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Existence and Uniqueness Results for a Nonstandard Variational-Hemivariational Inequalities with Application

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Abstract

This paper aims at establishing the existence and uniqueness of solutions for a nonstandard variational-hemivariational inequality. The solutions of this inequality are discussed in a subset K of a reflexive Banach space X. Firstly, we prove the existence of solutions in the case of bounded closed and convex subsets. Secondly, we also prove the case when K is compact convex subsets. Finally, we enhance the main results by the application of some differential inclusions.

Keywords : Set-valued operator; Clarke's generalized gradian; Generalized monotonicity; Variational-hemivariational inequality; Differential inclusion.

1 Introduction

It is well known that many problems in nonlinear analysis and optimization can be formulated as the variational inequality problems. As an important and useful generalization of variational inequality, hemivariational inequality was first introduced by P.D. Panagiotopoulos at the beginning of the 1980s (see [25] and [26]). Within a very short period of time, this theory witnessed a remarkable development in both pure and applied mathematics. It has been proved to be very efficient to describe a variety of some thing, such as mechanical problems, engineering sciences, economics, differential inclusion and optimal control (see [5, 6, 8, 12, 20, 22, 27, 30, 33]).

Generally, the applications of hemivariational inequality theory have been intensively studied by many authors (see [3, 4, 10, 16, 24, 27, 32]). Our study of new type of variational-hemivariational inequalities which arise from hemivariational inequalities if some constraints have to be taken into account.

In order to do it is very useful to understand several problems of mechanics and engineering for non-convex, and non-smooth energy functionals (see [18, 19, 21, 23, 28, 31]).

The main purpose of this work is to give a new contribution in this area. In particular, we establish the existence and uniqueness of solutions for new type of variational-hemivariational inequalities. It is worth mentioning that we do not deal with a classical technique to proof our results. Thus, several difficulties occur in finding an application to the main results, because the classical methods fail to be applied directly.

In order to achieve the aim, the study is divided into the following sections. In Section 2, we refer to some definitions and results that will assist us in the study. In Section 3, we prove

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the existence and uniqueness of solutions for the problem. The proof of the first result is based on arguments of lower quasi-hemicontinuous and α-monotone operators. However, the second result of this section relies essentially on the Schauder's fixed point Theorem. In the last section of this paper, we illustrate the applicability of our approach by a differential inclusion in the special case of our main results.

2 Preliminaries

Throughout this paper, unless stated otherwise, we always assume that E is Banach space and E^* is a topological dual space of E, while $\langle .,. \rangle$ and $\| \, . \, \|$ denote the duality pairing between E and E^* and norm in E^*, respectively.

For the convenience of the reader, we are going to review some definitions and results that will be used in our analysis.

Definition 2.1 *A functional $J : E \to \mathbb{R}$ is said to be locally Lipschitz if every point $u \in E$ possesses a neighborhood W such that*

$$|J(a) - J(b)| \leq M_u \|a - b\|_E \quad \forall a, b \in W$$

for a constant $M_u \geq 0$ which depends on W.

Definition 2.2 *Assume that $J : E \to \mathbb{R}$ is locally Lipschitz. The generalized derivative of J at the point $u \in E$ in the direction $z \in E$ is denoted by $J^0(u, z)$, i.e.,*

$$J^0(u; z) = \limsup_{\substack{a \to u \\ \lambda \searrow 0}} \frac{J(a + \lambda z) - J(a)}{\lambda}$$

Definition 2.3 *Assume that E is a Banach space and $J : E \to \mathbb{R}$ is locally Lipschitz functional. We say that J is regular (in the sense of Clarke) at $u \in E$ if for each $z \in E$ the one sided directional derivative $J^{'}(u, z)$ exists and $J^0(u, z) = J^{'}(u, z)$. We say that J is regular where J is regular at every point $u \in E$.*

Definition 2.4 *Let X be a Banach space. A mapping $\Lambda : X \to \mathbb{R}$ is said to be*

[(i)]lower semicontinuous (for short ,(l.s.c)) at $x_0 \in X$, if

$$\Lambda(x_0) \leq \liminf_n \Lambda(x_n)$$

upper semicontinuous (for short ,(u.s.c)) at $x_0 \in X$, if

$$\Lambda(x_0) \geq \limsup_n \Lambda(x_n)$$

for any sequence x_n of X such that $x_n \to x_0$.

2. **Proposition 2.1** *Let $J : E \to \mathbb{R}$ be a function on a Banach space E, which is locally Lipschitz of rank M_u near the point $z \in E$, then*

i) the $z \mapsto J^0(u, z)$ is subadditive, finite, positively homogeneous and satisfies

$$J^0(u, z) \leq M_u \parallel z \parallel;$$

ii) $J^0(u, z)$ is upper semicontiuous as a function of (u, z).

One can found it's proof in [9].

Definition 2.5 *The generalized gradient of J at $u \in E$, which is a subset of a dual space E^*, is defined by*

$$\partial J(u) = \Big\{ \xi \in E^* : \langle \xi, z \rangle \leq J^0(u; z), \forall z \in E \Big\}.$$

Monotone and generalized monotone play crucial role in several branches of mathematics such as variational analysis, engineering, optimization, and differentiability theory of convex functions, etc (see [2, 15, 17, 24, 34, 35]). Let us mention some of these generalizations of α-monotone (resp., uniformly monotone) operators which we shall use to prove as follows:

Definition 2.6 *[2] Let $T : E \multimap E^*$ be a set-valued and $\alpha : E \times E \to \mathbb{R}$ a bifunction. Then T is said to be α- monotone if $\langle x^* - y^*, x - y \rangle \geq \alpha(x, y)$, for all $x, y \in E$, $x^* \in T(x)$ and $y^* \in T(y)$.*

Definition 2.7 *[2, 34] Let $\alpha : E \times E \to \mathbb{R}$ be bifunction. A single-valued $T : E \to E^*$ is said to be*

[(i)]α- monotone, if
$\langle T(x) - T(y), x - y \rangle \geq \alpha(x, y)$, for all $x, y \in E$,
strong monotone, if there exists $\gamma > 0$ such that
$\langle T(x) - T(y), x - y \rangle \geq \gamma \|u - v\|^2$, for all $x, y \in E$.

In next definition B. Alleche and V. Radulscu [4] introduced a generalization of lower semicontinuous of set-valued function when the space E is a real topological Hausdorff vector space.

2. **Definition 2.8** *[4] Assume that* $T : E \multimap E^*$ *is a set-valued mapping. Then* T *is called* lower quasi-hemicontinuous *at* $x \in E$*, if whenever* $w \in E$ *and* $(\lambda_n)_n$ *a sequence in* $(0, 1)$ *such that* $\lim_{n\to\infty} \lambda_n = 0$*, there exists a sequence* $(w_n^*)_n$ *converging to some element* $x^* \in T(x)$ *such that* $w_n^* \in T(x + \lambda_n(w - x))$ *for every* n*. The set valued function* T *is said* lower quasi-hemicontinuous *on a subset* C *of* E *if* T *is lower quasi-hemicontinuous at every point of* C*.*

Similarly, one can define a single-valued T from the space E to E^* is *lower quasi-hemicontinuous* on E, if whenever $z \in E$ and $(\lambda_n)_n$ a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 0$, then $T(x+\lambda_n(z-x))$ converging to $T(x) \in E^* \quad \forall x \in E$.

Let us end this section with two theorems that will be used to prove our results. The first is the Schauder's fixed point Theorem (see [7]) while the second represents notions of a KKM mapping and the well-known intersection Lemma that is due to Ky Fan [12] will be needed.

Theorem 2.1 *[7]* Assume that K is a convex compact set in a Banach space E and that $G : K \to K$ is a continuous mapping. Then G has a fixed point in the set K.

Theorem 2.2 *[14]* Let K be a nonempty subset of a Hausdorff topological vector space E and let $\Lambda : K \multimap E$ be a KKM mapping. If $\Lambda(x)$ is closed in E for every $x \in K$ and compact for some $u_0 \in K$, then $\bigcap_{u\in K} \Lambda(u) \neq \phi$.

We understand the family of all the subsets of E is said to be a KKM mapping if for any finite subset $\{u_1, u_2, \cdots, u_n\}$ of K, $co\{u_1, u_2, \cdots, u_n\} \subset \bigcup_{i=1}^{n} \Lambda(u_i)$, where $co\{u_1, u_2, \cdots, u_n\}$ denotes the convex hull of $\{u_1, u_2, \cdots, u_n\}$.

3 Main results

Assume that K is a nonempty bounded, closed and convex subset of a real reflexive Banach space X. Our aim is to study the following inequality of nonlinear variational-hemivariational.

Find $v \in K$ and $v^* \in T(v)$ for every $u \in K$ such that

$$\langle v^*, u - v\rangle - H(v) + H(u) + J^0(Av; A\theta(v, u)) \geq 0. \tag{3.1}$$

We suppose that there exists a linear compact operator $A : X \to E$ and $\theta : X \times X \to X$ is single-valued function and that $J : E \to \mathbb{R}$ is a regular locally Lipschitz functional.

In order to solve problem (3.1), we assume that the following hypotheses fulfilled:

H$_1$: The mapping $\theta(\cdot, \cdot) : X \times X \to X$ satisfies the following conditions (i) $\theta(u, u) = 0$ for all $u \in X$;
(ii) $\theta(\cdot, u)$ is linear operator for all $u \in X$;
(iii) $\theta(v, u^m) \rightharpoonup \theta(v, u)$, whenever $u^m \rightharpoonup u$.

H$_2$: $\alpha : X \times X \to \mathbb{R}$ is a bifunction such that for all $u, v \in K$, $\lim_{n\to\infty} \frac{\alpha(v+\lambda_n(u-v),v)}{\lambda_n} = 0$ wherever, $(\lambda_n)_n$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $\limsup_n \alpha(u, v_n) \geq \alpha(u, v)$ whenever $(v_n)_n$ is a sequence in K converging to v.
H$_3$: T is α- monotone and lower quasi-hemicontinuous on K with respect to weak *-topology X^*.
H$_4$: $H : X \to \mathbb{R} \cup \{+\infty\}$ is a convex and l.s.c on K, $K \cap domH \neq H$, where $domH = \{x \in X : H(x) < +\infty\}$ is the effective domain of H.

Remark 3.1 *It is clear that the mapping* $u \mapsto J^0(v, \theta(v, u))$ *is convex for all* $v \in E$*. It follows from the convexity of* $J^0(u, v)$ *and linearity of* $\theta(., u)$*.*

Remark 3.2 *Since* A *is a linear compact operator we obtain that* Av_n *converges strongly to some* $Av \in K$*. Therefore,* $A\theta(v_n, u)$ *converges strongly to* $A\theta(v, u)$ *in which* $u \in K$*. By applying this fact, together with Proposition 2.1* (ii)*, one can get that*

$$\limsup_n J^0(Av_n; A\theta(v_n, u)) \leq J^0(Av; A\theta(v, u)). \tag{3.2}$$

We present an example of linear compact operator which satisfies hypothesis **H**$_1$.

Example 3.1 *Let* $A : X \to X$ *be a linear compact operator,* $r > 0, s \in X$*, and define a functinal* $g : X \to X$ *by* $g(x) := rA(x) + s$*. Let us* *define the functional* $\theta : X \times X \to X$ *as follows:*

$$\theta(v,u) := g(u) - g(v), \quad \forall u, v \in X.$$

In this case, $\theta(v,u)$ *satisfies the conditions* $(i), (ii)$ *and* (iii) *from* $\boldsymbol{H}_1$.

An example of a bifunction α that satisfy the conditions (H_2) and (H_3) is the following.

Example 3.2 *Letting* $\alpha(u,v) := \gamma\|u - v\|^2$, *where* $u \neq v$, $\gamma > 0$ *and* $r \geq 2$.
To satisfied the hypothsis H_2,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\alpha(v + \lambda_n(u-v), v)}{\lambda_n} \\
&= \lim_{n\to\infty} \frac{\gamma\|(v + \lambda_n(u-v) - v)\|^2}{\lambda_n}. \\
&= \lim_{n\to\infty} \gamma\lambda_n\|u - v\|^2 \\
&= 0.
\end{aligned}$$

Since the norm is continuous, then the property $\limsup_n \alpha(u, v_n) \geq \alpha(u,v)$ holds, whenever $(v_n)_n$ is a sequence in K converging to v.

To satisfied the hypothsis H_3. Letting $T = \Delta_p : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$, where $\Delta_p v = div(|\nabla v|^{p-2}\nabla v)$, $p > 1$ is a real constant, and Ω is a bounded domain of $\mathbb{R}^N, N \geq 1$ with smooth boundary $\partial\Omega$, where $W^{1,p}(\Omega)$ is Sobolev space and $W^{-1,q}(\Omega)$ its dual space, $\frac{1}{p} + \frac{1}{q} = 1$.
If whenever $z \in W_0^{1,p}(\Omega)$ and $\lim_{n\to\infty} \lambda_n = 0$ such that $\lambda_n \in (0,1)$, then it is clear that $\Delta_p(x + \lambda_n(z - x)) \rightharpoonup \Delta_p(x) \in W^{1,q}(\Omega)$. So, Δ_p is lower quasi-hemicontinuous at $x \in W_0^{1,p}(\Omega)$ because Δ_p is continuous (see [2] page 44).

Since the $p-$Laplacian is strong monotone, (see [1]), then for every $u, v \in W_0^{1,p}(\Omega)$,

$$\langle \Delta_p u - \Delta_p v, u - v\rangle \geq \gamma\|u - v\|^2.$$

Then T is α- monotone operator.

In what follows, the authors point out the fact that when K is a nonempty, closed, bounded and convex subset of a reflexive Banach space. However, by weakening of hypotheses of K we do to impose certain one of generalized monotonicity, so called α-monotonicity beside one of generalized continuity (see [4]) to establish the existence of at least one solution for nonstandard variational-hemivariational inequality.

Theorem 3.1 Let us consider the nonempty, closed, bounded and convex set $K \subset X$. If the conditions $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ and $\mathbf{H}_4$ are hold, then the problem (3.1) admits at least one solution.

Let us define the set-valued mapping $\Psi : K \multimap K$ as follows:

$$\begin{aligned}
\Psi(u) := \Big\{ & v \in K : \inf_{u^* \in T(u)} \langle u^*, u - v\rangle - \\
& H(v) + H(u) + J^0(Av; A\theta(v,u)) \geq \\
& \alpha(u,v) \Big\}.
\end{aligned} \tag{3.3}$$

For this mapping we verify the assumptions of Theorem 2.2.
Claim 1: $\Psi(u)$ is a KKM mapping. Arguing by contradiction let us assume that Ψ is not KKM. According to the definition of KKM mapping there exists a finite subset $\{u_1, u_2, \cdots, u_n\} \subset K$ and put $v_0 = \sum_{k=1}^{n} t_k u_k$ where $t_k \in (0,1)$ for every $k = \overline{1,n}$ and $\sum_{k=1}^{n} t_k = 1$, such that $v_0 \notin \bigcup_{k=1}^{n} \Psi(u_k)$. This is equivalent to

$$\begin{aligned}
&\inf_{u^* \in T(u_k)} \langle u^*, u_k - v_0\rangle - H(v_0) + H(u_k) + \\
&J^0(Av_0; A\theta(v_0, u_k)) < \alpha(u_k, v_0).
\end{aligned} \tag{3.4}$$

One can choose $u_k^* \in T(u_k)$, for every $k = \overline{1,n}$ in which

$$\begin{aligned}
&\langle u_k^*, u_k - v_0\rangle - H(v_0) + H(u_k) + \\
&J^0(Av_0; A\theta(v_0, u_k)) < \alpha(u_k, v_0).
\end{aligned} \tag{3.5}$$

On the other hand, T is α-monotone operator and thus, for every $k = \overline{1,n}$ we get

$$\begin{aligned}
\langle u_k^* - v_0^*, u_k - v_0\rangle \;\; & \geq \;\; \alpha(u_k, v_0) \\
& > \;\; \langle u_k^*, u_k - v_0\rangle - \\
& \qquad H(v_0) + H(u_k) + \\
& \qquad J^0(Av_0; A\theta(v_0, u_k)).
\end{aligned}$$

Then for every $v_0^* \in T(v_0)$,

$$\begin{aligned}
&\langle v_0^*, u_k - v_0\rangle - H(v_0) + H(u_k) + \\
&J^0(Av_0; A\theta(v_0, u_k)) < 0.
\end{aligned} \tag{3.6}$$

Using Remark 3.1, $\mathbf{H}_1(i)$ and $\mathbf{H}_4$, for every $v_0^* \in T(v_0)$,

$$\begin{aligned} 0 &= \langle v_0^*, v_0 - v_0 \rangle - H(v_0) + H(v_0) + \\ & J^0(Av_0; A\theta(v_0, v_0)) \\ &= \left\langle v_0^*, \sum_{k=1}^{n} t_k(u_k - v_0) \right\rangle - H(v_0) + \\ & H(\sum_{k=1}^{n} t_k u_k) + J^0(Av_0; A\theta(v_0, \sum_{k=1}^{n} t_k u_k)) \\ &\leq \sum_{k=1}^{n} t_k \Big[\langle v_0^*, u_k - v_0 \rangle - H(v_0) + \\ & H(u_k) + J^0(Av_0; A\theta(v_0, u_k)) \Big] \\ &< 0, \end{aligned}$$

which is a contradiction. Therefore, the set- valued mapping $u \mapsto \Psi(u)$ is a KKM mapping.

Claim 2: $\overline{\Psi(u)} = \Psi(u)$, for every $u \in K$. Assume that $v \in \overline{\Psi(u)}$ and $u \in K$. Let $(v_n)_n$ be a sequence in $\Psi(u)$ converging to v. Assume that $u^* \in T(u)$. Then for all $n \geq 1$, then

$$\langle u^*, u - v_n \rangle - H(v_n) + H(u) + J^0(Av_n; A\theta(v_n, u) \geq \alpha(u, v_n). \quad (3.7)$$

Taking into account $\mathbf{H}_4$ and Remark 3.2 for each $v \in K$, then

$$\begin{aligned} \alpha(u, v) &\leq \limsup_n \alpha(u, v_n) \\ &\leq \limsup_n [\langle u^*, u - v_n \rangle - H(v_n) + \\ & H(u) + \\ & J^0(Av_n; A\theta(v_n, u))] \\ &\leq \langle u^*, u - v \rangle - H(v) + H(u) + \\ & J^0(Av; A\theta(v, u)). \end{aligned}$$

Therefore, $v \in \Psi(u)$, and $\Psi(u)$ is a weakly closed subset of K, for each $u \in K$.

On the other hand, K is a weakly compact set as it is a bounded, convex and closed subset of the real reflexive Banach space X. Therefore, $\Psi(u)$ is a weakly compact subset of K, for each $u \in K$. Then by Theorem 2.2, we have $\bigcap_{u \in K} \overline{\Psi(u)} \neq \emptyset$.

Let $v_0 \in \bigcap_{u \in K} \overline{\Psi(u)}$. This implies that $v_0 \in K$ and for every $z \in K$, we have

$$\inf_{z^* \in T(z)} \langle z^*, z - v_0 \rangle - H(v_0) + H(z) + J^0(Av_0; A\theta(v_0, z)) \geq \alpha(z, v_0). \quad (3.8)$$

Assume that $u \in K$ is arbitrary and define $z_n = v_0 + \lambda_n(u - v_0)$ such that $(\lambda_n)_n$ is a sequence in $(0, 1)$ such that $\lim_{n \to \infty} \lambda_n = 0$. By using lower quasi-hemicontinuouity of T on K, we have $z_n^* \xrightarrow{w^*} v_0^* \in T(v_0)$ for each $z_n^* \in T(z_n)$.

Since $u \to J^0(\lambda v_0, \lambda u)$ is positively homogeneous, so from Remark 3.1 one can obtain that the mapping $u \mapsto J^0(Av, A\theta(v, u))$ is convex. Then the left side is as follows

$$\begin{aligned} & \lambda_n \langle z_n^*, u - v_0 \rangle - H(v_0) + \\ & H(v_0 + \lambda_n(u - v_0)) + \\ & \lambda_n J^0(Av_0; A\theta(v_0, u)) \\ & \quad + (1 - \lambda_n) J^0(Av_0; A\theta(v_0, v_0)) \\ & \leq \lambda_n \langle z_n^*, u - v_0 \rangle - H(v_0) + H(v_0) \\ & + \lambda_n [H(u) - H(v_0)] \\ & + \lambda_n J_n(Av_0; A\theta((v_0, u)) \\ & = \lambda_n [\langle z_n^*, u - v_0 \rangle - H(v_0) + H(u) \\ & + J^0(Av_0; A\theta(v_0, u))]. \end{aligned}$$

Hence,

$$\frac{\alpha(v_0 + \lambda_n(u - v_0), v_0)}{\lambda_n} \leq \langle z_n^*, u - v_0 \rangle - H(v_0) + H(u) + J^0(Av_0; A\theta(v_0, u)). \quad (3.9)$$

By approaching $n \to +\infty$, one can obtain

$$\langle v_0^*, u - v_0 \rangle - H(v_0) + H(u) + J^0(Av_0; A\theta(v_0, u)) \geq 0. \quad (3.10)$$

As for uniqueness of solutions authors present the next result.

Theorem 3.2 In addition to the hypotheses $\mathbf{H}_2$, $\mathbf{H}_3$ and $\mathbf{H}_4$, we assume that the following hypotheses are fulfilled:

$\boldsymbol{H}_5$: *there exists* $M > 0$ *such that* $\langle v_1^* - v_2^*, v_1 - v_2 \rangle \geq M\|v_2 - v_1\|^2$ *for all* $v_1, v_2 \in X$.
$\boldsymbol{H}_6$: *there exists a positive constant* $S \leq M$ *such that* $|J^0(x_0; v)| \leq \frac{S}{2}\|v\|$.
$\boldsymbol{H}_7$: $\|A\theta(v_1, v_2)\| \leq \|v_1 - v_2\|^2$.

Then (3.1) *has a unique solution.*

Towards to a contradiction, let us assume that $v_1, v_2 \in K$ are two solutions to (3.1). So, if write in (3.1) for v_1 with $u = v_2$, we have

$$\langle v_1^*, v_2 - v_1\rangle - H(v_1) + H(v_2) + J^0(Av_1; A\theta(v_2, v_1)) \geq 0, \quad (3.11)$$

and then for v_2 with $u = v_1$, we have

$$\langle v_2^*, v_1 - v_2\rangle - H(v_2) + H(v_1) + J^0(Av_2; A\theta(v_1, v_2)) \geq 0. \quad (3.12)$$

By multiplying each of the equations (3.11) and (3.12) by -1 and summing together, one can get

$$\begin{aligned} 0 &\geq \langle v_2^* - v_1^*, v_2 - v_1\rangle - J^0(Av_1; A\theta(v_2, v_1)) \\ &\quad -J^0(Av_2; A\theta(v_1, v_2)) \\ &\geq M\|v_2 - v_1\|^2 - |J^0(Av_1; A\theta(v_2, v_1))| - \\ &\quad |J^0(Av_2; A\theta(v_1, v_2))| \\ &\geq M\|v_2 - v_1\|^2 - \frac{S}{2}\|A\theta(v_1, v_2)\| \\ &\quad -\frac{S}{2}\|A\theta(v_1, v_2)\| \\ &\geq (M - S)\|v_2 - v_1\|^2. \end{aligned}$$

which shows that $\|v_2 - v_1\|^2 \leq 0$ since $M - S \geq 0$. Consequently, we have $v_1 = v_2 \in K$.

In the next result, we prove the problem (3.1) admits at least one solution in the case K is a compact convex subset of X without using any monotonicity conditions on T in a Banach space X. We shall assume that the following hypotheses fulfilled.

In order to prove our result, we need the following assertion:

H$_8$: T is $l.s.c$ on K with respect to weak *-topology X^*.

Theorem 3.3 *Assume that K is a nonempty compact convex subset of the Banach space X. If the hypotheses $\boldsymbol{H}_4$ and $\boldsymbol{H}_8$ hold, then the problem (3.1) admits at least one solution.*

Towards to a contradiction, we assume that problem (3.1) has no solution. Then, for each $v \in K$, there exists $u \in K$ such that

$$\sup_{v^* \in T(v)} \langle v^*, u - v\rangle - H(v) + H(u) + J^0(Av; A\theta(v, u)) < 0. \quad (3.13)$$

Let us define the set-valued mapping $\Gamma : K \multimap K$ as follows:

$$\Gamma(u) := \{v \in K : \inf_{v^* \in T(v)} \langle v^*, u - v\rangle - H(v) + H(u) + J^0(Av; A\theta(v, u)) \geq 0\}. \quad (3.14)$$

***Claim** 1. The set $\Gamma(u)$ is a nonempty and closed for each $u \in K$.*
Easily, $\Gamma(u)$ is nonempty since $u \in \Gamma(u)$ for each $u \in K$ according to definition of set Γ. Assume that $\{v_n\}_{n\geq 1} \subset \Gamma(u)$ is a sequence which converges weakly to v. We must prove that $v \in \Gamma(u)$, for each $n \geq 1$, and for each $v_n^* \in T(v_n)$, we have

$$\langle v_n^*, u - v_n\rangle - H(v_n) + H(u) + J^0(Av_n; A\theta(v_n, u)) \geq 0. \quad (3.15)$$

Let $v^* \in T(v)$ be fixed and let $v_n^* \in T(v_n)$. Using the lower semicontinuity of T and H, and Remark 3.2 , $v \subset \Gamma(u)$. Let us point out the fact that T is l.s.c at $x \in X$ if, and only if every generalized sequence $(v_n)_n$ converges to v and for every $v^* \in T(v)$, then there exists generalized sequence $(v_n^*)_n$ converges to v^* such that $v_n^* \in T(v_n)$ for every $n \in \mathbb{N}$ (see [29]). To do this, passing to limsup as $n \to \infty$ in (3.15) we obtain

$$\begin{aligned} 0 &\leq \limsup_n [\langle v_n^*, u - v_n\rangle - H(v_n) + H(u) \\ &\quad +J^0(Av_n; A\theta(v_n, u))] \\ &\leq \lim_n \langle v^*, u - v_n\rangle - \liminf_n H(v_n) + H(u) \\ &\quad + \limsup_n J^0(Av_n; A\theta(v_n, u)) \\ &\leq \langle v^*, u - v\rangle - H(v) + H(u) + \\ &\quad J^0(Av; A\theta(v, u)). \end{aligned}$$

Therefore, $v \in \Gamma(u)$, and $\Gamma(u)$ is a weakly closed subset of K.
According to (3.13) for each $v \in K$, there exists $u \in K$ such that $v \in [\Gamma(u)]^c = X - \Gamma(u)$. Therefore, the family $\{[\Gamma(u)]^c\}$ is an open covering of the compact set K, for each $u \in K$. This means that there exists a finite subset $\{u_1, u_2, \cdots, u_N\}$ of K such that $\{[\Gamma(u_r)]^c\}$ is a finite subcover of K for every $r = \overline{1, N}$.

Assume that $D_r(v) := dis(v; \Gamma(u_r))$ (i.e., the distance between v and the set $\Gamma(u_r)$) for every $r = \overline{1, N}$ and let $S_r : K \to [0, 1]$ be a function defined as follows:

$$S_r(v) := \frac{D_r(v)}{\sum_{i=1}^{N} D_i(v)}.$$

Notice that S_r is a Lipschitz continuous function for every $r = \overline{1, N}$, $S_r(v) \in [0, 1]$, for all $v \in K$

and $\sum_{r=1}^{N} S_r(v) = 1$. *Let* $M : K \to K$ *be a mapping defined by:*

$$M(v) := \sum_{r=1}^{N} S_r(v) u_r$$

***Claim** 2: The mapping* M *is continuous.*
To do this, we can obtain for any $v_1, v_2 \in K$ *the following estimation:*

$$\begin{aligned}
&\|M(v_1) - M(v_2)\| \\
&= \|\sum_{r=1}^{N} (S_r(v_1) - S_r(v_2))\ u_r\| \\
&\leq \sum_{r=1}^{N} \|u_r\| \|S_r(v_1) - S_r(v_2)\| \\
&\leq J_r \sum_{r=1}^{N} \|u_r\| \|v_1 - v_2\| \\
&\leq J\|v_1 - v_2\|.
\end{aligned}$$

This shows that M *is continuous map. Taking into account Theorem 2.1, there exists* $v_0 \in K$ *such that* $M(v_0) = v_0$.
Let us consider the functional $N : K \to \mathbb{R}$ *as follows:*

$$\begin{aligned}
N(v) &:= \sup_{v^* \in T(v)} \langle v^*, M(v) - v\rangle - H(v) \\
&+ H(M(v)) + J^0(Av; A\theta(v, M(v))).
\end{aligned}$$

By applying Proposition 2.1, Remark 3.1, the convexity of H *and the way the map* M *was constructed for each* $v \in K$, *one can obtain*

$$\begin{aligned}
N(v) &= \sup_{v^* \in T(v)} \left\langle v^*, \sum_{r=1}^{N} S_r(v)(u_r - v) \right\rangle \\
&\quad - H(v) + H(\sum_{r=1}^{N} S_r(v) u_r) \\
&\quad + J^0(Av; A\theta(v, \sum_{r=1}^{N} S_r(v) u_r)). \\
&\leq \sum_{r=1}^{N} S_r(v) \Big[\sup_{v^* \in T(v)} \langle v^*, u_r - v\rangle - \\
&\quad H(v) + H(u_r) + J^0(Av; A\theta(v, u_r)) \Big].
\end{aligned}$$

On the other hand, since $K \subset \bigcup_{r=1}^{N} [\Gamma(u_r)]^c$ *for every* $r = \overline{1, N}$, *there exists at least one index* $r_0 = \overline{1, N}$ *such that* $v \in [\Gamma(u_{r_0})]^c$. *This shows that* $N(v) < 0$ *for all* $v \in K$ *which contradicts the fact that* $N(v_0) = 0$.

Remark 3.3 Notice that the solutions of variational-hemivariational inequality on unbounded domains exist if we expand the conditions for the bounded domains with a coercivity condition. As, if we put some coercivity conditions, it will ensure that Theorem 3.1 or Theorem 3.3 will also satisfy when the set K is unbounded (for details, see [10], [11] and [13]).

4 Application to differential inclusion problems

It is worth mentioning that, there has been an increased interest in differential problems governed by higher order operators. In this section, we apply our main results, expressed in the previous section to a partial differential inclusion problem. Let us consider the usual Sobolev space as $W^{1,p}(\Omega)$ and Banach; $W^{-1,q}(\Omega)$ its dual space, where $\frac{1}{p} + \frac{1}{q} = 1$. The $p-$ Laplacian operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$, where $\Delta_p v = div(|\nabla v|^{p-2} \nabla v)$, $p > 1$ is a real constant, and Ω is a bounded domain of $\mathbb{R}^N, N \geq 1$ with smooth boundary $\partial\Omega$.

In order to highlight the application, we present below problem in the partial differential inclusions.

$$\begin{cases} -\Delta_p v - g(x) \in \partial J(v), & if \quad x \in \Omega \\ v = 0 & \quad\quad on\ \partial\Omega. \end{cases} \tag{4.16}$$

such that $g : \Omega \to \mathbb{R}$ is continuous function with compact support. For technical reasons, let us define $H : W_0^{1,p}(\Omega) \to \mathbb{R}$ as follows:

$$H(\xi) := \int_\Omega -g(x)\xi(x)dx.$$

Let K be a nonempty, closed, bounded and convex subset of Sobolev space $W_0^{1,p}(\Omega)$. In fact, our purpose is to find at least one solution of the following variational-hemivariational inequality problem under circumstances $\theta(v, u) := u - v$ and A is surjective: Find $v \in K$ as a weak solution of problem 4.1 such that

$$\begin{aligned}
&-\int_\Omega |\nabla v|^{p-2} \nabla v \nabla(u - v)dx + H(u) - \\
&H(v) + J^0(v; u - v)dx \geq 0,
\end{aligned} \tag{4.17}$$

for all $u \in K$.

To apply the first main result in differential inclusion, we must prove that any solution of (3.1) is solution to (4.17), show that the assumptions which depend in Theorem 3.1 are satisfied. To do this, the authors considered the letting $T := \Delta_p$ and $\alpha(u,v) := \gamma\|u-v\|^2$, where $u \neq v$ and $\gamma > 0$. Taking into account Example 3.2, then the bifunction α satisfies Hypothesis (H_2) and the operator $T := \Delta_p$ satisfies Hypothesis (H_3). It remains to prove that H is a convex and lower semi continuous function, we assume that $u_1, u_2 \in W_0^{1,p}(\Omega), t \in (0,1)$,

$$\begin{aligned}
&H(tu_1 + (1-t)u_2)\\
&= -\int_\Omega g(x)(tu_1(x) + (1-t)u_2(x))dx\\
&= t\Big[-\int_\Omega g(x)u_1(x)dx\Big]\\
&\quad + (1-t)\Big[-\int_\Omega g(x)u_2(x)dx\Big]\\
&= tH(u_1) + (1-t)H(u_2).
\end{aligned}$$

Also, if $u_n \rightharpoonup u \in W_0^{1,p}(\Omega)$

$$\begin{aligned}
&|H(u_n) - H(u)|\\
&= |-\int_\Omega g(x)(u_n(x) - u(x))dx|\\
&\leq \Big(\int_\Omega |g(x)|^q\Big)^{\frac{1}{q}} . \Big(\int_\Omega |u_n(x) - u(x)|^p\Big)^{\frac{1}{p}}\\
&\leq M\|u_n - u\|_{L^p}\\
&\leq M\|u_n - u\|\\
&\rightharpoonup 0.
\end{aligned}$$

Therefore, all conditions of Theorem 3.1 are achieved.

Similarly, one can apply the second main Theorem 3.3 in differential inclusion, that is because Δ_p is continuous (see [8] page 44), so $\mathbf{H}_8$ is satisfied.

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