



# A New Iterative Method For Solving Fuzzy Integral Equations

R. Ezzati <sup>\*†</sup>, S. Ziari <sup>‡</sup>, S. M. Sadatrasoul <sup>§</sup>

Received Date: 2016-02-10    Revised Date: 2016-08-23    Accepted Date: 2017-05-11

## Abstract

In the present work, by applying known Bernstein polynomials and their advantageous properties, we establish an efficient iterative algorithm to approximate the numerical solution of fuzzy Fredholm integral equations of the second kind. The convergence of the proposed method is given and the numerical examples illustrate that the proposed iterative algorithm are valid.

*Keywords* : Fuzzy Fredholm Integral Equation; Modulus of continuity; Partial modulus of continuity; fuzzy Bernstein polynomials.

## 1 Introduction

The concept of fuzzy integral was initiated by Dubois and Prade [11] and then investigated by Kaleva [21], Goetschel and Voxman [20], Nanda [23] and others. In [33], the Henstock integral of fuzzy-valued functions is defined, while the fuzzy Riemann integral and its numerical integration was investigated by Wu in [34]. In [7], the authors introduced some quadrature rules for the integral of fuzzy-number-valued mappings. Kaleva [21] proposed the existence and uniqueness of the solution of fuzzy differential equations using the Banach fixed point principle. Mordeson and Newman (see [22]) started the study of the subject of fuzzy integral equations. The Banach fixed point principle is the powerful tool to investigate of the existence and uniqueness of the solution

of fuzzy integral equations. The existence and uniqueness of the solution of fuzzy integral equations can be found in [5, 6, 16, 27, 28, 29]. In [19, 24], sufficient conditions are given, which under those conditions, solutions of fuzzy integral equations are bounded. In [14], the authors gave one of the applications of fuzzy integral for solving fuzzy Fredholm integral equation of the second kind. The iterative techniques are applied to fuzzy Fredholm integral equation of the second kind in [7, 15, 26]. Friedman et al. [16] presented a numerical algorithm to solve fuzzy Fredholm integral equations of the second kind based on successive approximations method. Also, Friedman et al. [17] investigated numerical procedures for solving fuzzy Fredholm integral equation of the second kind using the embedding method. Babolian et al. [4] used the Adomian decomposition method (ADM) to solve fuzzy Fredholm integral equation of the second kind. Abbassbandy et al. [1] obtained the solution of fuzzy Fredholm integral equations of the second kind by using the Nystrom method. In [8], the successive approximations method is used for nonlinear fuzzy Fredholm integral equations. Recently, Bica et al. [9] developed an iterative numerical method to

\*Corresponding author. ezati@kiaau.ac.ir, Tel: +989123618518.

<sup>†</sup>Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

<sup>‡</sup>Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

<sup>§</sup>Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

solve nonlinear fuzzy Hammerstein–Volterra integral equations with constant delay. In [10], the same method has been applied to the solutions that take values in the set of right-sided fuzzy numbers for a fuzzy Volterra integral equation with constant delay arising in epidemiology.

Recently, the authors used Bernstein polynomials (see [12]), Lagrange interpolation (see [13]), divided and finite differences (see [25]), Legendre wavelets (see [31]) and predictor-corrector procedures (see [32]) for fuzzy integral equations.

Here, we propose a numerical approach for solving linear fuzzy Fredholm integral equations of the second kind and obtain the error estimate in the approximation of the solution of such fuzzy integral equations. The rest of this paper is organized as follows: In Section 2, we review some elementary concepts of the fuzzy set theory and modulus of continuity. In Section 3, we drive the proposed method to obtain numerical solution of linear fuzzy Fredholm integral equations based on an iterative procedure. The error estimation of the proposed method is obtained in Section 4 in terms of uniform and partial modulus of continuity, proving the convergence of the method. Section 5 includes two numerical examples for the proposed method. Finally, Section 6 gives our concluding remarks.

## 2 Preliminaries

**Definition 2.1** [2]. A fuzzy number is a function  $u : R \rightarrow [0, 1]$  having the properties:

- (1)  $u$  is normal, that is  $\exists x_0 \in R$  such that  $u(x_0) = 1$ ,
- (2)  $u$  is fuzzy convex set i.e.  $u(\lambda x + (1 - \lambda)y) \geq \min \{u(x), u(y)\} \forall x, y \in R, \lambda \in [0, 1]$ ,
- (3)  $u$  is upper semi-continuous on  $R$ ,
- (4) the  $\overline{\{x \in R : u(x) > 0\}}$  is compact set.

The set of all fuzzy numbers is denoted by  $R_F$ . An alternative definition which yields the same  $R_F$  is given by [21].

**Definition 2.2** [17]. An arbitrary fuzzy number is represented, in parametric form, by an ordered

pair of functions  $(\underline{u}(r), \bar{u}(r)), 0 \leq r \leq 1$ , which satisfy the following requirements:

- (1)  $\underline{u}(r)$  is a bounded left continuous non-decreasing function over  $[0, 1]$ ,
- (2)  $\bar{u}(r)$  is a bounded left continuous non-increasing function over  $[0, 1]$ ,
- (3)  $\underline{u}(r) \leq \bar{u}(r)$ ,  $0 \leq r \leq 1$ .

The addition and scalar multiplication of fuzzy numbers in  $R_F$  are defined as follows:

- (1)  $(u \oplus v)(r) = (\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r))$ ,
- (2)  $(\lambda \odot u)(r) = \begin{cases} (\lambda \underline{u}(r), \lambda \bar{u}(r)) & \lambda \geq 0, \\ (\lambda \bar{u}(r), \lambda \underline{u}(r)) & \lambda < 0. \end{cases}$

**Definition 2.3** [3]. For arbitrary fuzzy numbers  $u = (\underline{u}(r), \bar{u}(r))$ ,  $v = (\underline{v}(r), \bar{v}(r))$  the quantity  $D(u, v) = \sup_{r \in [0, 1]} \max \{ |\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)| \}$

is the distance between  $u$  and  $v$ .

The following properties are hold [7]:

- (1)  $(R_F, D)$  is a complete metric space,
- (2)  $D(u \oplus w, v \oplus w) = D(u, v) \quad \forall u, v, w \in R_F$ ,
- (3)  $D(k \odot u, k \odot v) = |k| D(u, v) \quad \forall u, v \in R_F \quad \forall k \in R$ ,
- (4)  $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e) \quad \forall u, v, w, e \in R_F$ .

**Theorem 2.1** [2, 8].

- (1) The pair  $(R_F, \oplus)$  is a commutative semi-group with  $\tilde{0} = \chi_0$  zero element.
- (2) For fuzzy numbers which are not crisp, there is no opposite element (that is,  $(R_F, \oplus)$  cannot be a group).
- (3) For any  $a, b \in R$  with  $a, b \geq 0$  or  $a, b \leq 0$  and for any  $u \in R_F$ , we have  $(a + b) \odot u = a \odot u \oplus b \odot u$ . For arbitrary  $a, b \in R$ , this property is not fulfilled.
- (4) For any  $\lambda, \mu \in R$  and  $u \in R_F$ , we have  $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$ .
- (5) For any  $\lambda \in R$  and  $u, v \in R_F$ , we have  $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$ .

(6) The function of  $\|\cdot\|_F: R_F \rightarrow R$  by  $\|u\|_F = D(u, \tilde{0})$  has the usual properties of the norm, that is,  $\|u\|_F = 0$  if and only if  $u = \tilde{0}$ ,  $\|\lambda \odot u\|_F = |\lambda| \|u\|_F$  and  $\|u \oplus v\|_F \leq \|u\|_F + \|v\|_F$ .

(7)  $\| \|u\|_F - \|v\|_F \| \leq D(u, v)$  and  $D(u, v) \leq \| \|u\|_F + \|v\|_F \|$  for any  $u, v \in R_F$ .

**Definition 2.4** [3]. Let  $f : [a, b] \rightarrow R_F$  be a fuzzy real valued function, then function  $\omega_{[a,b]}(f, \cdot) : R_+ \cup \{0\} \rightarrow R_+$  defined by

$$\omega_{[a,b]}(f, \delta) = \sup\{D(f(x), f(y)) \mid x, y \in [a, b], |x - y| \leq \delta\}, \quad (2.1)$$

where  $R_+$  is the set of positive real numbers, is called the modulus of continuity of  $f$  on  $[a, b]$ .

Some properties of the modulus of continuity are given in below.

**Theorem 2.2** [7]. The following properties hold:

(1)  $D(f(x), f(y)) \leq \omega_{[a,b]}(f, |x - y|)$  for any  $x, y \in [a, b]$ ,

(2)  $\omega_{[a,b]}(f, \delta)$  is increasing function of  $\delta$ ,

(3)  $\omega_{[a,b]}(f, 0) = 0$ ,

(4)  $\omega_{[a,b]}(f, \delta_1 + \delta_2) \leq \omega_{[a,b]}(f, \delta_1) + \omega_{[a,b]}(f, \delta_2)$  for any  $\delta_1, \delta_2 \geq 0$ ,

(5)  $\omega_{[a,b]}(f, n\delta) \leq n\omega_{[a,b]}(f, \delta)$  for any  $\delta \geq 0$   $n \in N$ ,

(6)  $\omega_{[a,b]}(f, \lambda\delta) \leq (\lambda + 1)\omega_{[a,b]}(f, \delta)$  for any  $\delta, \lambda \geq 0$ ,

(7) If  $[c, d] \subseteq [a, b]$  then  $\omega_{[c,d]}(f, \delta) \leq \omega_{[a,b]}(f, \delta)$ .

**Definition 2.5** [21]. A fuzzy real number valued function  $f : [a, b] \rightarrow R_F$  is said to be continuous in  $x_0 \in [a, b]$ , if for each  $\varepsilon > 0$  there exist  $\delta > 0$  such that  $D(f(x), f(x_0)) < \varepsilon$ , whenever  $x \in [a, b]$  and  $|x - x_0| < \delta$ . We say that  $f$  is fuzzy continuous on  $[a, b]$  if  $f$  is continuous at each  $x_0 \in [a, b]$ , and denote the space of all such functions by  $C_F[a, b]$ .

**Definition 2.6** [3]. Let  $f : [a, b] \rightarrow R_F$ .  $f$  is fuzzy-Riemann integrable to  $I(f) \in R_F$  if for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any division  $P = \{[u, v]; \xi\}$  of  $[a, b]$  with the norms  $\Delta(p) < \delta$ , we have,

$$D\left(\sum_P^*(v - u) \odot f(\xi), I(f)\right) < \varepsilon, \quad (2.2)$$

where  $\sum^*$  denotes the fuzzy summation. In this case, it is denoted by

$$I(f) = (FR) \int_a^b f(t) dt.$$

In [20], the authors proved that if  $f \in C_F[a, b]$ , its definite integral exists, and also,

$$\overline{(FR) \int_a^b f(t; r) dt} = \int_a^b \overline{f}(t, r) dt,$$

$$\underline{(FR) \int_a^b f(t; r) dt} = \int_a^b \underline{f}(t, r) dt.$$

**Lemma 2.1** [18]. If  $f, g : [a, b] \subseteq R \rightarrow R_F$  are fuzzy continuous functions, then the function  $F : [a, b] \rightarrow R_+$  by  $F(t) = D(f(t), g(t))$  is continuous on  $A = [a, b]$ , and

$$D\left((FR) \int_a^b f(t) dt, (FR) \int_a^b g(t) dt\right) \leq \int_a^b D(f(t), g(t)) dt. \quad (2.3)$$

**Definition 2.7** (see [3], [18]) For  $f \in C_F[0, 1]$ , the Bernstein-type fuzzy polynomials for all  $x \in [0, 1]$  is as follows

$$B_n^{(F)}(f)(x) = \sum_{k=0}^n * f\left(\frac{k}{n}\right) \odot p_{n,k}(x), \quad n \in N$$

where  $P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$  and  $\sum^*$  means addition with respect to  $\oplus \in R_F$ .

It is obvious that  $P_{n,k}(x) \geq 0, \forall x \in [0, 1]$  and  $P_{n,0}(x), P_{n,1}(x), \dots, P_{n,n}(x)$  are linearly independent algebraic polynomials of degree  $\leq n$  and

$$\sum_{k=0}^n P_{n,k}(x) = 1$$

**Theorem 2.3** (see [3], [18]). If  $f \in C_F[0, 1]$ , then

$$\left(B_n^{(F)}(f)(x), f(x)\right) \leq \frac{3}{2} \omega_{[0,1]}(f, \frac{1}{\sqrt{n}}), \quad n \in N, \quad x \in [0, 1]; \quad (2.4)$$

i.e.,

$$\lim_{n \rightarrow \infty} D(B_n^{(F)}(f)(x), f(x)) = 0,$$

uniformly with respect to  $x \in [0, 1]$ .

### 3 Fuzzy integral equations

The linear fuzzy Fredholm integral equations of the second kind is as follows:

$$F(t) = f(t) \oplus \lambda \odot (FR) \int_a^b k(s, t) \odot F(s) ds, \quad (3.5)$$

where  $k(s, t)$  is an arbitrary crisp kernel function over the square  $a \leq s, t \leq b$ ,  $\lambda \geq 0$ , and  $F(t)$  is a fuzzy real valued function.

In [13], the authors presented sufficient conditions for the existence and unique solution of (2.4) as follows:

**Theorem 3.1** [17]. Let  $k(s, t)$  be continuous for  $a \leq s, t \leq b, \lambda > 0$ , and  $f(t)$  a fuzzy continuous of  $t, a \leq t \leq b$ . If

$$\lambda < \frac{1}{M(b-a)},$$

where

$$M = \max_{a \leq s, t \leq b} |k(s, t)|,$$

then the iterative procedure

$$\begin{aligned} F_0(t) &= f(t), \\ F_m(t) &= f(t) \oplus \lambda \odot (FR) \int_a^b k(s, t) \\ &\odot F_{m-1}(s) ds, \quad m \geq 1, \end{aligned} \quad (3.6)$$

converges to the unique solution of (2.4). Specifically,

$$D^*(F, F_m) \leq \frac{L^m}{1-L} D^*(F_0, F_1), \quad (3.7)$$

where  $L = \lambda M(b-a)$ , and  $D^*(f, g) = \sup_{a \leq t \leq b} D(f(t), g(t))$  denotes the uniform distance between fuzzy-number-valued functions.

Throughout this paper, we consider fuzzy Fredholm integral equation (4.11) with  $a = 0, b = 1$ . Here, we consider the linear fuzzy Fredholm integral equation (2.4) and uniform partition of the interval  $[0, 1]$  :

$$\Delta : 0 = t_0 < t_1 < \dots < t_{n-1} < t_n = 1, \quad (3.8)$$

with  $t_i = ih$  where  $h = \frac{1}{n}$ , then the following iterative procedure gives the approximate solution of (2.4) in point  $t$

$$\begin{aligned} y_0(t) &= f(t), \\ y_m(t) &= f(t) \oplus \lambda \odot \sum_{i=0}^n {}^*k(t_i, t) \\ &\odot y_{m-1}(t_i) \int_0^1 p_{n,i}(s) ds, \quad m \geq 1, \end{aligned} \quad (3.9)$$

where

$$p_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

**Remark 3.1** Since

$$\int_0^1 t^n (1-t)^m dx = \frac{m!n!}{(m+n+1)!}$$

we get

$$\int_0^1 p_{n,k}(s) ds = \frac{1}{n+1}.$$

### 4 Error estimation

Now, we obtain error estimate for given linear fuzzy Fredholm integral equations of the second kind (2.4).

**Theorem 4.1** Consider the linear fuzzy Fredholm integral equation (2.4) with continuous kernel  $k(s, t)$  having constant sign on  $[0, 1] \times [0, 1]$ ,  $f$  continuous on  $[0, 1]$  and also, if  $L = \lambda M \leq 1$  where  $M = \max_{s, t \in [0, 1]} |k(s, t)|$ , then iterative procedure (3.8) converges to the unique solution of (2.4),  $F$ , and its error estimate is as follows:

$$\begin{aligned} D^*(F, y_m) &\leq \frac{3L}{2(1-L)} \omega_{[0,1]}(f, \frac{1}{\sqrt{n}}) + \left[ \frac{L^{m+1}}{1-L} \right. \\ &\left. + \frac{L}{2M(1-L)^2} \left( 3\omega_s(k, h) + 2L\omega_t(k, \frac{1}{\sqrt{n}}) \right) \right] \|f\|_F, \end{aligned}$$

where

$$\|f\|_F = \sup_{a \leq t \leq b} \|f(t)\|_F$$

and

$$\omega_s(k, h) = \sup_{a \leq t \leq b} \{ \sup |k(s_1, t) - k(s_2, t)| : |s_1 - s_2| \leq h \},$$

and

$$\omega_t(k, h) = \sup_{a \leq s \leq b} \{ \sup |k(s, t_1) - k(s, t_2)| : |t_1 - t_2| \leq h \}.$$

**Proof.** Since  $F_m(t) = f(t) \oplus \lambda \odot (FR) \int_0^1 k(s, t) \odot F_{m-1}(s) ds, \forall t \in [0, 1]$ , we have:

$$\begin{aligned}
 D(F_m(t), y_m(t)) &= \\
 D(f(t), f(t)) &+ \lambda D\left((FR) \int_0^1 k(s, t) \odot F_{m-1}(s) ds, \sum_{i=0}^{n^*} k(t_i, t) \odot y_{m-1}(t_i) \odot \int_0^1 p_{n,i}(s) ds\right) \\
 &= \lambda D\left((FR) \int_0^1 k(s, t) \odot F_{m-1}(s) ds, \sum_{i=0}^{n^*} k(t_i, t) \odot y_{m-1}(t_i) \odot \int_0^1 p_{n,i}(s) ds\right) \\
 &\leq \lambda D\left(\int_0^1 k(s, t) \odot F_{m-1}(s) ds, \int_0^1 \sum_{i=0}^{n^*} p_{n,i}(s) k(s, t) \odot F_{m-1}(t_i) ds\right) \\
 &\quad + \lambda \int_0^1 \sum_{i=0}^n D\left(p_{n,i}(s) k(s, t) \odot F_{m-1}(t_i), p_{n,i}(s) k(s, t) \odot y_{m-1}(t_i)\right) ds \\
 &\quad + \lambda \int_0^1 \sum_{i=0}^n D\left(p_{n,i}(s) k(s, t) \odot y_{m-1}(t_i), p_{n,i}(s) k(t_i, t) \odot y_{m-1}(t_i)\right) ds.
 \end{aligned}$$

With suppose that  $M = \max_{s,t \in [0,1]} |k(s, t)|$  we have:

$$\begin{aligned}
 D(F_m(t), y_m(t)) &\leq \lambda M \int_0^1 D\left(F_{m-1}(s), \sum_{i=0}^{n^*} p_{n,i}(s) F_{m-1}(t_i)\right) ds + \lambda M \int_0^1 \sum_{i=0}^n |p_{n,i}(t)| \\
 &\quad D\left(F_{m-1}(t_i), y_{m-1}(t_i)\right) ds \\
 &\quad + \lambda \int_0^1 \sum_{i=0}^n |p_{n,i}(t)| |k(s, t) - k(t_i, t)| \\
 &\quad D\left(y_{m-1}(t_i), \tilde{0}\right) ds.
 \end{aligned}$$

Regarding to Theorem 2.3 and taking into account that  $\sum_{i=0}^n p_{n,i}(t) = 1$  and  $L = \lambda M$  and  $\|y_{m-1}\|_F = \sup_{0 \leq t \leq 1} D(y_{m-1}(t), \tilde{0})$  we have:

$$D(F_m(t), y_m(t)) \leq \frac{3L}{2} \omega_{[0,1]}(F_{m-1}, \frac{1}{\sqrt{n}})$$

$$\begin{aligned}
 &+ L.D\left(F_{m-1}(t_i), y_{m-1}(t_i)\right) \\
 &+ \lambda \|y_{m-1}\|_F \omega_s(k, h).
 \end{aligned}$$

Taking the supremum for  $0 \leq t \leq 1$  from above inequality we have:

$$D^*(F_m, y_m) \leq \frac{3L}{2} \omega_{[0,1]}(F_{m-1}, \frac{1}{\sqrt{n}})$$

$$+ L.D^*(F_{m-1}, y_{m-1}) + \frac{L}{M} \|y_{m-1}\|_F \omega_s(k, h),$$

where  $\omega_s(k, h)$  is the partial modulus of continuity with respect to s.

Considering inequality (3.9) we rewrite the following inequalities:

$$D^*(F_m, y_m) \leq \frac{3L}{2} \omega_{[0,1]}(F_{m-1}, \frac{1}{\sqrt{n}})$$

$$+ L.D^*(F_{m-1}, y_{m-1}) + \frac{L}{M} \|y_{m-1}\|_F \omega_s(k, h),$$

$$D^*(F_{m-1}, y_{m-1}) \leq \frac{3L}{2} \omega_{[0,1]}(F_{m-2}, \frac{1}{\sqrt{n}})$$

$$+ L.D^*(F_{m-2}, y_{m-2}) + \frac{L}{M} \|y_{m-2}\|_F \omega_s(k, h),$$

⋮

$$D^*(F_1, y_1) \leq \frac{3L}{2} \omega_{[0,1]}(f, \frac{1}{\sqrt{n}})$$

$$+ \frac{L}{M} \|f\|_F \omega_s(k, h). \tag{4.10}$$

Multiplying the above inequalities by  $1, L, L^2, \dots, L^{m-1}$ , respectively and summing them we have

$$D^*(F_m, y_m) \leq \frac{3L}{2} \left( \omega_{[0,1]}(F_{m-1}, \frac{1}{\sqrt{n}}) \right.$$

$$\left. + L \omega_{[0,1]}(F_{m-2}, \frac{1}{\sqrt{n}}) + \dots + L^{m-1} \omega_{[0,1]}(f, \frac{1}{\sqrt{n}}) \right)$$

$$+ \frac{L}{M} \omega_s(k, h) \left( \|y_{m-1}\|_F + L \|y_{m-2}\|_F \right.$$

$$\left. + \dots + L^{m-1} \|f\|_F \right). \tag{4.11}$$

Taking norm from (3.8) and considering the Remark 3.1 we obtain

$$\|y_i(t)\|_F \leq \|f(t)\|_F + L \|y_{i-1}(t)\|_F, \quad 1 \leq i \leq m.$$

Taking supremum the above inequality and then by successive substitutions on the obtained inequality and taking into account  $L = \lambda M \leq 1$  we have:

$$\|y_i\|_F \leq \frac{1}{1-L} \|f\|_F \quad 1 \leq i \leq m-1. \tag{4.12}$$

Also, according to the proof of Theorem (11) in [7] we have the following inequalities:

$$\omega_{[0,1]}(F_i, h) \leq \omega_{[0,1]}(f, h) + \frac{L}{M} \omega_t(k, h) \|F_{i-1}\|_F \quad 1 \leq i \leq m-1. \tag{4.13}$$

Also, taking norm from (3.5) we get

$$\|F_i(t)\|_F \leq \|f(t)\|_F + L \|F_{i-1}(t)\|_F, \quad 1 \leq i \leq m.$$

Taking supremum the above inequality and then by successive substitutions on the obtained inequality and taking into account  $L = \lambda M \leq 1$  we have:

$$\|F_i\|_F \leq \frac{1}{1-L} \|f\|_F \quad 1 \leq i \leq m-2. \tag{4.14}$$

Thus, substituting (4.13) into (4.14) we obtain:

$$\omega_{[0,1]}(F_i, h) \leq \omega_{[0,1]}(f, h) + \frac{L}{M(1-L)} \|f\|_F \omega_t(k, h) \quad 1 \leq i \leq m-1. \tag{4.15}$$

Finally, by substituting (4.12) and (4.15) into (4.11), we obtain the following inequality:

$$D^*(F_m, y_m) \leq \frac{3L}{2(1-L)} \omega_{[0,1]}(f, \frac{1}{\sqrt{n}}) + \frac{L \|f\|_F}{2M(1-L)^2} \left( 3\omega_s(k, h) + 2L\omega_t(k, \frac{1}{\sqrt{n}}) \right).$$

Considering the inequality (3.6) we obtain

$$D^*(F, y_m) \leq D^*(F, F_m) + D^*(F_m, y_m) \leq \frac{L^m}{1-L} D^*(F_1, F_0) + D^*(F_m, y_m).$$

Since

$$D(F_1(t), F_0(t)) = D\left(f(t) \oplus \lambda(FR) \int_0^1 k(s, t) \odot F_0(s) ds, F_0(t)\right)$$

$$\leq \lambda D\left((FR) \int_0^1 k(s, t) \odot F_0(s) ds, \tilde{0}\right),$$

we conclude that

$$D^*(F_1, F_0) \leq L \sup_{0 \leq s \leq 1} D(F_0(s), \tilde{0}) \leq L \|f\|_F.$$

Hence, we have:

$$D^*(F, y_m) \leq \frac{3L}{2(1-L)} \omega_{[0,1]}(f, \frac{1}{\sqrt{n}}) + \left[ \frac{L^{m+1}}{1-L} + \frac{L}{2M(1-L)^2} \left( 3\omega_s(k, h) + 2L\omega_t(k, \frac{1}{\sqrt{n}}) \right) \right] \|f\|_F. \quad \square$$

**Remark 4.1** Since  $L < 1$ , it follows that

$$\lim_{m \rightarrow \infty} L^{m+1} = 0.$$

In addition,

$$\lim_{h \rightarrow 0} \omega_{[0,1]}(f, h) = 0,$$

$$\lim_{h \rightarrow 0} \omega_s(k, h) = 0,$$

$$\lim_{h \rightarrow 0} \omega_t(k, h) = 0.$$

So,

$$\lim_{m \rightarrow \infty, h \rightarrow 0} D^*(F, y_m) = 0$$

that shows the convergence of the method.

## 5 Numerical examples

To illustrate the efficiency of the presented method in the previous section, we give two examples. Also, we compare the numerical solution obtained by using the proposed method with the exact solution.

**Example 5.1** Consider the following linear fuzzy Fredholm integral equations of the second kind:

$$F(t) = \left( e^t - \frac{0.2(e^{t+1} - 1)}{t + 1} \right) \odot \gamma \oplus 0.2 \odot \int_0^1 e^{st} \odot F(s) ds,$$

where  $\gamma = (r, 2 - r)$ .

The exact solution in this case is given by

$$F(t) = e^t \odot \gamma.$$

To compare the error with  $n = 10$ ,  $m = 12$  and  $n = 100$ ,  $m = 9$ , see Table 1.

**Table 1:** The accuracy on the level sets for Example 5.1 in  $t = 0.5$

r-level	n=10, m=12		n=100, m=9	
	$ \underline{F} - \underline{y}_m $	$ \overline{F} - \overline{y}_m $	$ \underline{F} - \underline{y}_m $	$ \overline{F} - \overline{y}_m $
0.00	0.000000	0.222423	0.000000	0.027002
0.25	0.027803	0.194620	0.003375	0.023627
0.50	0.055606	0.166817	0.006751	0.020252
0.75	0.083409	0.139014	0.010126	0.016876
1.00	0.111211	0.111211	0.013501	0.013501

**Table 2:** The accuracy on the level sets for Example 5.2 in  $t = 1.5$

r-level	n=10, m=9		n=100, m=9	
	$ \underline{F} - \underline{y}_m $	$ \overline{F} - \overline{y}_m $	$ \underline{F} - \underline{y}_m $	$ \overline{F} - \overline{y}_m $
0.00	0.000000	0.163788	0.000000	0.019717
0.25	0.020473	0.143314	0.002465	0.017253
0.50	0.040947	0.122841	0.004930	0.014788
0.75	0.061420	0.102367	0.007394	0.012323
1.00	0.081894	0.081894	0.009859	0.009859

**Example 5.2** Consider the following linear fuzzy Fredholm integral equations of the second kind:

$$\underline{f}(t, r) = rt - \frac{3}{26}rt^2 - \frac{3}{52}r$$

$$\overline{f}(t, r) = 2t - rt - \frac{3}{13}t^2 - \frac{3}{26} + \frac{3}{26}rt^2 + \frac{3}{52}r$$

and kernel

$$k(s, t) = \frac{(s^2 + t^2 - 2)}{13}$$

and  $a = 1, b = 2$ . The exact solution in this case is given by

$$\underline{F}(t, r) = rt$$

$$\overline{F}(t, r) = (2 - r)t.$$

Since the the fuzzy Fredholm integral equations with this method is defined only for  $t \in [0, 1]$ , so the transformation  $z = (b - a)t + a$  must be done. To compare the error with  $n = 10, m = 9$  and  $n = 100, m = 9$ , see Table 2.

## 6 Conclusions

In this paper, we proposed a numerical method to solve linear fuzzy Fredholm integral equations of the second kind based on iterative procedure using fuzzy Bernstein polynomials. Also, we have presented the error estimation for approximate solution of linear fuzzy Fredholm integral equations of the second kind, in terms of modulus of continuity. Illustrative numerical examples are included to demonstrate the accuracy of the proposed method. In the above presented numerical examples we see that the proposed method well perform for linear fuzzy integral equations and the convergence result, Theorem 4.1, is confirmed.

## References

- [1] S. Abbasbandy, E. Babolian, M. Alavi, Numerical method for solving linear fredholm fuzzy integral equations of the second kind, *Chaos Solutions an Fractals* 31 (2007) 138-146.
- [2] G. A. Anastassiou, S.G. Gal, On a fuzzy trigonometric approximation theorem of Weirstrass-type, *Journal of Fuzzy Mathematics*, Vol. 9, No. 3, Los Angeles, (2001) 701-708.
- [3] G. A. Anastassiou, Fuzzy mathematics : Approximation theory, *Springer-Verlag Berlin Heidelberg* (2010).
- [4] E. Babolian, H. Sadeghi Goghary, S. Abbasbandy, Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method, *Appl.Math.Comput.* 161 (2006) 733-744.
- [5] K. Balachandran, K. Kanagarajan, Existence of solutions of general nonlinear fuzzy Volterra-Fredholm integral equations, *J. Appl. Math. Stochastic Anal.* 3 (2005) 333-343.
- [6] K. Balachandran, P. Prakash, Existence of solutions of nonlinear fuzzy Volterra-Fredholm integral equations, *Indian J. Pure Appl. Math.* 33 (2002) 329-343.
- [7] B. Bede, S.G. Gal, Quadrature rules for integrals of fuzzy-number-valued functions, *Fuzzy Sets and Systems* 145 (2004) 359-380.
- [8] A.M. Bica, Error estimation in the approximation of the solution of nonlinear fuzzy Fredholm integral equations, *Information Sciences* 178 (2008) 1279-1292.
- [9] A. M. Bica, C. Popescu, Numerical solutions of the nonlinear fuzzy Hammerstein-Volterra delay integral equations, *Information Sciences* 223 (2013) 236-255.
- [10] A. M. Bica, One-sided fuzzy numbers and applications to integral equations from epidemiology, *Fuzzy Sets and Systems* 219 (2013) 27-48.
- [11] D. Dubois, H. Prade, Towards fuzzy differential calculus, *Fuzzy Sets and Systems* 8 (1982) 1-7, 105-116, 225-233.
- [12] R. Ezzati, S. Ziari, Numerical solution and error estimation of fuzzy Fredholm integral equation using fuzzy Bernstein polynomials, *Aust. J. Basic Appl. Sci.* 5 (9) (2011) 2072-2082.
- [13] M. A. Fariborzi Araghi, N. Parandin, Numerical solution of fuzzy Fredholm integral equations by the Lagrange interpolation based on the extension principle, *Soft Computing* 15 (2011) 2449-2456.
- [14] M. Friedman, M. Ma, A. Kandel, Numerical methods for calculating the fuzzy integral, *Fuzzy Sets and Systems* 83 (1996) 57-62.
- [15] M. Friedman, M. Ma, A. Kandel, On fuzzy integral equations, *Fund. Inform.* 37 (1999) 89-99.
- [16] M. Friedman, M. Ma, A. Kandel, Solutions to fuzzy integral equations with arbitrary kernels, *International Journal of Approximate Reasoning* 20 (1999) 249-262.
- [17] M. Friedman, M. Ma, A. Kandel, Numerical solutions of fuzzy differential and integral equations, *Fuzzy Sets and Systems* 106 (1999) 35-48.
- [18] S. G. Gal, Approximation theory in fuzzy setting, in: G.A. Anastassiou (Ed.), *Handbook of Analytic-Computational Methods in Applied Mathematics*, *Chapman & Hall, CRC Press, Boca Raton, London, New York, Washington DC, 2000* (Chapter 13).
- [19] D. N. Georgiou, I. E. Kougias, Bounded solutions for fuzzy integral equations, *Int. J. Math. Math. Sci.* 31 (2002) 109-114.
- [20] R. Goetschel, W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets and Systems* 18 (1986)31-43.
- [21] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems* 24 (1987) 301-317.
- [22] J. Mordeson, W. Newman, Fuzzy integral equations, *Inform. Sci.* 87 (1995) 215-229.



- [23] S. Nanda, On integration of fuzzy mappings, *Fuzzy Sets and Systems* 32 (1989) 95-101.
- [24] J. J. Nieto, R. Rodriguez-López, Bounded solutions for fuzzy differential and integral equations, *Chaos Solitons Fractals* 27 (2006) 1376-1386.
- [25] N. Parandin, M.A. Fariborzi Araghi, The numerical solution of linear fuzzy Fredholm integral equations of the second kind by using finite and divided differences methods, *Soft Computing* 15 (2010) 729-741.
- [26] J.Y. Park, S.Y. Lee, J.U. Jeong, The approximate solution of fuzzy functional integral equations, *Fuzzy Sets and Systems* 110 (2000) 79-90.
- [27] J. Y. Park, J. U. Jeong, On the existence and uniqueness of solutions of fuzzy Volterra-Fredholm integral equations, *Fuzzy Sets and Systems* 115 (2000) 425-431.
- [28] J. Y. Park, J. U. Jeong, A note on fuzzy integral equations, *Fuzzy Sets Systems* 108 (1999) 193-200.
- [29] J. Y. Park, H. K. Han, Existence and uniqueness theorem for a solution of fuzzy Volterra integral equations, *Fuzzy Sets Systems* 105 (1999) 481-488.
- [30] M. Puri, D. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.* 114 (1986) 409-422.
- [31] H. Sadeghi Goghary, M. Sadeghi Goghary, Two computational methods for solving linear Fredholm fuzzy integral equations of the second kind by Adomian method, *Appl. Math. Comput.* 161 (2005) 733-744.
- [32] M. Shafiee, S. Abbasbandy, T. Allahviranloo, Predictor-corrector method for nonlinear fuzzy Volterra integral equations, *Aust. J. Basic Appl. Sci.* 5 (12) (2011) 2865-2874.
- [33] C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued functions, *Fuzzy Sets and Systems* 120 (2001) 523-532.
- [34] H.-C. Wu, The fuzzy Riemann integral and its numerical integration, *Fuzzy Sets and Systems* 110 (2000) 1-25.



Reza Ezzati was born in Tabriz, Iran in 1974. He received his B.S. degree in pure mathematics in 1997 and M.S. degree in applied mathematics from Azerbaijan Teacher Education University and Tarbyat Modarres University of Tehran, respectively, and his Ph.D degree in applied mathematics from IAU- Science and Research Branch, Tehran, Iran in 2006. He is an Professor in the Department of Mathematics at Islamic Azad University, Karaj Branch, (Iran). He has published about 90 papers in international journals, and he also is the Associate Editor of Mathematical Sciences (a Springer Open Journal). He is the Member of Editorial Board of The Scientific Journal, Journal of Applied Mathematics. His current interests include fuzzy mathematics, especially, on numerical solution of fuzzy systems, fuzzy integral equations and fuzzy interpolation.



Shokrollah Ziari was born in 1971 in Iran. He received the B.S. degree in applied mathematics from Iran, Amir Kabir University of Technology in 1996, M.S., and Ph.D. degrees in applied mathematics from Islamic Azad University, Iran, in 1999 and 2013, respectively. He is currently an assistant Professor in department of mathematics, Firoozkooh Branch, Islamic Azad University, Firoozkooh, Iran. He follows researches on the fuzzy Integral equations, ranking fuzzy numbers and fuzzy transform.



Seyyed Mohsen Sadat Rasoul was born in Karaj, Iran, in 1977. He received his B.S. degree in pure mathematics in 1999 from Tehran Teacher Education University and M.S. degree and Ph.D. degree from IAU- karaj branch in 2012 and 2015, respectively. He is currently the provisional member of board at the CFU- karaj campus. He has published several papers in international journals on fuzzy integral equations, also he has authored about 10 books on mathematics educations. His current interests include fuzzy mathematics, illustration and problem solving.

## A New Iterative Method For Solving Fuzzy Integral Equations

R. Ezzati, S. Ziari, S. M. Sadatrasoul

### روش تکراری جدید جهت حل معادلات انتگرال فازی

#### چکیده:

در مقاله حاضر، با بکارگیری چند جمله ای های برنشتاین و خواص سودمندشان، یک الگوریتم تکراری کارا برای تقریب جواب عددی معادلات انتگرال فازی نوع دوم ایجاد نموده ایم. همگرایی روش پیشنهادی بیان شده و مثالهای عددی نشان می دهد که الگوریتم تکراری ارائه شده معتبر است.