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Image Compression Method Based on QR-Wavelet Transformation

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Abstract

In this paper, a procedure is reported that discuss how linear algebra can be used in image compression. The basic idea is that each image can be represented as a matrix. We apply linear algebra (QR factorization and wavelet transformation algorithms) on this matrix and get a reduced matrix out such that the image corresponding to this reduced matrix requires much less storage space than the original image.

Keywords : Image compression; Wavelets; Matrix factorization; PSNR.

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1 Introduction

ENtering the digital age and increasing growth of technology, we have to handle a vast amount of information that must be stored and retrieved in an efficient and effective manner, which often presents difficulties. As a result of it, various compression techniques are in demand which can help to reduce the size of data files. The image is actually a kind of redundant data i.e. it contains the same information from certain perspective of view. By using data compression techniques, it is possible to remove some of the redundant information contained in images.
The vital purpose of compression techniques is reducing information redundancy for minimizing transmission bandwidth and archiving costs [18, 16]. So the size in bytes of a graphics file is minimized without degrading the quality of the image to an unacceptable level. The reduction in file size allows more images to be stored in a certain amount of disk or memory space and reduces the time necessary for images to be sent over the internet or downloaded from web pages. Performance of compression techniques depends on some criteria such as quality of image, compression ratio (speed of compression and computational complexity), memory resources, and power consumption [17].
Compression techniques can be either lossless or loosy [15]. If removing the redundancy is a reversible process and has no information loss, this is called lossless compression. On the other hand, if it has an information loss, it will be called lossy compression technique. In lossless compression approach, decompressed image is identical to the original one [19]. In lossy compression, data loss is incurred and the compressed image is not usually the same as the original one, but forms a close approximation to the original image and compression ratio is very high. Therefore, some form of distortion measure is required. Distortion mea-

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sure is a mathematical quantity that specifies how close is an approximation image to the original image. The most commonly used distortion measures in image compression are MSE and PSNR that will be described in Section 4.

The goal of this research is to find out a transformation based compression approach which have low computational complexity and be able to concentrate the signal energy in the smallest number of parameters. Transform based techniques are based on converting input vector X (may be image) through transform T into another form Y which is less correlated than X. Transform T does not compress any data; the compression comes from processing and quantization of Y components. For this target there is a description of discrete cosine transform and discrete wavelet tyransform (DWT) [8]. Wavelets provide a mathematical way of encoding information in such a way that it is layered according to level of details. This layering facilitates approximations at various intermediate stages. These approximations can be stored using a lot less space than the original data. Here a low complex 2D image compression method using wavelets and the QR matrix decomposition is presented.

2 Background

2.1 QR decomposition

There is a class of compression schemes that are based purely on linear algebra and are completely insensitive to analytical origin of the operator. This class consists of the singular value decomposition (SVD) [7], the so-called QR, QLP factorizations [13] and several others. The QR matrix decomposition appears in fields related directly with algebra, such as linear equations, least squares problems, constrained least squares problems, the pseudo-inverse of a matrix with linearly independent columns and the inverse of a nonsingular matrix. Also its usefulness in applications concerning image processing has been evaluated. Among these appliations we can mention patron recognition, secret communiaction of digital images, quantization, and compression of images as much as of video sequences [1].

In QR decomposition any matrix A of size $m \times n$ can be factored as

$$A = QR,$$

where Q is an $m \times n$ matrix satisfying $QQ^T = I$ and R is an $n \times n$ upper triangular matrix. If A is non-singular, this factorisation is unique.

In [14], Naderahmadian and Hosseini proved that if the columns of matrix A are correlated then the absolute value of elements of the first row of matrix R is larger than the absolute value of the other rows. Based on this fact, small change in the first row of matrix R will not lead to image distortion, if A is an image matrix.

Theorem 2.1 *[5] Suppose that A is an $m \times n$ matrix, $l = \min(m, n)$, and k is an integer such that $1 \le k \le l$. Then there exists a factorization*

$$AP = QR, \tag{2.1}$$

where P is an $n \times n$ permutation matrix, Q is an $m \times l$ matrix with orthonormal columns, and R is an $l \times n$ upper triangular matrix. Furthermore, splitting Q and R,

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \qquad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \tag{2.2}$$

in such that Q_{11} and R_{11} are $k \times k$ matrices, Q_{21} is $(m-k) \times k$, Q_{12} is $k \times (l-k)$, Q_{22} is $(m-k) \times (l-k)$, R_{12} is $k \times (n-k)$ and R_{22} is $(l-k) \times (n-k)$, results the following inequalities

$$\sigma_k(R_{11}) \ge \sigma_k(A) \frac{1}{\sqrt{1 + k(n-k)}}, \tag{2.3}$$

$$\sigma_1(R_{22}) \le \sigma_{k+1}(A) \sqrt{1 + k(n-k)}, \tag{2.4}$$

$$\|R_{11}^{-1} R_{12}\|_F \le \sqrt{k(n-k)}, \tag{2.5}$$

where $\{\sigma_i\}_{i=1}^{r}$ are singular values of the matrix A with $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$.

Remark 2.1 Let $\varepsilon = \sigma_{k+1}(A)$ be a very small number. Therefore, the inequality (2.4) implies that A can be well approximated by a low rank matrix such that

$$\|A - \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} [R_{11} | R_{12}] P^* \|_2 \le \varepsilon \sqrt{1 + k(n-k)}. \tag{2.6}$$

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While Theorem (2.1) asserts the existence of QR factorization with the properties (2.3), (2.4), (2.5) , it says nothing about the cost of constructing such a factorization numerically. The following theorem asserts that a factorization that satisfies bounds that weaker than (2.3), (2.4), (2.5) by a factor of $\sqrt{n}$ can be computed in $O(mn^2)$ operations.

Theorem 2.2 *[5] For an $m \times n$ matrix A that satisfies the inequalities*

$$\sigma_k(R_{11}) \geq \sigma_k(A)\frac{1}{\sqrt{1+nk(n-k)}}, \qquad (2.7)$$

$$\sigma_1(R_{22}) \leq \sigma_{k+1}(A)\sqrt{1+nk(n-k)}, \qquad (2.8)$$

$$\|R_{11}^{-1}R_{12}\|_F \leq \sqrt{nk(n-k)}, \qquad (2.9)$$

a factorization of the form (2.1) can be computed in $O(mn^2)$ operations.

Remark 2.2 The complexity $O(mn^2)$ in Theorem (2.2) is a worst-case bound. Typically, the number of opeations required is similar to the time required for simple pivoted Gram-Schmidt algorithm; $O(mnk)$.

Theorem (2.1) tells us how to obtain approximations of small rank for a matrix. This result has been used to compress an image in the following way.
An image (matrix) A of size $m \times n$ has initially $m \times n$ entires to store. So if we consider A_k defined as

$$A_k = \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} [R_{11}|R_{12}]P^*, \qquad (2.10)$$

instead of A, then we have an approximation of A which can be stored with $k(m+n)+mn$ values. Clearly, a compromise between the precision of approximation and desired compression ratio must be achieved. The compression algorithm is competitive when with a small value of k we get already a good quality of the resulting image.
All popular image compression techniques work on the sub-block of original image instead of compressing the whole image at once. If a portion of the image is simple, then only a smaller k needs to be used to achieve satisfactory approximation. On the other hand, if the image is complex, then larger k would have to be used in order to maintain the image quality. To see the effect of this rank selection scheme, we can observe the fact that the ranks used for each sub-block are indeed correlated with the complexity of the image. So, to obtain a better adaptation to the concrete characteristics of a given image we apply the QR algorithm to the matix by blocks. A further reduction in the ranks used can be achieved by subtracting the mean of the original image before perfromaning the QR decomposition. Then the mean is added back to the QR construction to obtain the the reconstructed image.

2.2 Wavelet transform

Wavelets are a more general way to represent and analyze multiresolution images that can also be applied to 1D signals. Wavelets are a class of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation and translation parameters a and b are vary continuously, the following family of continuous wavelets are obtained

$$\psi_{ab}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-a}{a}\right), \; a, b \in R, \; a \neq 0.$$

When the parameters a and b are restricted to discrete values as $a = 2^{-k}, b = n2^{-k}$, then, we have the following family of discrete wavelets

$$\psi_{kn}(t) = 2^{\frac{k}{2}}\psi(2^k t - n), \; k, n \in \mathbb{Z},$$

where the function ψ, the mother wavelet, satisfies $\int_{\mathbb{R}} \psi(t)dt = 0$. We are interested in the case where ψ_{kn} constitutes an orthonormal basis of $L^2(\mathbb{R})$. A systematic way to do this is by means of multiresolution analysis (MRA).
In 1910, Haar [6] constructed the first orthonormal basis of compactly supported wavelets for $L^2(R)$. It has the form $\{2^{j/2}\psi(2^j t - k) : j, k \in \mathbb{Z}\}$ where the fundamental wavelet ψ is constructed as follows: Construct a compactly supported scaling function φ by the two-scale scaling relation $\varphi(t) = \varphi(2t) + \varphi(2t-1)$ together with the normalization constraint $\int \varphi(t)dt = 1$. A solution of this recursion that represents φ in $L^2(\mathbb{R})$ is $\chi[0, 1)$. Then $\psi(t) = \varphi(2t) - \varphi(2t-1)$. The Haar wavelets are piecewise continuous and have discontinuities at certain dyadic rational numbers.
In a seminal papers; Daubechies [3, 2], constructed the first orthonormal basis of continuous

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compactly supported wavelets for $L^2(\mathbb{R})$. They have led to a significant literature and development, both in theoretical and applied arenas.
Later in 1989, Mallat [9] studied the properties of multiresolution approximation and proved that it is characterized by a 2π-periodic function. From any MRA, one can derive a function $\psi(t)$ called a wavelet such that $\{2^{j/2}\psi(2^j t-k):\ j,k\in\mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. The MRA showed the full computational power that this new basis for $L^2(R)$ possessed. In the same year, Mallat [10] applied MRA for analyzing the information content of the images.
Note that a system : $k\in\mathbb{Z}$ is called a Riesz basis if it is obtained from an orthonormal basis by means of a bounded invertible operator [20].

Definition 2.1 *The increasing sequence $\{V_k\}_{k\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ with scaling function $\varphi\in V_0$ is called MRA if*

(i) $\bigcup_k V_k$ *is dense in* $L^2(\mathbb{R})$ *and* $\bigcap_k V_k=\{0\}$,

(ii) $f(t)\in V_k$ *iff* $f(2^{-k}t)\in V_0$,

(iii) $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ *is a Riesz basis for* V_0.

Note that (iii) implies that the sequence $\{2^{k/2}\varphi(2^k t-n)\}_n$ is an orthonormal basis for V_k. Let $\psi(t)$ be the mother wavelet, then

$$\psi(t)=\sum_{n\in\mathbb{Z}}a_n\varphi(2t-n),$$

and $\{2^{k/2}\psi(2^k t-n)\}_{k,n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$ under suitable conditions.
Over the past few years, a variety of powerful and sophisticated wavelet-based schemes for image compression have been developed and implemented. Wavelet based transform represent a signal with good resolution in time and frequency using a set of basis functions called wavelets [11]. The 2D wavelets used in image compression are separable functions. Their implementation can be obtained by first applying low pass filter on rows to produce L and H subbands, then apply high pass filter on columns to produce four subbands LL (approximate subband), LH, HL, and HH (detail subbands). Then, in the second level, each of these four subbands is self-decomposed into four subbands LL2, LH2, HL2, HH2, and so on. It can be decomposed into $3,4,\ldots$ levels. At each level, we just store the differences (residuals) between the image at that level and the predicted image from the next level and we can reconstruct the image by just adding up all the residuals. One of the advantages of wavelets method is that the residuals are easier to store. Also, wavelet coding schemes at higher compression avoid blocking artifacts and are better matched to the HVS (Human Visual System) characteristics.

3 Proposed Technique

As we said in the previous section wavelet transformation can help to improve the compression capablities of the QR algorithm. Unifying the featured aforementioned concepts, we construct our algorithm as the following multistage process.

(i) Image A is read.

(ii) Consider $X=A-mean$.

(iii) X is converted into a column vector Y of size $mn\times 1$ and Y is resorted as increasing order to obtain Y'. Meanwhile record the original position of the element of Y', it will be used in step (x).

(iv) Y' is reshaped back to $m\times n$ image X'. Then X' transformed using wavelet transformation which decomposes image into 4 different frequency bands as explained in Section (*2.2*).

(v) Step 4 is applied on upto 3 Level decomposition on LL block. We get one approximation image of size $m/8\times n/8$ and nine details images of different size (3 images of size $m/8\times n/8$, 3 images of size $m/4\times n/4$ and 3 images of size $m/2\times n/2$).

(vi) Approximation image (LL subband) resulting from iteration 3 is divided into non-overlapping blocks $\{b_{ij}\}$ of size $m_1\times n_1$.

(vii) Apply QR decomposition to each block to obtain two components Q^{ij} and R^{ij}, where they can spilit such as equation (2.2).

(viii) Perform A_k^{ij} as said in equation (2.10) for each $\{b_{ij}\}$ and merge all of these blocks together to obtain X'' as opposite to step (vi).

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(ix) Perform inverse wavelet transform to X'' and other subbands HL , LH and HH obtained in step (v).

(x) X'' is converted into a column vector of size $mn \times 1$ and by using the position vector in step iii, rearrange this vecor. Then reshape back final vector to $m \times n$ image matrix. By adding back the mean value to this matrix we can obtain compressed image X^C.

At this algorithm, we can choose different values of k and compare the results with each other and select the best ones. In this paper we choose the rank of each block as the value of k.

4 Experimental Analysis

In this section, the feasibility and robustness of the proposed image compression method are analysed by simulation experiments. All simulation experiments are conducted on a personal computer using MATLAB version R2015b. We apply the QR and QR-wavelet methods to some normal images such as the Cameraman image (256 × 256, gray scale) and Moon image (256 × 256, gray scale), with the purpose of studying the pros and cons of these methods.
There is a need for specifying methods that can judge image quality after reconstruction process and measure the amount of distortion due to compression process as minimal image distortion means better quality. There are two types of image quality measures, subjective quality measurement and objective quality measurement [12]. Subjective quality measurement is established by asking human observers to report and judge image or video quality according to their experience, and these measures would be relative or absolute. Absolute measures classify image quality not regarding to any other image but according to some criteria of television allocations study organization. On the other hand, relative measures compare image against another and choose the best one.
Objective measures are mathematical measures that measure the amount of image distortion and image quality. Those measures are

(1) Mean Square Error (MSE) which is defined as

$$MSE = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(I(i,j) - I^c(i,j))^2 \tag{4.11}$$

where I is original image and I^c is compressed image with the same size $m \times n$.

(2) Peak-Signal-To-Noise ratio (PSNR) which measures the size of error relative to peak value of $I(i,j)$ (for 8 bit pixel $\max I(i,j)$ equals 255) of the signal and it is given by

$$PSNR = 20\log_{10}\frac{\max I(i,j)}{\sqrt{MSE}}. \tag{4.12}$$

(3) Compression Ratio (CR) is the ratio of the storage space required to store original image to that required to store a compressed image compression ratio is a term that is being used to describe and measured with bits per pixel (bpp) as described as follows

$$CR = \frac{\text{Entries of the original image}}{\text{Entries stored in the compressed image}}.$$

When reconstructed image is close to original one, this means that MSE between two images is low. On the other hand, higher PSNR means better image quality. CR can be used to judge how compression efficiency is, as higher CR means better compression.

Figures 1 and 2 show the results of the compression with QR-wavelet method used for 'cameraman' and 'moon' images with block size 16. Figures 3 and 4 show the absolute difference between the original image and the reconstructed image with block size 16.
In tables 1 and 2, degree of compression is measured using compression ratio, MSE and PSNR values. From figures and tables we conclude that in QR method larger size of the blocks conclude more compression ratio (i.e. less storage space is required) but image quality deteriorates (i.e. larger MSE and smaller PSNR values). In QR-wavelet method smaller size of the blocks conclude more compression ratio (i.e. less storage space is required) but image quality deteriorates

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Table 1: Image quality and Compression ratio for cameraman image.

Size of blocks	QR decomposition			QR-wavelet method (Haar wavelet)		
	CR	MSE	PSNR	CR	MSE	PSNR
4	2	0.0038	72.2250	48.7619	7.1783e-05	89.5022
8	4	0.0104	67.8862	41.7959	1.1914e-04	87.3017
16	8	0.0225	64.5499	35.9298	1.8799e-05	95.3211

Table 2: Image quality and Compression ratio for moon image.

Size of blocks	QR decomposition			QR-wavelet method (Haar wavelet)		
	CR	MSE	PSNR	CR	MSE	PSNR
4	2	0.0044	71.6726	57.6901	822.9787	18.9428
8	4	0.0118	67.3961	62.0606	563.9517	20.5843
16	8	0.0246	64.1883	58.5143	563.1445	20.5905

(i.e. larger MSE and smaller PSNR values). Thus, it is necessary to strike a balance between storage space required and image quality for good image compression. Generally, choice of block size depends on the application. For instance, in some applications, if image quality is important then higher values of block size are chosen but sometimes storage space is more important than image quality , in that case lower block size values are taken.

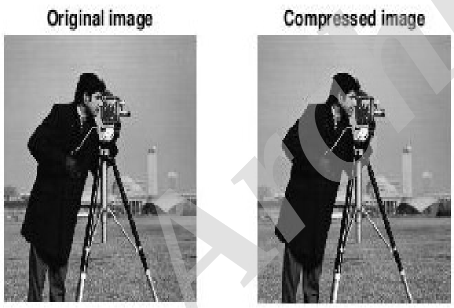

Figure 1: QR-wavelet method with size of blocks=16

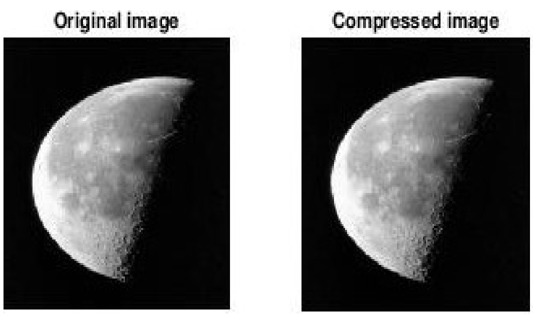

Figure 2: QR-wavelet method with size of blocks=16

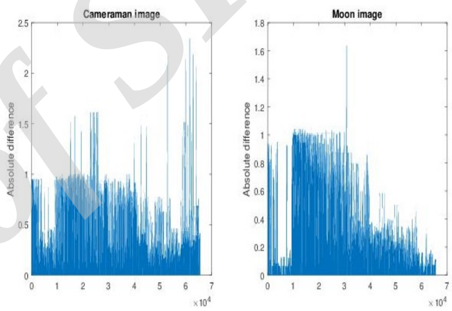

Figure 3: Absolute difference between the original image and the reconstructed images by QR-wavelet method with size of blocks 16 for 'cameraman'

5 Conclusion

In this paper, we present a compression scheme that allows us to modify the QR method with wavelet method in such way that a larger compression is attained. From the results it is derived that QR-wavelet technique achieved higher PSNR value compared to QR method. So proposed technique have better visual decompressed image or less loss of compressing embedding compared to QR based compression. The reason behind better decompressed image quality of proposed technique is that compression with wavelets is scalable as the transform process can be applied to an image as many times as wanted and hence very high compression ratios can be achieved. Wavelet based compression allows parametric gain control for image softening and sharpening. Wavelet-based coding is more

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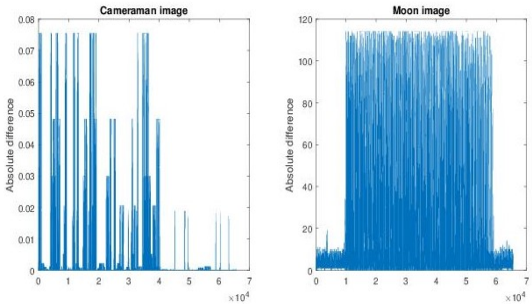

Figure 4: Absolute difference between the original image and the reconstructed images by QR-wavelet method with size of blocks 16 for 'moon'

robust under transmission and decoding errors, and also facilitates progressive transmission of images. Wavelet compression is very efficient at low bit rates and provide an efficient decomposition of signals prior to compression.

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