



# Planarity of Intersection Graph of submodules of a Module

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## Abstract

Let  $R$  be a commutative ring with identity and  $M$  be an unitary  $R$ -module. The intersection graph of an  $R$ -module  $M$ , denoted by  $\Gamma(M)$ , is a simple graph whose vertices are all non-trivial submodules of  $M$  and two distinct vertices  $N_1$  and  $N_2$  are adjacent if and only if  $N_1 \cap N_2 \neq 0$ . In this article, we investigate the concept of a planar intersection graph and maximal submodules of an  $R$ -module. In particular, we show that if  $\Gamma(M)$  is a planar graph, then  $M \cong M_1 \oplus M_2$  for a multiplication  $R$ -module  $M$  with  $|Max(M)| \neq 1$ .

*Keywords* : Interval methods; Multiplication modules; Planar Graph; Module Theory; Torsion Graphs.

## 1 Introduction

It is well known that graph is a very useful tool to model problems originated in all most all areas of our life. In this article, we concentrate our discussion on intersection graphs. Let  $S = \{S_i : i \in I\}$  be an arbitrary family of sets. The intersection graph  $\Gamma(S)$  of  $S$  is the graph whose vertices are  $S_i$ ,  $i \in I$  and there is an edge between two distinct vertices  $S_i$  and  $S_j$  if and only if  $S_i \cap S_j \neq \emptyset$ . It is more interesting to study the intersection graphs  $\Gamma(S)$  when the elements of  $S$  have an algebraic structure. These studies allow us to obtain characterization and representation of the classes of algebraic structure in terms of graphs and vice versa.

Let  $R$  be a commutative ring with identity and  $M$  be a unitary  $R$ -module. The idea of the intersection graph of semigroups was introduced by Bosak in [5]. Inspired by his work, Csákány and

Pollák in [8], studied the graph of subgroups of a finite group. The intersection graph of ideals of a ring, was considered by Chakrabarty, Ghosh, Mukherjee and Sen in [7]. Recently, Akbari, Tavallaei and Khaiashi in [1], introduced and investigated the intersection graph of submodules of a module.

In this paper, we investigate the concept of intersection graph of a module. The intersection graph of an  $R$ -module  $M$ , denoted by  $\Gamma(M)$ , is defined to be the undirected simple graph whose vertices are all non-trivial submodules of  $M$  and two distinct vertices are adjacent if and only if the corresponding submodules of  $M$  have nonzero intersection. This study helps to illuminate the structure of  $M$ , for example, if  $\Gamma(M)$  is a planar graph, then  $M$  is both Noetherian and Artinian.

Recall that a simple graph is finite if its vertices set is finite, and we use the symbol  $|\Gamma(M)|$  to denote the number of vertices in graph  $\Gamma(M)$ . Also, a graph  $G$  is connected if there is a path between any two distinct vertices. The distance,  $d(x, y)$  between connected vertices  $x, y$  is the length of the shortest path from  $x$  to  $y$ , ( $d(x, y) = \infty$  if there is no such path). An isolated vertex is a

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vertex that has no edges incident to it. A complete  $r$ -partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes  $m$  and  $n$  is denoted by  $K_{m,n}$ . A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use  $K_n$  for the complete graph with  $n$  vertices. The complement  $\overline{G}$  of  $G$  is the graph with vertex set  $V(\overline{G}) = V(G)$ , and  $E(\overline{G}) = \{uv : uv \notin E(G)\}$ . The complement of a complete graph is the null graph. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A remarkably simple characterization of planar graphs was given by Kuratowski in [5], p.153. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of  $K_5$  or  $K_{3,3}$ .

An  $R$ -module  $M$  is a multiplication module if for every  $R$ -submodule  $K$  of  $M$  there is an ideal  $I$  of  $R$  such that  $K = IM$ . Note that  $I \subseteq [N : M]$ , hence  $N = IM \subseteq [N : M]M \subseteq N$ . So  $N = [N : M]M$ . An  $R$ -module  $M$  is called a cancellation module if  $IM = JM$  for any ideals  $I$  and  $J$  of  $R$  implies that  $I = J$ . Also, an  $R$ -module  $M$  is a weak-cancellation module if  $IM = JM$  for any ideals  $I$  and  $J$  of  $R$  implies that  $I + Ann(M) = J + Ann(M)$ . Finitely generated multiplication modules are weak cancellation, Theorem 3 [2]. Let  $P$  be a maximal ideal of  $R$ . An  $R$ -module  $M$  is called  $P$ -torsion if for each  $m \in M$  there exists  $p \in P$  such that  $(1 - p)m = 0$ . On the other hand,  $M$  is called  $P$ -cyclic if there exists  $x \in M$  and  $q \in P$  such that  $(1 - q)M \subseteq Rx$ . Theorem 1.2 [6] showed that an  $R$ -module  $M$  is multiplication if and only if for every maximal ideal  $P$  of  $R$  either  $M$  is  $P$ -torsion or  $P$ -cyclic.

In this paper, we study the number of maximal and minimal prime submodule of multiplication modules. It is shown that if  $\Gamma(M)$  is a planar graph, then  $|Max(M)| \leq 4$  and  $|Min(M)| \leq 4$ . Also, we show that, if  $M$  is a multiplication  $R$ -module with  $|Max(M)| \neq 1$  and  $\Gamma(M)$  is a planar graph, then  $M \cong M_1 \oplus M_2$ .

Throughout the paper,  $Max(M)$  is a set of the maximal submodules  $H$  of  $M$ , we use symbol  $|Max(M)|$  to denote the number of maximal submodule of  $M$ . As a consequence of Theorem 2.5 [6], for any non-zero multiplication  $R$ -module  $Max(M) \neq \emptyset$ . Also,  $Min(M)$  is a set of the min-

imal prime submodules  $N$  of  $M$ . let  $J(R)$  be the Jacobson radical of  $R$  and

$$J(M) := \bigcap_{H \in Max(M)} H.$$

We follow standard notation and terminology from graph theory [5] and module theory [3].

## 2 Planar intersection graph

This section is concerned with some basic and important results in the theory of planar torsion graphs over a module.

**Lemma 2.1** *Let  $M$  be an  $R$ -module. If  $\Gamma(M)$  is a planar graph, then  $M$  is both Noetherian and Artinian.*

**Proof.** Let  $N_1 \subset N_2 \subset N_3 \subset N_4 \subset N_5 \subset \dots$  be a chine of nontrivial proper submodule of  $M$ . Then vertices  $N_i, 1 \leq i \leq 5$  form  $K_5$  as an induced subgraph, which is a contradiction. So every chain of nontrivial proper submodule of  $M$  is stationary. Therefore  $M$  is both Noetherian and Artinian.

**Lemma 2.2** *Let  $M$  be a multiplication  $R$ -module and  $N$  be a prime submodule of  $M$ . If  $\bigcap_{i=1}^n N_i \subseteq N$ , where  $N_i$  be a submodule of  $M$ , then there is  $1 \leq i \leq n$  such that  $N_i \subseteq N$ .*

**Proof.** Let  $\bigcap_{i=1}^n N_i \subseteq N$ , where  $N_i$  be a submodule of  $M$ . Then  $[N_1 : M][N_2 : M] \dots [N_n : M]M \subseteq N$ . Since  $N$  is a prime submodule of  $M$ , there is  $1 \leq i \leq n$  such that  $[N_i : M] \subseteq [N : M]$ . Therefore  $N_i \subseteq N$ .

**Lemma 2.3** *Let  $M$  be a  $Q$ -cyclic  $R$ -module for all maximal ideal  $Q$  of  $R$ . Then  $[N : M]$  is a prime ideal of  $R$  for any proper submodule  $N$  of  $M$  if and only if  $[N : M]M$  is a prime submodule of  $M$ .*

**Proof.** Let  $[N : M]$  be a prime ideal of  $R$ . Clearly  $[N : M]M$  is a proper submodule of  $M$ . Suppose  $ax \in [N : M]M$  such that  $a \notin [N : M]$ , for some  $a \in R$  and  $x \in M$ . Let  $k = \{r \in R | rx \in [N : M]M\}$ . If  $k \neq R$ , then there is a maximal ideal  $Q$  of  $R$  such that  $k \subseteq Q$ . Since  $M$  is a  $Q$ -cyclic  $R$ -module,  $(1 - q)M \subseteq Rm$  for some  $q \in Q$  and  $m \in M$ . Hence  $(1 - q)ax \in (1 - q)[N : M]M \subseteq [N : M]m$ . So  $(1 - q)x = sm$  and  $(1 - q)ax = \alpha m$  for some  $s \in R$  and  $\alpha \in [N : M]$ . Thus  $(\alpha - sa)m = 0$ . It is clear that  $(1 -$

$q)ann(m) \subseteq Ann(M)$ . Therefore  $(1-q)(\alpha-sa) \in Ann(M) \subseteq [N : M]$ . Then  $(1-q)sa \in [N : M]$ . Hence  $(1-q) \in k \subseteq Q$ , which is a contradiction. This contradiction implies that  $k = R$  and so  $x \in [N : M]M$ . Therefore  $[N : M]M$  is a prime submodule of  $M$ .

Conversely, let  $N$  be a prime submodule of  $M$ . Thus  $[N : M]$  is a proper ideal of  $R$ . Suppose  $st \in [N : M]$ . So  $sM \subseteq N$  or  $tM \subseteq N$ . Therefore  $[N : M]$  is a prime ideal of  $R$ .

**Theorem 2.1** *Let  $M$  be a  $Q$ -cyclic  $R$ -module for all maximal ideal  $Q$  of  $R$ . If  $\Gamma(M)$  is a planar graph, then  $|Min(M)| \leq 3$ .*

**Proof.** Let  $\Gamma(M)$  be a planar graph. Suppose  $|Min(M)| \geq 4$  and  $N_1, N_2, \dots, N_4$  be distinct minimal submodules of  $M$ , such that  $N_1 \cap N_2 \cap N_3 = 0$ . Then  $[N_1 : M][N_2 : M][N_3 : M]M \subseteq N_4$ . Hence  $[N_1 : M][N_2 : M][N_3 : M] \subseteq [N_4 : M]$ . It is clear that  $[N_4 : M]$  is a prime ideal of  $R$ . So  $[N_i : M]M \subseteq [N_4 : M]M \subseteq N_4$ , for some  $1 \leq i \leq 3$ . By Lemma 2.3,  $[N : M]M$  is a prime submodule of  $M$ . Also, since  $N$  is a minimal prime submodule of  $M$ ,  $[N : M]M = N$ . Therefore  $N_i = N_4$  for some  $1 \leq i \leq 3$ , which is a contradiction. Hence  $N_1 \cap N_2 \cap N_3 \neq 0$ . Therefore, vertices  $N_1 \cap N_2, N_1 \cap N_3, N_2 \cap N_3, N_1, N_2$  and  $N_3$  form  $K_6$  as an induced subgraph, which is a contradiction. Consequently  $|Min(M)| \leq 3$ .

**Corollary 2.1** *Let  $M$  be a multiplication  $R$ -module. If  $\Gamma(M)$  is a planar graph, then  $\bigcap_{N \in Min(M)} N = 0$ .*

**Proposition 2.1** *Let  $M$  be a multiplication  $R$ -module. If  $\Gamma(M)$  is a planar graph, then  $1 \leq |Max(M)| \leq 3$ .*

**Proof.** Let  $\Gamma(M)$  be a planar graph. Suppose  $|Max(M)| \geq 4$  and  $H_1, H_2, \dots, H_4$  be distinct maximal submodules of  $M$ , such that  $H_1 \cap H_2 \cap H_3 = 0$ . Then  $H_1 \cap H_2 \cap H_3 \subseteq H_4$ . Since every maximal submodule of multiplication modules is prime, by Lemma 2.2,  $H_i \subseteq H_4$ , for some  $1 \leq i \leq 3$ . But  $H_i$  is a maximal submodule of  $M$  implies that  $H_i = H_4$  for some  $1 \leq i \leq 3$ , which is a contradiction, hence  $H_1 \cap H_2 \cap H_3 \neq 0$ . Therefore, vertices  $H_1 \cap H_2, H_1 \cap H_3, H_2 \cap H_3, H_1, H_2$  and  $H_3$  form  $K_6$  as an induced subgraph, which is a contradiction. Consequently  $1 \leq |Max(M)| \leq 3$ .

**Corollary 2.2** *Let  $M$  be a multiplication  $R$ -module. If  $\Gamma(M)$  is a planar graph, then  $J(M) = 0$ .*

**Proposition 2.2** *Let  $M = M_1 \times M_2$  be an  $R$ -module. Then  $\Gamma(M)$  is planar if and only if  $\Gamma(M_1)$  or  $\Gamma(M_2)$  is empty and another is null.*

**Proof.** Let  $\Gamma(M)$  be a planar graph. Suppose that  $\Gamma(M_1)$  and  $\Gamma(M_2)$  are not empty. So there exist nontrivial proper submodules  $N_1$  of  $M_1$  and  $N_2$  of  $M_2$ . Therefore  $0 \times N_2, 0 \times M_2, N_1 \times M_2, N_1 \times N_2$  and  $M_1 \times N_2$  form  $K_5$  as an induced subgraph, which is a contradiction. Hence one of  $\Gamma(M_1)$  or  $\Gamma(M_2)$  is empty. Let  $\Gamma(M_2)$  be empty. Now we show that  $\Gamma(M_1)$  is null. If  $N_1$  is a proper nontrivial submodule of  $M_1$  such that it is adjacent to  $H_1$  for some  $H_1 \in V(\Gamma(M_1))$ , then  $N_1 \cap H_1 \neq 0$ . So  $N_1 \times 0, H_1 \times 0, M_1 \times 0, N_1 \times M_2$  and  $H_1 \times M_2$  form  $K_5$  as an induced subgraph, which is a contradiction. This contradiction implies that  $\Gamma(M_1)$  is null.

**Corollary 2.3**  *$\Gamma(M_1 \times M_2 \times M_3)$  is planar if and only if  $M_i$  is a simple  $R_i$ -module for  $i \in \{1, 2, 3\}$ .*

**Proof.** Let  $\Gamma(M_1 \times M_2 \times M_3)$  be a planar graph and  $M_1$  not simple. So there exists  $0 \neq N < M_1$ . Then  $N \times M_2 \times M_3, 0 \times M_2 \times M_3, N \times M_2 \times 0, M_1 \times M_2 \times 0$  and  $M_1 \times 0 \times M_3$  form  $K_5$  as an induced subgraph of  $\Gamma(M)$ , which is a contradiction. Therefore  $M_i$  is a simple  $R_i$ -module for  $i \in \{1, 2, 3\}$ .

**Proposition 2.3** *Let  $M$  be a multiplication  $R$ -module with  $|Max(M)| = 3$ . If  $\Gamma(M)$  is planar, then  $M \cong M_1 \oplus M_2$  where  $M_1$  and  $M_2$  are simple.*

**Proof.** Let  $|Max(M)| = 3$  and  $H_i, 1 \leq i \leq 3$  be distinct maximal submodules of  $M$ . By Corollary 2.2,  $H_1 \cap H_2 \cap H_3 = 0$ . If  $H_2 \cap H_3 = 0$ , then  $H_2 \cap H_3 \subseteq H_1$  and by Lemma 2.2,  $H_1 = H_2$  or  $H_1 = H_3$ , which is a contradiction. Hence  $M = H_1 \oplus H_2 \cap H_3$ . By Proposition 2.2, one of  $\Gamma(H_1)$  or  $\Gamma(H_2 \cap H_3)$  is null another is empty. Suppose that  $\Gamma(H_1)$  be null. If  $H_1$  is not a simple submodule of  $M$ . Then there is a nontrivial submoddule  $N_1$  of  $H_1$  such that  $N_1$  is adjacent to  $N_1 + H_2 \cap H_3$ . So  $\Gamma(H_1)$  is not null, which is a contradiction. Thus  $H_1$  and  $H_2 \cap H_3$  are simple.

**Lemma 2.4** *Let  $M$  be a faithful finitely generated multiplication  $R$ -module. Then  $J(R)M = J(M)$ .*

**Proof.** Let  $M$  be a faithful finitely generated multiplication  $R$ -module and  $H$  be a maximal submodule of  $M$ . By Theorem 3.1 of [6],  $hM \neq M$  for all maximal ideal  $h$  of  $M$ . Also, by Theorem 2.5 of [6],  $H = hM$  for some maximal ideal  $h$  of  $M$ . On the other hand by Theorem 1.6 of [6],  $J(M) = \bigcap_{H \in \text{Max}(M)} H = \bigcap_{h \in \text{Max}(R)} (hM) = (\bigcap_{h \in \text{Max}(R)} h)M = J(R)M$

**Theorem 2.2** *Let  $M$  be a faithful multiplication  $R$ -module with  $|\text{Max}(M)| = 2$ . Then  $\Gamma(M)$  is a planar graph if and only if  $M \cong [H_1 : M]^4 M \oplus [H_2 : M]^4 M$  such that  $\Gamma([H_1 : M]^4 M)$  or  $\Gamma([H_1 : M]^4 M)$  is empty another is null, where  $H_1, H_2$  are maximal submodule of  $M$ .*

**Proof.** Let  $H_1$  and  $H_2$  be distinct maximal submodules of  $M$ . Suppose that  $[H_1 : M]^4 M + [H_2 : M]^4 M \neq M$ . By Theorem 2.5 of [6], there is a maximal submodule  $H$  of  $M$  such that  $[H_1 : M]^4 M + [H_2 : M]^4 M \subseteq H$ . Since  $|\text{Max}(M)| = 2$ , we have  $H = H_1$  or  $H = H_2$ . It follows that  $[H_1 : M]^4 M \subseteq H_2$  or  $[H_2 : M]^4 M \subseteq H_1$ . Thus  $H_1 = H_2$ , which is a contradiction. So  $M = [H_1 : M]^4 M + [H_2 : M]^4 M$ . Assume  $[H_1 : M]^4 M \cap [H_2 : M]^4 M \neq 0$ . Hence  $H_1 \cap H_2 \neq 0$ . On the other hand By Theorem 1.6 [6],  $[H_1 : M]^i M \cap [H_2 : M]^i M = ([H_1 : M]^i \cap [H_2 : M]^i)M$ , for all positive integer  $i$ . Since  $M$  is a cyclic faithful multiplication module, by Lemma 2.4, we have  $J(R)M = J(M)$ . Now Nakayama's lemma follows that  $([H_1 : M]^4 \cap [H_2 : M]^4)M \subset \dots \subset ([H_1 : M] \cap [H_2 : M])M \subset H_1$ . Hence  $\Gamma(M)$  contains an induced subgraph  $K_5$ , which is a contradiction. Therefore  $[H_1 : M]^4 M \cap [H_2 : M]^4 M = 0$ . Consequently  $M \cong [H_1 : M]^4 M \oplus [H_2 : M]^4 M$  and by Proposition 2.2, the result follows.

**Proposition 2.4** *Let  $M$  be a multiplication  $R$ -module with  $|\text{Max}(M)| = 1$ . If  $\Gamma(M)$  is a planar graph, then  $|M| \leq 5$  or  $[H : M]^5 M = 0$  where  $H$  is a maximal submodule of  $M$ .*

**Proof.** Suppose  $M$  be a faithful multiplication  $R$ -module. If  $\Gamma(M)$  is a planar graph, then by Lemma 2.1,  $M$  is finitely generated and by Lemma 2.4,  $R$  is a local ring with unique maximal ideal  $[H : M]$ . By Nakayama's lemma, we have  $[H : M]^i M \neq [H : M]^j M$  for all positive integer  $i \neq j$ . Since  $\Gamma(M)$  is a planar graph, then  $[H : M]^5 M = 0$ . If  $M$  is not faithful, then  $\Gamma(M)$  is a complete graph. Hence  $|M| \leq 5$ .

Now we obtain the central results of this section.

**Corollary 2.4** *Let  $M$  be a multiplication  $R$ -module with  $|\text{Max}(M)| \neq 1$ . If  $\Gamma(M)$  is a planar graph, then  $M \cong M_1 \oplus M_2$ .*

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## Planarity of Intersection Graph of submodules of a Module

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هامنی (مسطح) بودن گراف اشتراک زیر مدول های یک مدول

### چکیده:

فرض کنید  $R$  یک حلقه جابجای یکدار و  $M$  یک  $R$ -مدول یکدار باشد. گراف اشتراک زیر مدول های یک مدول با نماد  $\square(M)$  نمایش داده می شود. گراف  $\square(M)$  گرافی ساده بدون جهت است که مجموعه رئوس آن مجموعه تمام زیر مدول های غیر بدیهی مدول  $M$  است و دو رأس متمایز  $H$  و  $N$  مجاورند اگر و تنها اگر  $H \cap N \neq 0$ . در این مقاله ما رابطه بین گزاره های مدول تئوری مدول  $M$  و گزاره های گراف تئوری  $\square(M)$  را بررسی می کنیم. همچنین حالت هایی را مطالعه می کنیم که گراف اشتراک زیر مدول های یک مدول هامنی (مسطح) است.