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Estimating Upper and Lower Bounds For Industry Efficiency With Unknown Technology

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Abstract

With a brief review of the studies on the industry in Data Envelopment Analysis (DEA) framework, the present paper proposes inner and outer technologies when only some basic information is available about the technology. Furthermore, applying Linear Programming techniques, it also determines lower and upper bounds for directional distance function (DDF) measure, overall and allocative efficiency in industry level. Finally, the results are illustrated using a Cobb-Douglas function.

Keywords : Data envelopment analysis (DEA); Industry; Unknown technology; Efficiency bound.

1 Introduction

Efficiency analysis at the firms level has been worked out by plenty of authors with respect to different aspects of production theory. Recently, the concept of industry, as defined by the aggregation of all observed firms, has drawn many research interests. Particularly, in Production theory, some leading economist devotes their researches to the tradeoffs between industry and firm efficiency analysis. A dominant nonparametric tool to address this subject is Data envelopment analysis (DEA), first introduced by [18] and extended to multiple input multiple output case by [7]. [18] has introduced the structural efficiency concept to assess the performance of an industry in DEA framework. He argued that, an industry holds the performance of its own best firms. He utilized the weighted average of all individual technical efficiency indices (the outputs as weights) as a measure of industry efficiency. [8] employed Farrell's definition in order to assess the industry technical efficiency, it attempts to bring together the various definitions of efficiency found in the economic literature, integrate the notion of inefficiency into standard microeconomic theory, and analyze the results in terms of macroeconomic variables.

In terms of technology, [22], with a focus on short-run industry technologies, studied the industry technology. [14] extended Johansen's production model with a single output and firm specific inputs. In order to compare the industry and firm models, they restated them in a gen-

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eral programming form and ignore issues of functional form. [19] employed two measures for the industry efficiency based on Farrells work [18]. Farrell's measures of efficiency are generalized to nonhomogeneous production functions. Several new measures of efficiency have been introduced and applied to the Swedish milk processing industry. [28] pinpointed an undesirable result due to the connection between average and structural efficiency, as the efficiency of an average unit has an indeterminate relationship with the average of the individual efficiency scores. Also, he suggested the shadow price model, introduced by [25], as a useful model to express and describe the meaning of structural efficiency.

[2] carried out the initial condition for aggregation of inputs/outputs efficiency measures. They showed that if either the aggregated index or its inverse is a linear weighted average of firm efficiency indices or their inverses, the aggregation is definable in the industry level as firm efficiency indices. They argued that neither of common efficiency measures such as those introduced by [12, 18, 17, 29] met the requirements nor therefore they cannot be used to address the efficiency aggregation problem. They finally offered the directional distance function which has the aggregation condition.

[4] followed the question of efficiency indices aggregation, which is posed by [2] and carried out some necessary and sufficient conditions on aggregation problem and introduced an approach for measuring the industry efficiency based on directional distance function (DDF). They also introduced a lower bound for industrial allocative efficiency by quantity data in firm level, without using price data. As mentioned before, vast ranges of DEA modeling at industry level are promoted based on full access to the technical and price data, while the limitations on exact technology and price data in empirical cases are undeniable. The importance of this limitation persuaded the researchers to find some new approaches to deal with the nature of such common economic problems. For this, [25], by imposing some initial assumptions on the firm technologies such as sameness, constant return to scale (CRS) and convexity, have introduced an efficiency measure for a group of firms. They have also, in absence of firms price data, suggested an approach to estimate the revenue maximum shadow price vector to compute the allocative efficiency of individual firms. It is noteworthy that the access to the complete information of industry technology in their work is a pivotal assumption.

Also, in terms of shortage of information on firms price data, we can refer to the [9] which involved the existing DEA tools for profit analysis with considering both cases, unknown and known price data at firms and industry levels. Also, they studied firm efficiency behaviors for which they promote a methodology with utilizing interior prices. [24] tried to exposit the efficiency measure in aggregation problems without information on price data. [24] by considering some assumptions such as known and identical firm technologies, define some appropriate indices based on directional distance function. They suggested a lower bound for industry allocative efficiency with incomplete information on firms price data.

[21] have provided a review on various conditions in accessibility/inaccessibility to the technology and cost information in order to compute the profit efficiency at the firm level. They focused on four main cases: known prices and technology, known prices and unknown technology, unknown prices and known technology, unknown prices and technology. In either case, they suggested an approach to determine profit efficiency. In addition, they showed that, unlike the industry profit inefficiency measure, defined as the sum of firms profit inefficiencies, we cannot demonstrate the technical and allocative inefficiency measures as a direct sum of corresponding inefficiency measures at firm level. To illustrate this, they utilized the CobbDouglas function to construct the firm technologies and showed that the technical inefficiency at industry level may be due to allocative inefficiency at firm level and vice versa. Finally, they applied a TopDown approach and introduced an approximate for profit efficiency measure at industry level with unknown technology and exact price information. In case of unknown prices and known technologies, using shadow prices, an approximate for industry profit efficiency is introduced. In fact, their work well corroborated the importance of studying the var-

ious access levels to the main firm information (cost and technology) that are subject to the foremost restrictions in empirical applications.

[21] analyzed profit efficiency and estimated a lower bound for profit efficiency with incomplete technology information. The other researches in this area do not deal with estimating overall or allocative efficiency scores with unknown technology. Cherchye and Puyenbroeck [10] in their work identified the natural tools in DEA framework to deal with the lack of technical and price information in empirical assessments. They provided a good substrate in profit efficiency debate in firm level in case of unknown or uncertain prices data using DEA.

In case of unknown firm technology, [1] introduced inner and outer bound technologies which the inner bound is achieved from a DEA model involving all observations and the outer bound is obtained from a subset of data points that passes the Varians weak axiom of profit maximization (WAPM) test. [20] used both DEA and dual nonparametric frontiers and considered undesirable outputs. In their approach the outer bound is provided from the dual frontier while the DEA frontier provides the inner bound for the unknown underlying technology. In addition, [23] by considering unknown price information and using a dual characterization of the directional distance function suggested a lower and upper bounds on technical inefficiency.

Most of the mentioned articles work with known technologies, but, forasmuch as, the technology is usually unknown in empirical studies it is necessary to present a procedure to analyze the DMUs with this property. Also, determining the performance of industry firm is useful and usable for comparing the performance of an industry in different provinces of one country or different countries or in different times. Imagine that a country has 20 provinces that each of them has 100 shoe manufacturing workshop. For comparing the performance of the provinces it is not necessary to evaluate all 2000 workshops, it is sufficient to determine the industry firm of each province and then compare them. In this paper, we determine the performance of industry level with unknown technology. We propose an inner and outer estimation for the underlying technology at industry level regarding the restrictions on firm technology and data accesses in aggregation framework. Consequently, an interval for industry directional distance function (DDF) measure, industry overall efficiency and industry allocative efficiency is provided when the technology of firms are not known precisely. To do so, we apply the advantageous of the directional distance function as formed by generalizing Shephard distance function ([3, 5, 6]).

The contributions of this paper are as follows:

- Presenting the inner and outer technology for firms with unknown technology
- Presenting an interval for directional distance function (DDF) measure of industry firm with unknown technology
- Presenting the lower and upper bounds for overall and allocative efficiency score of industry firm with unknown technology
- A numerical example shows the applicability and efficiency of our method for industry firm of some agricultural systems

The organization of the paper is as follows: In section 2, we provide a background of the most important works around the industry efficiency analysis concerning the lack of complete technology information. Then, we promote our work in case of unknown technology and accordingly we propose an inner and an outer bound for industry technology in section 3. Section 4 is devoted to determine an interval for DDF, overall, and allocative efficiency measures by giving some theorems. In section 5, we examine accuracy of proposed intervals for DDF, allocative, and overall industry measures. Finally, section 6 concludes the paper.

2 Background

Industries growth is known as an evidence for economic growth in a country and would fairly fill in the gap between developed and developing countries. Industry growth in many important industries, like automotive, food, apparel, etc. causes a consumer or even an importer country convert to a producer or even an exporter one. Thus, it is worthy to present a comparison between different industries in one country or a special industry in different countries to know performance of industries to improve. The contribution of this paper is applicable for analyzing and comparing performance of a special industry in different countries, when the technology is unknown. So, industries with high performance in one country and low performance in another one can be recognized to be exported to the latter one in order to improve its economic indicators. In the literature, the term industry refers to a group of firms with common activity. In order to compare the country performances, we need firms data of each country. This resembles a multilayered efficiency analysis. However, the strategy of this paper is aggregation of underlying firms in each country into a single unit, called industry, which provides a panorama of the industry status in that country. As already mentioned in the previous section, the inability of suggested measure for industry efficiency, by [18], led to provide a consistent method in industry debate. [25] in the first step of their seminal work argued that the reallocation of firms' input has not been considered in the industry efficiency measure suggested by [18]. They also challenged the incompatibility of the industry efficiency measure introduced by [19] and stressed that the mentioned methods compute the industry efficiency by comparing the average of firms only with firms efficient frontier and therefore their model doesnt measure the industry productive efficiency. [25] by generalizing [18] and [19], developed the industry efficiency analyzing framework. They considered a production group consists of a number of firms and argued that assumption on firm technologies need to be considered in constructing technology of the group.

Following the most common notations, consider an industry composed of K firms where the firm k consumes $x^k \in \mathbb{R}^{+M}$ to produce $y^k \in \mathbb{R}^{+N}$, $k = 1,\ldots,K$. Let $T^k = \{(x^k, y^k) : x^k \text{ can produce } y^k\}$ denotes the technology which firm k belongs to for $k = 1,\ldots,K$. Then, the industry technology would be given by:

$$T^I = \Big\{\sum_{k=1}^{K}(x^k, y^k) : (x^k, y^k) \in T^k, k = 1,\ldots,K\Big\} \quad (2.1)$$

Li and Ng [25] under some initial assumptions on firm's technology; including convexity and sameness of firm's technology showed that, the firm's technology is a convex cone (constant return to scale technology) if and only if:

$$T^I = \sum_{k=1}^{K} T^k = T \quad (2.2)$$

They also proposed three measures for technical, overall, and allocative efficiencies in industry level and indicated that the efficiency in firm level does not guarantee the efficiency in industry level. Therefore, they showed that the inefficiency in industry level is due to inside or outside firm inefficiencies and it is caused by the reallocation of resources among firms which form the industry. Hence, the interactive performance of firms in an industry is important for the performance of the industry (as an individual firm) in a set of homogeneous enterprises. Using a decomposition of industry efficiency measures into the weighted technical efficiency, weighted allocative efficiency, and industry reallocative efficiency, they revealed that the industry revenue efficiency can be estimated without complete prices information.

3 Industry with unknown technology

With the assumption that the industry technology is unknown and includes firms, the ultimate aim of this section is to prepare a discussion for introducing bounds for DDF and other efficiency scores like overall and allocative efficiencies. They will be used in the next section for performance evaluation of the industry unit in lack of complete information on its technology.

Let us limit the firm technologies to those which satisfy following general axioms.

A1 Although the firm technologies are unknown, all firms operate under the same technology T; (sameness)

A2 The observed activity (x^k, y^k) for $k = 1, \ldots, K$ belongs to T;

A3 T is strongly disposable for inputs and outputs;

A4 T is a closed set;

A5 T is a convex set.

Note that there exist many linear and nonlinear technologies for which all the above assumptions are satisfied.

Assumptions (A2)-(A6) put the variable returns to scale (VRS) technology as a subset of our unknown technology set T. Assumptions (A1) and (A2) distinguish our paper from most of the others in estimating underlying production technology set. The large number of the industry researches deal with industries with known technology. Based on these assumptions, we propose to apply lower and upper bounds for efficiencies of unknown technology industries.

To introduce our proposed approach, let $x^I = \sum_{k=1}^K x^k$ and $y^I = \sum_{k=1}^K y^k$ denote the industry inputs and outputs vectors, respectively. The industry technology, denoted by T^I, is defined as the aggregation of technologies of K firms (see [25]):

$$T^I = \sum_{k=1}^K T$$

Since T is assumed to be an unknown technology satisfying minimal assumptions (A1)-(A5), the aggregation of K firm technologies (the industry technology) would be unknown. To achieve paper objective, some properties of these two unknown technologies are investigated:

Theorem 3.1 $T_{VRS} \subseteq T \subseteq T_{CRS}$

Proof. Evidently, $T_{VRS} \subseteq T$ due to the assumptions (A2). To show the second inclusion, since T_{CRS} satisfies assumptions (A3)-(A6) as well as CRS assumption which enlarges the technology, we have $T \subseteq T_{CRS}$. This completes the proof.

Li and Ng [25] have shown that for a convex technology T

$$T^I = \sum_{k=1}^K T = KT \tag{3.3}$$

Therefore, they deduced that

$$\sum_{k=1}^K T_{VRS} = KT_{VRS}, \quad \text{and} \quad \sum_{k=1}^K T_{CRS} = T_{CRS}$$

In following theorem, an alternative aspect of their proposition is given:

Theorem 3.2 *Assume that K individual firms operate under the same technology T. T is convex if and only if*

$$\sum_{k=1}^K T = KT.$$

Proof. Assuming T is convex and let $(x, y), (u, v) \in T$. By convexity of T, $\lambda(x, y) + (1 - \lambda)(u, v) \in T$ for $0 \leq \lambda \leq 1$. To prove $\sum_{k=1}^K T = KT$, it is shown that

$$\sum_{k=1}^K T \subseteq KT \quad \text{and} \quad \text{KT} \subseteq \sum_{k=1}^K T$$

By definition of $\sum_{k=1}^K T$,

$$\begin{aligned}(w, z) &:= \sum_{k=1}^K [\lambda(x, y) + (1 - \lambda)(u, v)] \\ &\in \sum_{k=1}^K T\end{aligned}$$

Note that (x, y) and (u, v) has been chosen arbitrarily and calculating their convex combinations by variation of λ in $[0, 1]$, implies that (w, z) to be any point in $\sum_{k=1}^K T$. Therefore,

$$\begin{aligned}(w, z) &= \sum_{k=1}^K [\lambda(x, y) + (1 - \lambda)(u, v)] \\ &= K [\lambda(x, y) + (1 - \lambda)(u, v)] \in KT\end{aligned}$$

This proves the first inclusion above. To show the second inclusion, by definition of KT, it is seen that:

$$(w, z) := K [\lambda(x, y) + (1 - \lambda)(u, v)] \in KT$$

On the other hand,

$$\sum_{k=1}^K [\lambda(x, y) + (1 - \lambda)(u, v)] \in \sum_{k=1}^K T$$

Thus, $(w, z) \in \sum_{k=1}^{K} T$ and this completes the necessary condition.

To prove the sufficient condition, assume that $\sum_{k=1}^{K} T = KT$, which is the special case of

$$\sum_{k=1}^{K}(\alpha_k T) = T \sum_{k=1}^{K} \alpha_k$$

when $\alpha_k = 1$. Assume (x, y) and (u, v) are any points in T. It is clear that

$$K(x,y) = \sum_{k=1}^{K}(x,y) \in \sum_{k=1}^{K} T = KT$$
$$K(u,v) = \sum_{k=1}^{K}(u,v) \in \sum_{k=1}^{K} T = KT$$

Therefore,

$$\lambda[K(x,y)] \in \lambda KT$$
$$(1-\lambda)[K(u,v)] \in (1-\lambda)KT$$

which implies:

$$\lambda[K(x,y)] + (1-\lambda)[K(u,v)] \in KT$$

Thus,

$$\lambda(x,y) + (1-\lambda)(u,v) \in T$$

This completes the proof.

Now, we present the following main result that gives an inner and an outer technology set for the unknown industry technology using the foregoing two theorems 3.1 and 3.2.

Theorem 3.3 *Assuming an industry composed of K individual firms whose technologies satisfy (A1)-(A5) and denoted by T. If $T^I = \sum_{k=1}^{K} T = KT$ denotes the industry technology, then*

$$KT_{VRS} \subseteq KT = T^I \subseteq T_{CRS}$$

Proof. It immediately follows from theorems 3.1 and 3.2.

4 Different measures for industry with unknown technology

The common measures cannot cope with the efficiency analysis under aggregation circumstances. [4] applied directional distance function, as defined by [16], as an appropriate measure to evaluate the industry efficiency:

Definition 4.1 *The function $D_T : \mathbb{R}^{+M+N} \times (-\mathbb{R}^{+M} \times \mathbb{R}^{+N}) \mapsto \mathbb{R}^{+}$ defined by*

$$D_T(x, y; g) := \sup\left\{\delta \in \mathbb{R}^{+} : (x, y) + \delta g \in T\right\} \tag{4.4}$$

is called directional distance function, hereof, abbreviated as DDF, in the direction $g = (g^x, g^y) \in \mathbb{R}^{+M} \times \mathbb{R}^{+N}$ in which T is the production technology. In fact, it expands outputs in direction g^y and contracts inputs in direction g^x.

It can be easily verified that DDF is compatible with addition, viz:

$$D_T\left(\sum_{k=1}^{K}(x^k, y^k); g\right) = \sum_{k=1}^{K} D_T(x^k, y^k; g) \tag{4.5}$$

Following [4], for the industry technology set, $T^I = \sum_{k=1}^{K} T^k$ where under the aforementioned assumptions denoted as T and according to [25] we have $T^I = T$.

While the firm technology satisfies CRS, [4] carried out some necessary and sufficient conditions for the linearity of the DDF in addition to the aggregation of efficiency in any group of firms. Besides, they defined the industry technical efficiency as:

$$ITE := D_{T^I}\left(\sum_{k=1}^{K}(x^k, y^k); g\right) \tag{4.6}$$

Likewise, the sum of firm technical efficiency is defined as:

$$FTE := \sum_{k=1}^{K} D_{T^k}(x^k, y^k; g) \tag{4.7}$$

Utilizing the DDF additive property, they also showed that $FTE \leq ITE$.

In aspect of profit analysis, given the price vector $(p, w) \in \mathbb{R}^{+M+N}$, the profit function would be defined as in [15] and [16] as follows:

$$\Pi(p, w) := \sup\left\{p^t y - w^t x : (x, y) \in T\right\} \tag{4.8}$$

where the superscript "t" stands with the transpose vector. If the overall efficiency is given as [6]:

$$OE(x, y, p, w; g^x, g^y) := \frac{\Pi(p, w) - (p^t y - w^t x)}{p^t g^y + w^t g^x} \tag{4.9}$$

The Mahler inequality [26] would be deduced as:

$$D_T(x, y; g) \leq OE(x, y, p, w; g^x, g^y) \quad (4.10)$$

[13] has considered the gap between overall and technical efficiencies as the allocative efficiency. In other words,

$$AE(x, y, p, w; g^x, g^y) = OE(x, y, p, w; g^x, g^y) - D_T(x, y; g) \quad (4.11)$$

In aggregation framework, we use the following definitions by Briec, Dervaux, and Leleu [4]: Overall efficiency index for industry:

$$IOE := OE\left(\sum_{k=1}^{K}(x^k, y^k), p, w; g\right) \quad (4.12)$$

Allocative efficiency index for industry:

$$IAE := IOE - ITE \quad (4.13)$$

and we have:

$$IOE = \sum_{k=1}^{K} OE(x^k, y^k, p, w; g) \quad (4.14)$$

This means that the overall efficiency index would be aggregated directly to compute the industry overall efficiency. Finally, they proposed a lower bound for sum of firm allocative efficiencies, denoted by FAE, as the difference between two technical efficiency indices:

$$ITE - FTE \leq FAE \quad (4.15)$$

It is clear that $ITE = 0$ is a necessary and sufficient condition for FAE to achieve its lower bound. [24] emphasized on unknown price information and known firm technologies. By applying the above mentioned inequality for FAE, computed via technical indices, they estimate the sum of firm allocative efficiencies with no price data. These authors also promoted some DEA models in computational aspect for ITE and FTE based on directional distance function which will be used in our approach in performance evaluation with unknown technology. The following theorems provide intervals for the DDF measure, overall and allocative efficiency of an industry with unknown technology.

Theorem 4.1 *Let $x^I = \sum_{k=1}^{K} x^k$ and bmy$^I = \sum_{k=1}^{K} y^k$ denote the industry inputs and outputs, respectively. For given direction g we have:*

$$D_{KT_{VRS}}(x^I, y^I; g) \leq D_{T^I}(x^I, y^I; g) \leq D_{T_{CRS}}(x^I, y^I; g) \quad (4.16)$$

Proof. It simply followed from definition 4.1 and the fact that $KT_{VRS} \subseteq T^I \subseteq T_{CRS}$.

Theorem 4.2 *Given the price vector $(p, w) \in \mathbb{R}^{+M+N}$, the direction $g = (g^x, g^y) \in -\mathbb{R}^{+M} \times \mathbb{R}^{+N}$ and the technologies T_1 and T_2 with $T_1 \subseteq T_2$, for any $(x_o, y_o) \in T_1$ we have:*

$$OE_{T_1}(x_o, y_o, p, w; g^x, g^y) \leq OE_{T_2}(x_o, y_o, p, w; g^x, g^y) \quad (4.17)$$

Proof. Following the fact $T_1 \subseteq T_2$, from Eq. 4.8, we have:

$$\begin{aligned}\Pi_1(p, w) &= \sup\{p^t y - w^t x : (x, y) \in T_1\} \\ &\leq \sup\{p^t y - w^t x : (x, y) \in T_2\} \\ &= \Pi_2(p, w)\end{aligned}$$

Hence,

$$\Pi_1(p, w) - (p^t y_o - w^t x_o) \leq \Pi_2(p, w) - (p^t y_o - w^t x_o)$$

Since $0 \leq p^t g^y - w^t g^x$, the result follows.

Corollary 4.1 *Let $x^I = \sum_{k=1}^{K} x^k$ and $y^I = \sum_{k=1}^{K} y^k$ denote the industry inputs and outputs, respectively. For a given price vector (p, w) and direction $g = (g^x, g^y)$ we have:*

$$\begin{aligned}OE_{KT_{VRS}}&(x^I, y^I, p, w, g^x, g^y) \\ &\leq OE_{T^I}(x^I, y^I, p, w, g^x, g^y) \\ &\leq OE_{KT_{CRS}}(x^I, y^I, p, w, g^x, g^y) \quad (4.18)\end{aligned}$$

Eqs. 4.16 and 4.18 suggest lower and upper bounds for industry allocative efficiency:

Theorem 4.3 *Let $x^I = \sum_{k=1}^{K} x^k$ and $y^I = \sum_{k=1}^{K} y^k$ denote the industry inputs and outputs, respectively. For a given price vector (p, w) and direction $g = (g^x, g^y)$ we have:*

$$\begin{aligned}&OE_{KT_{VRS}}(x^I, y^I, p, w, g^x, g^y) \\ &- D_{KT_{CRS}}(x, y; g) \leq AE_{T^I}(x^I, y^I, p, w, g^x, g^y) \\ &\leq OE_{T_{CRS}}(x^I, y^I, p, w, g^x, g^y) \\ &- D_{KT_{VRS}}(x, y; g) \quad (4.19)\end{aligned}$$

Proof. The result follows from Eqs. 4.16 and 4.18 along with the compatibility with addition property of ordering relation $\leq$ on real numbers.

Remark 4.1 Eq. 4.16 implies that the upper bound in Eq. 4.19 is nonnegative. However, this is not the case for the lower bound. Since, the allocative efficiency is always nonnegative, the lower bound can be considered as

$$\max\{OE_{KT_{VRS}}(x^I, y^I, p, w; g^x, g^y) - D_{T_{CRS}}(x, y; g), 0\}.$$

5 Illustrative examples

In this section, our results will be analyzed by two numerical examples. To explain our results more clear, first we assume the known industry technology is existed and it is made by K identical technologies (T) which their frontier are composed by Cobb-Douglas function. The Cobb- Douglas function form of production function is generally generated as $Y = AL^{\alpha}C^{\beta}$ by [11] where Y is total production, L is labor input, C is capital input, A is total factor productivity, α, and β are the percentage change of output divided by the percentage change of an input of labor and capital, interpreted as:

- If $\alpha + \beta = 1$, the production function has constant returns to scale technology.
- If $\alpha + \beta < 1$, the production function has decreasing returns to scale technology.
- If $\alpha + \beta > 1$, the production function has increasing returns to scale technology.

The Cobb- Douglas function for a single input to produce single output is $y = x^{\alpha}$, ($\alpha > 0$). In our example the K identical technologies which make industry technology are assumed as follows:

$$T^K := \left\{(x, y) : y \leq (x-1)^{0.5}\right\}$$

It is obvious that the frontier of above technology is made by Cobb- Douglas function, and it includes all the assumptions (A1)-(A5) for unknown technology. x is a single input used to specify the maximum level of output y (Just to compare T, T_{CRS}, and T_{VRS}, graphically, we switched on the single input/single output case.). $y \leq (x-1)^{0.5}$ is a decreasing returns to scale (DRS) technology. To visualize and describe our results easily $(g^x, g^y) = (-x_o, 0)$ is our chosen direction.

Example 5.1 *In this example, to show the results more accurately, we assume that the technology is known and we show that the achieved efficiencies of industry unit are between the ones which were introduced as the bounds. Let there are 6 hypothetical firms with one input and one output, and they are included in a technology which is made by $y \leq (x-1)^{0.5}$. Table 1 shows data of 6 firms and industry firm.*

Table 1: Data of 6 firms

Firm	Input	Output
1	1.5	0.6
2	2	1
3	3.5	1.2
4	5	1
5	5.8	0.5
6	4.5	0.5
Industry unit	22.3	4.8

The frontiers of T which is made by Cobb-Douglas function, T_{CRS} and T_{VRS} which are made by the observed firms are shown in figure 1. It is obvious that T is a convex technology, so

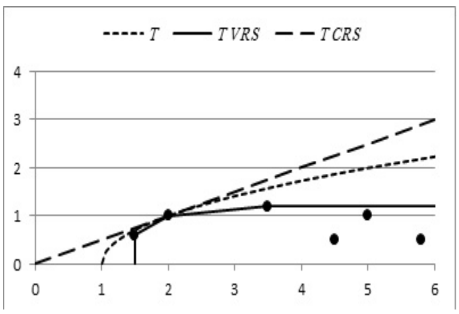

Figure 1: T, T_{CRS}, and T_{VRS}

based on theorem 3.2, $\sum_{k=1}^{K} T = KT$. The frontiers of $KT = T^I$, T_{CRS}, and KT_{VRS} are shown in figure 2. The industry firm is determined in figure 2, as well. Table 2 shows the DDF measure of industry firm.

As we expected, the DDF measure of industry firm in its technology is between the DDF

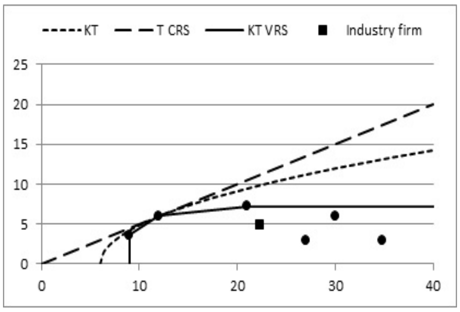

Figure 2: KT, T_{CRS}, and KT_{VRS}

Table 2: DDF measure in 3 different technologies

Firm	KT_{VRS}	KT	T_{CRS}
Industry unit	0.5921	0.5587	0.5695

measure of industry firm in KT_{VRS}, and T_{CRS}. Now, we want to evaluate the overall efficiency score of industry firm. First of all, we should calculate the profit efficiency score of industry firm and then handle equation 4.9. The Cobb-Douglas profit function for single input and output assuming that $0 \leq \alpha < 1$ is as follows:

$$\begin{aligned}\Pi(p,w) &= \sup\{py - wx : (x,y) \in T\} \\ &= \sup\{px^{\alpha} - wx : (x,y) \in T\}\end{aligned}$$

The first order condition depends on x show:

$$\begin{aligned}\frac{d\Pi(p,w)}{dx} &= p\alpha x^{\alpha-1} - w = 0 \\ &\Rightarrow p\alpha x^{\alpha-1} = w\end{aligned}$$

Thus, x (critical point) will be obtained by the above equation and then $y = x^{\alpha}$ can be reached, afterwards, with the placement of the obtained x and y in $px^{\alpha} - wx$, the maximum profit will be attained (see [27]). Table 3 shows the overall efficiency of industry firm which is evaluated by equation 4.9. Let $w = 1$ and $p = 5$.

Table 3: Overall efficiency score in 3 different technology

Firm	KT_{VRS}	KT	T_{CRS}
Industry unit	0.7309	1.1411	1.4238

As it has been shown the overall efficiency score of industry firm in its technology is between the efficiency score of industry firm in KT_{VRS} and T_{CRS}. To obtain overall efficiency of industry firm in its technology (KT), first we should maximize $py - wx$ for $Y = 6y$, $X = 6x$, and $Y = \sqrt{X-1}$. As mentioned before, applying the derivative of function is away to determine the maximum of the function.

Remark 5.1 In the profit model which we use, we obtain the input and output of under evaluated firm through the points which can dominate under evaluated firms.

$$\begin{aligned}pY - wX &= 5(6y) - 1(6x) = 5(6\sqrt{x-1}) - 6x \\ (pY - wX)' &= \left(5(6\sqrt{x-1}) - 6x\right)' \\ &= 6\left(\frac{5}{2\sqrt{x-1}-1}\right) = \frac{30 - 12\sqrt{x-1}}{2\sqrt{x-1}} = 0 \\ 30 - 12\sqrt{x-1} &= 0 \Rightarrow \sqrt{x-1} = 2.5 \\ &\Rightarrow x = 7.25 \Rightarrow 6x = 43.5\end{aligned}$$

To evaluate the overall efficiency of industry firm in KT, we should set $6x \leq 22.3$ or equivalently $x \leq 3.71667$ with respect to the earlier remark. On the other hand the domain of $Y = \sqrt{X-1}$ is $[1, +\infty)$. Since the critical point is not laid between 1 and 3.71667, we do not need to compute the value of function at its critical point. The maximum profit of $pY - cX$ is 27.1469 which is obtained in $X = 6x = 22.3$ and $Y = 6y = 9.88938$. So,

$$\begin{aligned}OE_{T^I}&(x^I, y^I, p, w; g^x, g^y) \\ &= \frac{\Pi(p,w) - (py - wx)}{pg^y + wg^x} \\ &= \frac{27.1469 - (24 - 22.3)}{5 \times 0 + 1 \times 22.3} \\ &= 1.141117\end{aligned}$$

Table 4 shows the allocative efficiency of an industry unit in its technology (KT) which is evaluated by equation 4.11, and the bounds which are obtained for it in theorem 4.3.

Example 5.2 *Wheat is considered as one of the main primary food of Iranians and the most important agricultural commodities in Iran in terms of production and consumption. Producing wheat is so important in terms of income, nutrition and employment of people. In consumption side, the per capita consumption for bread wheat in Iran*

Table 4: Allocative efficiency score in KT and its bounds

Firm	Lower bound	KT	Upper bound
Industry firm	0.1614	0.5824	0.8946

Table 5: Descriptive statistics on a data set of 30 provinces

	Consumed seed	Cultivated area	Wheat production
Min	1009	7700	11611
Max	160755	821189	1179322
Average	41817.23	248918.4	446263

Table 6: Descriptive statistics on a data set of 30 provinces

bounds for DDF measure, overall efficiency, allocative efficiency of industry unit		
Firm	KT_{VRS}	T_{CRS}
DDF measure	0.482638	0.587636
Overall efficiency	1.1260667	2.7980873
Allocative efficiency	0.6434287	2.2104513

is about 160 kilograms which is higher than most of the other countries. Great demand for wheat in Iran and the difficulties arisen to meet the demand made the government to import wheat. So, Iran is one of the largest wheat importers in the world. Iran governments encourage farmers to produce more wheat and they devote some programs to increase wheat production. This section analyzes wheat farming efficiency in provinces of Iran in 2008–2009 crop year which is started on 22 September 2008 and ended on 22 September 2009. In the mentioned time, Iran consisted of 30 provinces which were managed by the government. Forasmuch as producing technology depends on some factors such as demands, price of demands, cost of production, natural resources and etc. and these factors vary from province to province, the technology may be unknown. So, the performance of the provinces cannot be analyzed and compare with each other. In this example, we consider 30 provinces of Iran with two inputs and one output. Inputs are consumed seed (based on ton) and cultivated area (based on hectare) and the output is wheat production (based on ton). Table 5 summarizes descriptive statistics on a data set of 30 under evaluated provinces on Iran wheat farming in 2008-2009 crop year http://www.maj.com/.

Table 6 shows the lower (the second column) and higher (third column) bounds for the DDF measure, overall efficiency and allocative efficiency of Industry unit. $(g^x, g^y) = (-x_o, 0)$ is our chosen direction and price of consumed seed, cultivated area and wheat production are assumed 5, 3 and 4, respectively.

As it is shown in the table 6, if firm technologies satisfy all the conditions of (A1)–(A5), regardless of its type, the minimum and maximum amount can be presented for DDF measure, overall efficiency and allocative efficiency of industry unit. So, the technology of the firms can be compliance with the mentioned conditions such as NIRS (non-increasing returns to scale), NDRS (non-decreasing returns to scale), etc. or even a combination of some technologies or the technologies which are still unknown.

6 Conclusion

Evaluating the performance of an industry unit in known technologies for known or unknown price is an interesting subject for researchers to study, but there are a few papers which handle unknown technologies to evaluate an industry performance. In real world, technology of systems is not available and determinable always, so, it is necessary to deal with unknown technologies and determine the performance of systems with unknown technology. The current paper presented inner and outer technology estimations for an unknown industry. Industry firm is a firm which is an important one for comparing a large number of systems in different places or in different times. Using the achieved inner and outer technologies, a bounded range for DDF and the overall and allocative efficiency of the industry unit were calculated. So, we suggested a method to evaluate industry firm with unknown technology. Finally, in illustrative examples section a Cobb-Douglas production function is used to make an industry technology and verify the results. Then, an example about agriculture is presented to show the application of the paper in real industries. Dealing with unknown technology and unknown prices can be a good suggestion for future researches to evaluate cost, revenue and profit efficiency of industry level with unknown data.

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