcap_{N \in Min(M)} N = 0.$

Proposition 2.1 Let M be a multiplication R-module. If $\Gamma(M)$ is a planar graph, then $1 \leq |Max(M)| \leq 3$.

Proof. Let $\Gamma(M)$ be a planar graph. Suppose $|Max(M)| \ge 4$ and H_1, H_2, \ldots, H_4 be distinct maximal submodules of M, such that $H_1 \cap H_2 \cap H_3 = 0$. Then $H_1 \cap H_2 \cap H_3 \subseteq H_4$. Sine every maximal submodule of multiplication modules is prime, by Lemma 2.2, $H_i \subseteq H_4$, for some $1 \le i \le 3$. But H_i is a maximal submodule of M implies that $H_i = H_4$ for some $1 \le i \le 3$, which is a contradiction, hence $H_1 \cap H_2 \cap H_3 \ne 0$. Therefore, vertices $H_1 \cap H_2$, $H_1 \cap H_3$, $H_2 \cap H_3$, H_1 , H_2 and H_3 form K_6 as an induced subgraph, which is a contradiction. Consequently $1 \le |Max(M)| \le 3$.

Corollary 2.2 Let M be a multiplication R-module. If $\Gamma(M)$ is a planar graph, then J(M) = 0.

Proposition 2.2 Let $M = M_1 \times M_2$ be an *R*module. Then $\Gamma(M)$ is planar if and only if $\Gamma(M_1)$ or $\Gamma(M_2)$ is empty and another is null.

Proof. Let $\Gamma(M)$ be a planar graph. Suppose that $\Gamma(M_1)$ and $\Gamma(M_2)$ are not empty. So there exist nontrivial proper submodules N_1 of M_1 and N_2 of M_2 . Therefore $0 \times N_2$, $0 \times M_2$, $N_1 \times M_2$, $N_1 \times N_2$ and $M_1 \times N_2$ form K_5 as an induced subgraph, which is a contradiction. Hence one of $\Gamma(M_1)$ or $\Gamma(M_2)$ is empty. Let $\Gamma(M_2)$ be empty. Now we show that $\Gamma(M_1)$ is null. If N_1 is a proper nontrivial submodule of M_1 such that it is adjacent to H_1 for some $H_1 \in V(\Gamma(M_1))$, then $N_1 \cap H_1 \neq 0$. So $N_1 \times 0$, $H_1 \times 0$, $M_1 \times 0$, $N_1 \times M_2$ and $H_1 \times M_2$ form K_5 as an induced subgraph, which is a contradiction. This contradiction implies that $\Gamma(M_1)$ is null.

Corollary 2.3 $\Gamma(M_1 \times M_2 \times M_3)$ is planar if and only if M_i is a simple R_i -module for $i \in \{1, 2, 3\}$.

Proof. Let $\Gamma(M_1 \times M_2 \times M_3)$ be a planar graph and M_1 not simple. So there exists $0 \neq N < M_1$. Then $N \times M_2 \times M_3$, $0 \times M_2 \times M_3$, $N \times M_2 \times 0$, $M_1 \times M_2 \times 0$ and $M_1 \times 0 \times M_3$ form K_5 as an induced subgraph of $\Gamma(M)$, which is a contradiction. Therefore M_i is a simple R_i module for $i \in \{1, 2, 3\}$.

Proposition 2.3 Let M be a multiplication Rmodule with |Max(M)| = 3. If $\Gamma(M)$ is planar, then $M \cong M_1 \oplus M_2$ where M_1 and M_2 are simple.

Proof. Let |Max(M)| = 3 and H_i , $1 \le i \le 3$ be distinct maximal submodules of M. By Corollary 2.2, $H_1 \cap H_2 \cap H_3 = 0$. If $H_2 \cap H_3 = 0$, then $H_2 \cap H_3 \subseteq H_1$ and by Lemma 2.2, $H_1 = H_2$ or $H_1 = H_3$, which is a contradiction. Hence M = $H_1 \oplus H_2 \cap H_3$. By Proposition 2.2, one of $\Gamma(H_1)$ or $\Gamma(H_2 \cap H_3)$ is null another is empty. Suppose that $\Gamma(H_1)$ be null. If H_1 is not a simple submodule of M. Then there is a nontrivial submodule N_1 of H_1 such that N_1 is adjacent to $N_1 + H_2 \cap H_3$. So $\Gamma(H_1)$ is not null, which is a contradiction. Thus H_1 and $H_2 \cap H_3$ are simple.

Lemma 2.4 Let M be a faithful finitely generated multiplication R-module. Then J(R)M = J(M). 228

Proof. Let M be a faithful finitely generated multiplication R-module and H be a maximal submodule of M. By Theorem 3.1 of [6], $hM \neq M$ for all maximal ideal h of M. Also, by Theorem 2.5 of [6], H = hM for some maximal ideal h of M. On the other hand by Theorem 1.6 of [6], $J(M) = \bigcap_{H \in Max(M)} H = \bigcap_{h \in Max(R)} (hM) =$ $(\bigcap_{h \in Max(R)} h)M = J(R)M$

Theorem 2.2 Let M be a faithful multiplication R-module with |Max(M)| = 2. Then $\Gamma(M)$ is a planar graph if and only if $M \cong [H_1 : M]^4 M \oplus [H_2 : M]^4 M$ such that $\Gamma([H_1 : M]^4 M)$ or $\Gamma([H_1 : M]^4 M)$ is empty another is null, where H_1, H_2 are maximal submodule of M.

Proof. Let H_1 and H_2 be distinct maximal submodules of M. Suppose that $[H_1: M]^4 M + [H_2:$ $M^{4}M \neq M$. By Theorem 2.5 of [6], there is a maximal submodule H of M such that $[H_1 :$ $M^{4}M + [H_{2}: M]^{4}M \subseteq H.$ Since |Max(M)| = 2, we have $H = H_1$ or $H = H_2$. It follows that $[H_1 : M]^4 M \subseteq H_2 \text{ or } [H_2 : M]^4 M \subseteq H_1.$ Thus $H_1 = H_2$, which is a contradiction. So $M = [H_1 : M]^4 M + [H_2 : M]^4 M$. Assume $[H_1 : M]^4 M$. $M^{4}_{1}M \cap [H_{2}: M]^{4}_{1}M \neq 0$. Hence $H_{1} \cap H_{2} \neq 0$. On the other hand By Theorem 1.6 [6], $[H_1]$: $M^{i}M \cap [H_2:M]^{i}M = ([H_1:M]^{i} \cap [H_2:M]^{i})M,$ for all positive integer i. Since M is a cyclic faithful multiplication module, by Lemma 2.4, we have J(R)M = J(M). Now Nakayama's lemma follows that $([H_1 : M]^4 \cap [H_2 : M]^4)M \subset \ldots \subset$ $([H_1:M] \cap [H_2:M])M \subset H_1$. Hence $\Gamma(M)$ contains an induced subgraph K_5 , which is a contradiction. Therefore $[H_1:M]^4 M \cap [H_2:M]^4 M =$ 0. Consequently $M \cong [H_1:M]^4 M \oplus [H_2:M]^4 M$ and by Proposition 2.2, the result follows.

Proposition 2.4 Let M be a multiplication Rmodule with |Max(M)| = 1. If $\Gamma(M)$ is a planar graph, then $|M| \le 5$ or $[H : M]^5 M = 0$ where His a maximal submodule of M.

Proof. Suppose M be a faithful multiplication R-module. If $\Gamma(M)$ is a planar graph, then by Lemma 2.1, M is finitely generated and by Lemma 2.4, R is a local ring with unique maximal ideal [H : M]. By Nakayama's lemma, we have $[H : M]^i M \neq [H : M]^j M$ for all positive integer $i \neq j$. Since $\Gamma(M)$ is a planar graph, then $[H : M]^5 M = 0$. If M is not faithful, then $\Gamma(M)$ is a complete graph. Hence $|M| \leq 5$. Now we obtain the central results of this section.

Corollary 2.4 Let M be a multiplication Rmodule with $|Max(M)| \neq 1$. If $\Gamma(M)$ is a planar graph, then $M \cong M_1 \oplus M_2$.

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هامنی (مسطح) بودن گراف اشتراک زیر مدول های یک مدول

چکيده:

فرض کنیدR یک حلقه جابجای یکدار و M یک R- مدول یکدار باشد.گراف اشتراک زیر مدول های یک مدول با نماد (M) نمایش داده می شود. گراف (M) گرافی ساده بدون جهت است که مجموعه رئوس آن مجموعه تمام زیر مدول (M) مای غیر بدیهی مدول M است و دو راس متمایز N و H مجاورند اگر وتنها اگر $0 \neq N \cap N$. در این مقاله ما رابطه بین گزاره های غیر بدیهی مدول تئوری مدول M و گزاره های گراف تئوری (M) را بررسی می کنیم. همچنین حالت هایی را مطالعه می کنیم که گراف اشتراک زیر مدول های یک مدول با نماد مدون جهت است که مجموعه رئوس آن مجموعه تمام زیر مدول مای عبین بدیهی مدول M است و دو راس متمایز N و H مجاورند اگر وتنها اگر $0 \neq N \cap N \neq 0$. در این مقاله ما رابطه بین گزاره های مدول تئوری مدول M است.

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