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Research Article

Leap Zagreb Indices of Some Graph Operations

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Abstract

The first leap Zagreb index LM_1 of a (molecular) graph, is the sum of squares of the second degrees of vertices (number of their second neighbors), and the second leap Zagreb index LM_2 is the sum of the products of the second degrees of pairs of adjacent vertices, and the third leap Zagreb index LM_3 is the sum of the product of the degree and second degree of the vertices. In this paper, we determine the first, second and third leap Zagreb indices of some graph operations.

Keywords : Zagreb indices; Leap Zagreb indices; Graph operations; Supply chains; Consumers.

1 Introduction

In this paper G is a simple and connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *degree* of a vertex v in G is the number of edges incident to v and denoted by $d(v/G)$. The *distance* $d_G(u, v)$ between any two vertices u and v of a graph G is equal to the length of a shortest path connecting them. For a vertex $v \in V(G)$ and a positive integer k, the *open k-neighborhood* of v in the graph G, denoted by $N_k(v/G)$, is defined as $N_k(v/G) = \{u \in V(G) : d(u, v) = k\}$. The *$k$-distance degree* of a vertex v in G, denoted by $d_k(v/G)$ is the number of k-neighbors of the vertex v in G, i.e., $d_k(v/G) = |N_k(v/G)|$. It is clear that $d_1(v/G) = d(v/G)$ for every $v \in V(G)$.

In chemical graph theory, a graphical invariant is a number related to a graph which is structurally invariant. These invariant numbers are also known as the topological indices. The well-known Zagreb indices are one of the oldest graph invariants firstly introduced by Gutman and Trinajstić [?], where they examined the dependence of total π-electron energy on molecular structures, and this was elaborated in [?]. For a (molecular) graph G, the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$, defined as:

$$\begin{aligned} M_1(G) &= \sum_{v\in V(G)} d(v/G)^2 \\ &= \sum_{uv\in E(G)} [d(u/G) + d(v/G)] \end{aligned}$$

and

$$M_2(G) = \sum_{uv\in E(G)} d(u/G)d(v/G).$$

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For properties of the two Zagreb indices see [?, ?, ?, ?] and the papers cited therein. In recent years, some novel invariants of Zagreb indices have been put forward, such as Zagreb coindices [?, ?, ?], reformulated Zagreb indices [?, ?], Zagreb hyperindex [?, ?], multiplicative Zagreb indices [?, ?], multiplicative sum Zagreb index [?, ?], and multiplicative Zagreb coindices [?], etc. The Zagreb coindices are defined as:

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} [d_1(u/G) + d_1(v/G)]$$

and

$$\overline{M_2}(G) = \sum_{uv \notin E(G)} d_1(u/G)d_1(v/G).$$

Recently Naji, Soner and Gutman [?], extended the concept of Zagreb index to analogous invariants based on the second vertex degree as leap Zagreb indices. For a graph G, the first, second, and third leap Zagreb indices are:

$$LM_1(G) = \sum_{v \in V(G)} d_2(v/G)^2$$

$$LM_2(G) = \sum_{uv \in E(G)} d_2(u/G)d_2(v/G)$$

$$LM_3(G) = \sum_{v \in V(G)} d(v/G)d_2(v/G).$$

The authors [?] investigated basic properties of these invariants and established some bounds on leap Zagreb indices in terms of Zagreb indices, and in terms of the order and the size of the graph.

Graph operations play an important role in chemical graph theory. Some chemically important graphs can be obtained from some graphs by different graph operations, such as some nanotorus or Hamming graph, that is Cartesian product of complete graphs. Many authors computed some indices for some graph operations (see, for instance [?, ?, ?, ?, ?, ?, ?] and the references cited therein).

In this paper, we compute the first, second, and third leap Zagreb indices of some graph operations.

2 The corona, disjunction, symmetric difference, composition and cartesian product of graphs

In this section, we compute the first, second, and third leap Zagreb indices of the corona, disjunction, symmetric difference, composition and cartesian product of graphs. For convenience, we assume that $D_2(G) = \sum_{u \in V(G)} d_2(u/G)$.

2.1 The corona product of graphs

The corona product $G_1 \circ G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph obtained by one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the i-th vertex of G_1 to every vertex in i-th copy of G_2. Obviously,

$$|V(G_1 \circ G_2)| = |V(G_1)| + |V(G_1)||V(G_2)|$$

and

$$|E(G_1 \circ G_2)| =$$

$$|E(G_1)| + |V(G_1)||E(G_2)| + |V(G_1)||V(G_2)|$$

.

We begin with the following decisive lemma related to 2-distance degree properties of a vertex in the corona product of two graphs. The proof of that is immediate, so omitted.

Lemma 2.1. *Let G_i be a graph of order n_i for $i = 1, 2$. Then*

$$d_2(u/G_1 \circ G_2) = d_2(u/G_1) + n_2 d(u/G_1)$$

if

$$u \in V(G_1),$$

and

$$d_2(u/G_1 \circ G_2) = n_2 - 1 - d(u/G_2) + d(x/G_1)$$

if

$$u \in V(G_2), x \in V(G_1) \text{ and } ux \in E(G_1 \circ G_2)$$

.

Theorem 2.1. *Let G_i be a graph of order n_i and size ε_i for $i = 1, 2$. Then*

(a) $LM_1(G_1 \circ G_2) =$

$$LM_1(G_1) + n_2^2 M_1(G_1) + 2n_2 LM_3(G_1) + n_1 n_2 (n_2-1)^2 + n_2 M_1(G_1) + n_1 M_1(G_2) - 8\varepsilon_1\varepsilon_2 + 4(n_2-1)(n_2\varepsilon_1 - n_1\varepsilon_2).$$

(b) $LM_2(G_1 \circ G_2) =$

$$LM_2(G_1) + n_2^2 M_2(G_1) + 2\varepsilon_1 n_2^2 (n_2-1) - 4\varepsilon_1\varepsilon_2 n_2 + n_2^2 M_1(G_1) + [n_2(n_2-1) - 2\varepsilon_2] D_2(G_1) + n_2 LM_3(G_1) n_1\varepsilon_2 (n_2-1)^2 l + \varepsilon_2 M_1(G_1) - n_1(n_2-1) M_1(G_2) + 4(n_2-1) \varepsilon_1\varepsilon_2 - 2\varepsilon_1 M_1(G_2) + n_1 M_2(G_2) + n_2 \sum_{v_iv_j \in E(G_1)} [d(v_i/G_1) d_2(v_j/G_1) + d_2(v_i/G_1) d(v_j/G_1)].$$

(c) $LM_3(G_1 \circ G_2) =$

$$LM_3(G_1) + n_2 M_1(G_1) + 2\varepsilon_1 n_2^2 + n_2 D_2(G_1) + 2n_2\varepsilon_1 + (2n_1\varepsilon_2 + n_1 n_2)(n_2-1) - n_1 M_1(G_2) + 4\varepsilon_1\varepsilon_2 - 2n_1\varepsilon_2.$$

Proof. Suppose that $V(G_1) = \{v_1, \ldots, v_{n_1}\}$ and

$$V(G_{2i}) = \{u_{i1}, \ldots, u_{in_2}\}, 1 \le i \le n_1$$

such that G_{2i} is the i-th copy of G_2. By Lemma **??** we have

$$LM_1(G_1 \circ G_2) = \sum_{u \in V(G_1 \circ G_2)} (d_2(u/G_1 \circ G_2))^2$$

$$= \sum_{i=1}^{n_1} (d_2(v_i/G_1 \circ G_2))^2 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} (d_2(u_{ij}/G_1 \circ G_2))^2$$

$$= \sum_{i=1}^{n_1} (d_2(v_i/G_1) + n_2 d(v_i/G_1))^2 + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} (n_2 - 1 - d(u_{ij}/G_2) + d(v_i/G_1))^2$$

$$= \sum_{i=1}^{n_1} [d_2(v_i/G_1)^2 + 2n_2 d_2(v_i/G_1) d(v_i/G_1) + n_2^2 d(v_i/G_1)^2] + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} [(n_2-1)^2 + d(u_{ij}/G_2)^2 + d(v_i/G_1)^2 - 2(n_2-1) d(u_{ij}/G_2) + 2(n_2-1) d(v_i/G_1) - 2d(v_i/G_1) d(u_{ij}/G_2)]$$

$$= LM_1(G_1) + 2n_2 LM_3(G_1) + n_2^2 M_1(G_1) + n_1 n_2 (n_2-1)^2 + n_1 M_1(G_2) + n_2 M_1(G_1) - 8\varepsilon_1\varepsilon_2 + 4(n_2-1)(n_2\varepsilon_1 - n_1\varepsilon_2).$$

$$LM_2(G_1 \circ G_2) =$$

$$\sum_{uv \in E(G_1 \circ G_2)} d_2(u/G_1 \circ G_2) d_2(v/G_1 \circ G_2)$$
$$= \sum_{v_i v_j \in E(G_1)} d_2(v_i/G_1 \circ G_2) d_2(v_j/G_1 \circ G_2)$$
$$+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} d_2(v_i/G_1 \circ G_2) d_2(u_{ij}/G_1 \circ G_2) +$$
$$\sum_{i=1}^{n_1} \sum_{u_{ij} u_{ik} \in E(G_{2i})} d_2(u_{ij}/G_1 \circ G_2) d_2(u_{ik}/G_1 \circ G_2)$$
$$= \sum_{v_i v_j \in E(G_1)} (n_2 d(v_i/G_1) + d_2(v_i/G_1))(n_2 d(v_j/G_1)$$
$$+ d_2(v_j/G_1)) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (n_2 d(v_i/G_1) + d_2(v_i/G_1))$$
$$(n_2 - 1 - d(u_{ij}/G_2) + d(v_i/G_1)) +$$
$$\sum_{i=1}^{n_1} \sum_{u_{ij} u_{ik} \in E(G_{2i})} (n_2 - 1 - d(u_{ij}/G_2)$$

$$+ d(v_i/G_1))(n_2 - 1 - d(u_{ik}/G_2) + d(v_i/G_1))$$

$$= \sum_{v_i v_j \in E(G_1)} [n_2^2 d(v_i/G_1) d(v_j/G_1)$$

$$+ n_2 (d(v_i/G_1) d_2(v_j/G_1) + d(v_j/G_1) d_2(v_i/G_1)) +$$

$$d_2(v_i/G_1) d_2(v_j/G_1)] +$$
$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} [n_2(n_2 - 1) d(v_i/G_1)$$
$$- n_2 d(v_i/G_1) d(u_{ij}/G_2) + n_2 d(v_i/G_1)^2$$

$$+ (n_2 - 1) d_2(v_i/G_1) - d(u_{ij}/G_2)$$

$$d_2(v_i/G_1) + d(v_i/G_1)) d_2(v_i/G_1)] +$$

$$\sum_{i=1}^{n_1} \sum_{u_{ij} u_{ik} \in E(G_{2i})} [(n_2 - 1)^2$$
$$- (n_2 - 1)(d(u_{ij}/G_2) + d(u_{ik}/G_2))$$

$$- d(v_i/G_1)(d(u_{ij}/G_2) + d(u_{ik}/G_2))$$

$$+ 2(n_2 - 1) d(v_i/G_1) + d(u_{ij}/G_2) d(u_{ik}/G_2)$$

$$+ d(v_i/G_1)^2] = n_2^2 M_2(G_1) + LM_2(G_1) + n_2$$

$$\sum_{v_i v_j \in E(G_1)} [d(v_i/G_1) d_2(v_j/G_1) + d_2(v_i/G_1) d(v_j/G_1)]$$

$$+ 2\varepsilon_1 n_2^2 (n_2 - 1) - 4\varepsilon_1 \varepsilon_2 n_2$$

$$+ n_2^2 M_1(G_1) + [n_2(n_2 - 1) - 2\varepsilon_2] D_2(G_1) +$$

$$n_2 LM_3(G_1) + n_1 \varepsilon_2 (n_2 - 1)^2 - n_1(n_2 - 1)$$

$$M_1(G_2) - 2\varepsilon_1 M_1(G_2) +$$

$$4(n_2 - 1)\varepsilon_1 \varepsilon_2 + n_1 M_2(G_2) + \varepsilon_2 M_1(G_1).$$

$$LM_3(G_1 \circ G_2) = \sum_{u \in V(G_1 \circ G_2)} d(u/G_1 \circ G_2)$$
$$d_2(u/G_1 \circ G_2) = \sum_{i=1}^{n_1} d(v_i/G_1 \circ G_2)$$
$$d_2(v_i/G_1 \circ G_2) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2}$$
$$d(u_{ij}/G_1 \circ G_2) d_2(u_{ij}/G_1 \circ G_2)$$
$$= \sum_{i=1}^{n_1} (n_2 + d(v_i/G_1))(n_2 d(v_i/G_1)$$
$$+ d_2(v_i/G_1)) + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (d(u_{ij}/G_2) + 1)$$
$$(n_2 - 1 - d(u_{ij}/G_2) + d(v_i/G_1))$$
$$= \sum_{i=1}^{n_1} [n_2^2 d(v_i/G_1) + n_2 d_2(v_i/G_1)$$
$$+ n_2 d(v_i/G_1)^2 + d(v_i/G_1) d_2(v_i/G_1)] +$$
$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} [(n_2 - 1) d(u_{ij}/G_2) - d(u_{ij}/G_2)^2$$
$$+ d(u_{ij}/G_2) d(v_i/G_1) +$$
$$n_2 - 1 - d(u_{ij}/G_2) + d(v_i/G_1)]$$
$$= 2\varepsilon_1 n_2^2 + n_2 D_2(G_1) + n_2 M_1(G_1)$$
$$+ LM_3(G_1) + (2n_1 \varepsilon_2 + n_1 n_2)(n_2 - 1)$$
$$- n_1 M_1(G_2) + 4\varepsilon_1 \varepsilon_2 - 2n_1 \varepsilon_2 + 2n_2 \varepsilon_1.$$

□

2.2 Disjunction product of graphs

The disjunction product $G_1 \vee G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1)\times V(G_2)$ and $(u,x)(v,y)$ is an edge of $G_1 \vee G_2$ if $uv \in E(G_1)$ or $xy \in E(G_2)$. Obviously, $|E(G_1 \vee G_2)| = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|^2$.

Lemma 2.2. [?] Let G_i be a graph of order n_i for $i = 1,2$. Then $d_{G_1\vee G_2}((u,x),(v,y)) = \begin{cases} 1 & uv \in E(G_1) \text{ or } xy \in E(G_2) \\ 2 & \text{otherwise.} \end{cases}$

Using Theorem **??** and a simple verification, we obtain the following result.

Lemma 2.3. Let G_i be a graph of order n_i for $i = 1,2$. Then $d_2((u,x)/G_1 \vee G_2) = n_1n_2 - d((u,x)/G_1 \vee G_2) - 1$.

Lemma 2.4. [?] Let G and H be graphs. Then

$$M_1(G \vee H) = (|V(G)||V(H)|^2 - 4|E(H)||V(H)|)M_1(G) + M_1(H)M_1(G) + (|V(H)||V(G)|^2 - 4|E(G)||V(G)|)M_1(H) + 8|E(G)||E(H)||V(G)||V(H)|$$

Theorem 2.2. Let G_i be a graph of order n_i and size ε_i for $i = 1,2$. Then

$$(a)\ LM_1(G_1 \vee G_2) = M_1(G_1 \vee G_2) + n_1n_2(n_1n_2-1)^2 - 4(n_1n_2-1)(\varepsilon_1 n_2^2 + \varepsilon_2 n_1^2).$$

$$(b)\ LM_2(G_1 \vee G_2) = M_2(G_1 \vee G_2) - (n_1n_2-1)M_1(G_1 \vee G_2) + (n_1n_2-1)^2(\varepsilon_1 n_2^2 + \varepsilon_2 n_1^2).$$

$$(c)\ LM_3(G_1 \vee G_2) = 2(n_1n_2-1)(\varepsilon_1 n_2^2 + \varepsilon_2 n_1^2) - M_1(G_1 \vee G_2).$$

Proof. By Lemma **??** and Theorem **??**, we deduce that

$$LM_1(G_1 \vee G_2) = \sum_{(u,x)\in V(G_1\vee G_2)} (d_2((u,x)/G_1 \vee G_2))^2 = \sum_{(u,x)\in V(G_1\vee G_2)} (n_1n_2 - d((u,x)/G_1 \vee G_2) - 1)^2 = M_1(G_1 \vee G_2) + n_1n_2(n_1n_2-1)^2 - 4(n_1n_2-1)(\varepsilon_1 n_2^2 + \varepsilon_2 n_1^2).$$

$$LM_2(G_1 \vee G_2) = \sum_{(u,x)(v,y)\in E(G_1\vee G_2)} d_2((u,x)/G_1 \vee G_2)\, d_2((v,y)/G_1 \vee G_2) = \sum_{(u,x)(v,y)\in E(G_1\vee G_2)} (n_1n_2 - d((u,x)/G_1 \vee G_2) - 1)(n_1n_2 - d((v,y)/G_1 \vee G_2) - 1)$$
$$= M_2(G_1 \vee G_2) - (n_1n_2-1)M_1(G_1 \vee G_2) + (n_1n_2-1)^2(\varepsilon_1 n_2^2 + \varepsilon_2 n_1^2).$$

$$LM_3(G_1 \vee G_2) = \sum_{(u,x)\in V(G_1\vee G_2)} d((u,x)/G_1 \vee G_2)\, d_2((u,x)/G_1 \vee G_2) = \sum_{(u,x)\in V(G_1\vee G_2)} d((u,x)/G_1 \vee G_2)(n_1n_2 - d((u,x)/G_1 \vee G_2) - 1)$$
$$= 2(n_1n_2-1)(\varepsilon_1 n_2^2 + \varepsilon_2 n_1^2) - M_1(G_1 \vee G_2).$$

□

2.3 The symmetric difference of graphs

The symmetric difference $G_1 \oplus G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1 \oplus G_2) = V(G_1)\times V(G_2)$ and $(u,x)(v,y)$ is an edge of $G_1 \oplus G_2$ if $uv \in E(G_1)$ or $xy \in E(G_2)$ but not both. Obviously,

$$|E(G_1 \oplus G_2)| = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|^2 - 4|E(G_1)||E(G_2)|$$

.

Lemma 2.5. [?] Let G_i be a graph of order n_i for $i = 1,2$. Then $d_{G_1\oplus G_2}((u,x),(v,y)) = \begin{cases} 1 & \text{if } uv \in E(G_1) \text{ or } xy \in E(G_2) \\ & \text{but not both} \\ 2 & \text{otherwise.} \end{cases}$

By Theorem **??**, the following result is attained.

Lemma 2.6. Let G_i be a graph of order n_i for $i = 1,2$. Then $d_2((u,x)/G_1 \oplus G_2) = n_1n_2 - d((u,x)/G_1 \oplus G_2) - 1$.

Lemma 2.7. [?] Let G and H be graphs. Then

$$M_1(G \oplus H) = (|V(G)||V(H)|^2 - 8|E(H)||V(H)|)M_1(G) + 4M_1(G)M_1(H) + (|V(H)||V(G)|^2 - 8|E(G)||V(G)|)M_1(H) + 8|E(G)||E(H)||V(G)||V(H)|$$

Theorem 2.3. Let G_i be a graph of order n_i and size ε_i for $i = 1,2$. Then

$$(a)\ LM_1(G_1 \oplus G_2) =$$

$$M_1(G_1 \oplus G_2) + n_1n_2(n_1n_2 - 1)^2 -$$

$$4(n_1n_2 - 1)(\varepsilon_1 n_2^2 + \varepsilon_2 n_1^2 - 4\varepsilon_1\varepsilon_2).$$

$$(b)\ LM_2(G_1 \oplus G_2) =$$

$$M_2(G_1 \oplus G_2) - (n_1n_2 - 1)M_1(G_1 \oplus G_2) +$$

$$(n_1n_2 - 1)^2(\varepsilon_1 n_2^2 + \varepsilon_2 n_1^2 - 4\varepsilon_1\varepsilon_2).$$

$$(c)\ LM_3(G_1 \oplus G_2) =$$

$$2(n_1n_2 - 1)(\varepsilon_1 n_2^2 + \varepsilon_2 n_1^2 - 4\varepsilon_1\varepsilon_2)$$

$$-M_1(G_1 \oplus G_2).$$

Proof. The proof is similar to that described in the proof of Theorem **??** and so omitted. □

2.4 The composition of graphs

The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $(u, x)(v, y)$ is an edge of G if $(uv \in E(G_1))$ or $(xy \in E(G_2)$ and $u = v)$. Obviously, $|E(G)| = |E(G_1)||V(G_2)|^2 + |E(G_2)||V(G_1)|$.

Lemma 2.8. [?] Let G_1 and G_2 be graphs. If G_1 is connected, $|V(G_1)| > 1$ and $G = G_1[G_2]$, then for every vertex $(u, x), (v, y) \in V(G_1[G_2])$ we have: $d_{G_1[G_2]}((u, x), (v, y)) = \begin{cases} d_{G_1}(u, v) & \text{if } u \neq v \\ 1 & \text{if } u = v \text{ and } xy \in E_2 \\ 2 & \text{if } u = v \text{ and } xy \notin E_2. \end{cases}$

By Theorem **??**, the proof of the following Lemma is immediate, so omitted.

Lemma 2.9. Let G_i be a graph of order n_i for $i = 1, 2$. If G_1 is connected and $n_1 > 1$, then $d_2((u, x)/G_1[G_2]) = n_2 + n_2 d_2(u/G_1) - d(x/G_2) - 1$.

Theorem 2.4. Let G_i be a graph of order n_i and size ε_i for $i = 1, 2$. If G_1 is connected and $n_1 > 1$, then

$$(a)\ LM_1(G_1[G_2]) =$$

$$n_2^3 LM_1(G_1) + n_1n_2(n_2 - 1)^2$$

$$+ n_1M_1(G_2) - 4\varepsilon_2 n_1(n_2 - 1)$$

$$+ (2n_2^2(n_2 - 1) - 4n_2\varepsilon_2)D_2(G_1).$$

$$(b)\ LM_2(G_1[G_2]) =$$

$$\varepsilon_1 n_2^2(n_2 - 1)^2 - \varepsilon_1(n_2 - 1)$$

$$[2M_1(G_2) + 2\overline{M_1}(G_2) + 4\varepsilon_2]$$

$$+ n_2^3(n_2 - 1)LM_3(G_1)$$

$$+ \varepsilon_1[2M_2(G_2) + 2\overline{M_2}(G_2)$$

$$+ M_1(G_2)] + n_2^4 LM_2(G_1) + \varepsilon_2 n_1$$

$$(n_2 - 1)^2 - n_1(n_2 - 1)$$

$$M_1(G_2) + n_2^2\varepsilon_2 LM_1(G_1) + (2n_2\varepsilon_2(n_2 - 1)$$

$$- n_2M_1(G_2))D_2(G_1) + n_1M_2(G_2) -$$

$$n_2 \sum_{x,y \in V(G_2)} \sum_{uv \in E(G_1)} [d_2(u/G_1)d(y/G_2)$$

$$+ d_2(v/G_1)d(x/G_2)].$$

$$(c)\ LM_3(G_1[G_2]) =$$

$$n_2^3 LM_3(G_1) + 2n_2^3\varepsilon_1 -$$
$$4n_2\varepsilon_1\varepsilon_2 - 2n_2^2\varepsilon_1 - 2n_1\varepsilon_2 +$$
$$2n_1n_2\varepsilon_2 - n_1M_1(G_2) +$$
$$2n_2\varepsilon_2 D_2(G_1)$$

Proof. By Lemma **??** we have

$$LM_1(G_1[G_2]) = \sum_{(u,x)\in V(G_1[G_2])} d_2^2((u,x)$$

$$/G_1[G_2]) = \sum_{(u,x)\in V(G_1[G_2])} (n_2 + n_2 d_2(u/G_1)$$

$$- d(x/G_2) - 1)^2 = \sum_{u\in V(G_1)} \sum_{x\in V(G_2)} [(n_2-1)^2 - 2(n_2-1) d(x/G_2) + d(x/G_2)^2 + n_2^2 d_2(u/G_1)^2 + 2n_2(n_2-1)d_2(u/G_1) - 2n_2 d_2(u/G_1)$$

$$d(x/G_2)] = n_2^3 LM_1(G_1) + n_1 n_2 (n_2-1)^2 + n_1 M_1(G_2) - 4\varepsilon_2 n_1(n_2-1) + (2n_2^2(n_2-1) - 4n_2\varepsilon_2)D_2(G_1).$$

$$LM_2(G_1[G_2]) = \sum_{(u,x)(v,y)\in E(G_1[G_2])} d_2((u,x)/G_1[G_2]) d_2((v,y)/G_1[G_2])$$
$$= \sum_{x,y\in V(G_2)} \sum_{uv\in E(G_1)} (n_2 + n_2 d_2(u/G_1) - d(x/G_2) - 1)(n_2 + n_2 d_2(v/G_1)$$

$$- d(y/G_2) - 1) + \sum_{u=v\in V(G_1)} \sum_{xy\in E(G_2)} (n_2 + n_2 d_2(u/G_1) - d(x/G_2) - 1)$$

$$(n_2 + n_2 d_2(u/G_1) - d(y/G_2) - 1) = \sum_{x,y\in V(G_2)} \sum_{uv\in E(G_1)} [(n_2-1)^2 + n_2(n_2-1)(d_2(u/G_1) + d_2(v/G_1)) + n_2^2 d_2(u/G_1) d_2(v/G_1) - (n_2-1)(d(x/G_2) + d(y/G_2)) - n_2 d_2(u/G_1) d(y/G_2) -$$

$$n_2 d_2(v/G_1) d(x/G_2) + d(x/G_2)d(y/G_2)] + \sum_{u=v\in V(G_1)} \sum_{xy\in E(G_2)} [(n_2-1)^2 + 2n_2(n_2-1)d_2(u/G_1) + n_2^2 d_2(u/G_1)^2 -$$
$$(n_2-1)(d(x/G_2) + d(y/G_2)) - n_2 d_2(u/G_1)(d(x/G_2) + d(y/G_2)) + d(x/G_2)d(y/G_2)]$$
$$= \varepsilon_1 n_2^2 (n_2-1)^2 + n_2^3 (n_2-1) LM_3(G_1) - \varepsilon_1 (n_2-1)[2M_1(G_2) + 2\overline{M_1}(G_2) + 4\varepsilon_2] + n_2^4 LM_2(G_1) - n_2 \sum_{x,y\in V(G_2)} \sum_{uv\in E(G_1)} [d_2(u/G_1)d(y/G_2) + d_2(v/G_1)d(x/G_2)] + \varepsilon_1[2M_2(G_2) + 2\overline{M_2}(G_2) + M_1(G_2)] + \varepsilon_2 n_1 (n_2-1)^2 + (2n_2\varepsilon_2(n_2-1) - n_2 M_1(G_2))D_2(G_1) + n_2^2 \varepsilon_2 LM_1(G_1) - n_1(n_2-1)M_1(G_2) + n_1 M_2(G_2).$$

$$LM_3(G_1[G_2]) = \sum_{(u,x)\in V(G_1[G_2])} d((u,x)/G_1[G_2]) d_2((u,x)/G_1[G_2]) = \sum_{(u,x)\in V(G_1[G_2])} (n_2 d(u/G_1) + d(x/G_2))(n_2 + n_2 d_2(u/G_1) - d(x/G_2) - 1)$$
$$= \sum_{u\in V(G_1)} \sum_{x\in V(G_2)} [n_2^2 d(u/G_1) d_2(u/G_1) + n_2(n_2-1)d(u/G_1) - n_2 d(x/G_2) d(u/G_1) +$$

$$n_2 d(x/G_2) d_2(u/G_1) + (n_2-1)d(x/G_2) - d(x/G_2)^2] = n_2^3 LM_3(G_1) + 2n_2^3\varepsilon_1 - 4n_2\varepsilon_1\varepsilon_2 - 2n_2^2\varepsilon_1 - 2n_1\varepsilon_2 +$$

$$2n_1 n_2 \varepsilon_2 - n_1 M_1(G_2) + 2n_2\varepsilon_2 D_2(G_1).$$

□

2.5 The Cartesian product of graphs

The cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and $(u,x)(v,y)$ is an edge of $G_1 \times G_2$ if $uv \in E(G_1)$ and $x = y$, or $u = v$ and $xy \in E(G_2)$. Obviously, $|E(G_1 \times G_2)| = |E(G_1)||V(G_2)| + |V(G_1)||E(G_2)|$.

Lemma 2.10. [?] Let G_i be a graph of order n_i for $i = 1, 2$. Then

$$d_{G_1\times G_2}((u,x),(v,y)) = d_{G_1}(u,v) + d_{G_2}(x,y).$$

The proof of the following Lemma is immediate, so omitted.

Lemma 2.11. Let G_i be a graph of order n_i for $i = 1, 2$. Then $d_2((u, x)/G_1 \times G_2) = d_2(u/G_1) + d_2(x/G_2) + d(u/G_1)d(x/G_2)$.

Theorem 2.5. Let G_i be a graph of order n_i and size ε_i for $i = 1, 2$. Then

$$(a)\ LM_1(G_1 \times G_2) = n_2LM_1(G_1) + n_1LM_1(G_2) + M_1(G_1)M_1(G_2) + 4\varepsilon_2LM_3(G_1) + 4\varepsilon_1LM_3(G_2) + 2D_2(G_1)D_2(G_2).$$

$$(b)\ LM_2(G_1 \times G_2) = n_2LM_2(G_1) + \varepsilon_1LM_1(G_2) + M_1(G_1) LM_3(G_2) + M_2(G_1)M_1(G_2) + LM_3(G_1)D_2(G_2) + LM_3(G_2)D_2(G_1) + n_1LM_2(G_2) + \varepsilon_2LM_1(G_1) + M_1(G_2) LM_3(G_1) + M_2(G_2)M_1(G_1) + 2\varepsilon_2 \sum_{uv\in E(G_1)}[d_2(u/G_1)d(v/G_1) + d_2(v/G_1)d(u/G_1)] + 2\varepsilon_1 \sum_{xy\in E(G_2)}[d_2(x/G_1)d(y/G_1) + d_2(y/G_1)d(x/G_1)].$$

$$(c)\ LM_3(G_1 \times G_2) = n_2LM_3(G_1) + n_1LM_3(G_2) + 2\varepsilon_1M_1(G_2) + 2\varepsilon_2M_1(G_1)\ 2\varepsilon_2D_2(G_1) + 2\varepsilon_1D_2(G_2).$$

Proof. By Lemma **??** we have

$$LM_1(G_1 \times G_2) = \sum_{(u,x)\in V(G_1\times G_2)} (d_2((u, x)/G_1 \times G_2))^2 = \sum_{(u,x)\in V(G_1\times G_2)} (d_2(u/G_1) + d_2(x/G_2) + d(u/G_1)d(x/G_2))^2$$
$$= \sum_{u\in V(G_1)} \sum_{x\in V(G_2)} [d_2(u/G_1)^2 + d_2(x/G_2)^2 + d(u/G_1)^2d(x/G_2)^2 + 2d_2(u/G_1)d_2(x/G_2) + 2d_2(u/G_1)d(u/G_1)d(x/G_2) + 2d_2(x/G_2)d(u/G_1)d(x/G_2)]$$
$$= n_2LM_1(G_1) + n_1LM_1(G_2) + M_1(G_1)M_1(G_2) + 2D_2(G_1)D_2(G_2) + 4\varepsilon_2LM_3(G_1) + 4\varepsilon_1LM_3(G_2).$$

$$LM_2(G_1 \times G_2) = \sum_{(u,x)(v,y)\in E(G_1\times G_2)} d_2((u, x)/G_1 \times G_2)d_2((v, y)/G_1 \times G_2)$$
$$= \sum_{x=y\in V(G_2)} \sum_{uv\in E(G_1)} [(d_2(u/G_1) + d_2(x/G_2) + d(u/G_1)d(x/G_2)) (d_2(v/G_1) + d_2(x/G_2) + d(v/G_1)d(x/G_2))$$
$$+ \sum_{u=v\in V(G_1)} \sum_{xy\in E(G_2)} [(d_2(u/G_1) + d_2(x/G_2) + d(u/G_1)d(x/G_2)) (d_2(u/G_1) + d_2(y/G_2) + d(u/G_1)d(y/G_2))]$$
$$= \sum_{x=y\in V(G_2)} \sum_{uv\in E(G_1)} [d_2(u/G_1)d_2(v/G_1) + d_2(x/G_2)(d_2(u/G_1) + d_2(v/G_1)) + d(x/G_2)(d_2(u/G_1)d(v/G_1) + d_2(v/G_1) d(u/G_1)) + d_2(x/G_2)^2 + d(x/G_2)d_2(x/G_2)(d(u/G_1) + d(v/G_1)) + d(x/G_2)^2d(u/G_1)d(v/G_1)] +$$
$$\sum_{u=v\in V(G_1)} \sum_{xy\in E(G_2)} [d_2(x/G_2)d_2(y/G_2) + d_2(u/G_2)(d_2(x/G_2) + d_2(y/G_2)) + d(u/G_1)(d_2(x/G_2)d(y/G_2) + d_2(y/G_2) d(x/G_2)) + d_2(u/G_1)^2 + d(u/G_1) d_2(u/G_1)(d(x/G_2) + d(y/G_2)) + d(u/G_1)^2 d(x/G_2)d(y/G_2)] = n_2LM_2(G_1) + LM_3(G_1)D_2(G_2) + \varepsilon_1LM_1(G_2) + M_1(G_1)LM_3(G_2) + M_2(G_1)M_1(G_2) + 2\varepsilon_2 \sum_{uv\in E(G_1)}[d_2(u/G_1)d(v/G_1) + d_2(v/G_1) d(u/G_1)] + n_1LM_2(G_2) + LM_3(G_2)D_2(G_1) + \varepsilon_2LM_1(G_1) + M_1(G_2)LM_3(G_1) + M_2(G_2) M_1(G_1) + 2\varepsilon_1 \sum_{xy\in E(G_2)}[d_2(x/G_1)d(y/G_1) + d_2(y/G_1)d(x/G_1)].$$

$$LM_3(G_1 \times G_2) = \sum_{(u,x)\in V(G_1\times G_2)} d((u, x)/G_1 \times G_2)d_2((u, x) /G_1 \times G_2) = \sum_{(u,x)\in V(G_1\times G_2)} (d(u/G_1) + d(x/G_2))(d_2(u/G_1) + d_2(x/G_2) + d(u/G_1)d(x/G_2)) = \sum_{u\in V(G_1)} \sum_{x\in V(G_2)} [d_2(u/G_1) d(u/G_1) + d(u/G_1)d_2(x/G_2) + d(u/G_1)^2 d(x/G_2) + d(x/G_2)d_2(u/G_1) + d_2(x/G_2) d(x/G_2) + d(u/G_1)d(x/G_2)^2]$$
$$= n_2LM_3(G_1) + 2\varepsilon_1D_2(G_2) + 2\varepsilon_2M_1(G_1) + 2\varepsilon_2D_2(G_1) + n_1LM_3(G_2) + 2\varepsilon_1M_1(G_2).$$

□

3 Mycielskian graphs

For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with the disjoint union $V \cup X \cup \{x\}$ as its vertex set and $E \cup \{v_i x_j \mid v_i v_j \in E\} \cup \{x x_j \mid 1 \le j \le n\}$ as its edge set, where $V = \{v_1, v_2, \ldots, v_n\}$ and $X = \{x_1, x_2, \ldots, x_n\}$.

We will use the following results.

Lemma 3.1. [?] Let $\mu(G)$ be the Mycieliskian of G. Then for each $v \in V(\mu)$,

$$d(v/\mu(G)) = \begin{cases} n & \text{if } v = x \\ 1 + \deg_G(v_i) & \text{if } v = x_i \\ 2d(v_i/G). & \text{otherwise.} \end{cases}$$

Lemma 3.2. [?] In the Mycieliskian $\mu(G)$ of G, the distance between two vertices $u, v \in V(\mu(G))$ are given as follows,

$$d_{\mu(G)}(u, v) = \begin{cases} 1 & \text{if } u = x, v = x_i \\ 2 & \text{if } u = x, v = v_i \\ 2 & \text{if } u = x_i, v = x_j \\ d_G(v_i, v_j) & \text{if } u = v_i, v = v_j, \\ & d_G(v_i, v_j) \le 3 \\ 4 & \text{if } u = v_i, v = v_j, \\ & d_G(v_i, v_j) \ge 4 \\ 2 & \text{if } u = v_i, v = x_j, i = j \\ d_G(v_i, v_j) & \text{if } u = v_i, v = x_j, i \ne j, \\ & d_G(v_i, v_j) \le 2 \\ 3 & \text{if } u = v_i, v = x_j, i \ne j, \\ & d_G(v_i, v_j) \ge 3. \end{cases}$$

Specially, the diameter of the Mycialiskian graph is at most four.

The proof of the following result is obvious, so omitted.

Lemma 3.3. Let G be a graph of order n. Then

$$d_2(u/\mu(G)) = \begin{cases} n & \text{if } u = x \\ d_2(v_i/G) + n - 1 & \text{if } u = x_i \\ 2d_2(v_i/G) + 2 & \text{if } u = v_i. \end{cases}$$

Theorem 3.1. Let G be a graph of order n and size ε. Then

$$(a)\ LM_1(\mu(G)) = 5LM_1(G) + n^3 - n^2 + 5n + 2(n+3)D_2(G).$$

$$(b)\ LM_2(\mu(G)) = 6LM_2(G) + 6LM_3(G) + 4\varepsilon + 2(n-1)\varepsilon + n^3 - n^2 + nD_2(G) + 2(n-2)\textstyle\sum_{v_iv_j \in E(G)} d_2(v_i/G).$$

$$(c)\ LM_3(\mu(G)) = 5LM_3(G) + 2(n-1)\varepsilon + 8\varepsilon + 2n^2 - n + D_2(G).$$

Proof. By Lemma **??** we have

$$LM_1(\mu(G)) = \sum_{u \in V(\mu(G))} (d_2(u/\mu(G)))^2$$

$$= n^2 + \sum_{i=1}^{n} (d_2(v_i/\mu(G)))^2 + \sum_{i=1}^{n} (d_2(x_i/\mu(G)))^2$$

$$= n^2 + \sum_{i=1}^{n} (2d_2(v_i/G) + 2)^2 + \sum_{i=1}^{n} (d_2(v_i/G) + n - 1)^2$$

$$= 5LM_1(G) + n^3 - n^2 + 5n + 2(n+3)D_2(G).$$

$$LM_2(\mu(G)) = \sum_{uv\in E(\mu(G))} d_2(u/\mu(G))$$
$$d_2(v/\mu(G))$$
$$= \sum_{xx_i\in E(\mu(G))} d_2(x/\mu(G))d_2(x_i/\mu(G))$$
$$+ \sum_{v_iv_j\in E(\mu(G))} d_2(v_i/\mu(G))d_2(v_j/\mu(G))$$
$$+ \sum_{v_ix_j\in E(\mu(G))} d_2(v_i/\mu(G))d_2(x_j/\mu(G))$$
$$= \sum_{i=1}^{n} n(d_2(v_i/G) + n - 1)$$
$$+ \sum_{v_iv_j\in E(G)} (2d_2(v_i/G) + 2)(2d_2(v_j/G)$$
$$+ 2) + \sum_{v_iv_j\in E(G)} (2d_2(v_i/G) + 2)(d_2(v_j/G)$$
$$+ n - 1)$$
$$= 6LM_2(G) + 6LM_3(G) + 4\varepsilon + 2(n-1)\varepsilon$$
$$+ n^3 - n^2 + nD_2(G) + 2(n-2)$$
$$\sum_{v_iv_j\in E(G)} d_2(v_i/G).$$

$$LM_3(\mu(G)) = \sum_{u\in V(\mu(G))} d(u/\mu(G))d_2(u/\mu(G))$$
$$= n^2 + \sum_{i=1}^{n} d(v_i/\mu(G))d_2(v_i/\mu(G))$$
$$+ \sum_{i=1}^{n} d(x_i/\mu(G))d_2(x_i/\mu(G))$$
$$= n^2 + \sum_{i=1}^{n} 2d(v_i/G)(2d_2(v_i/G) + 2)$$
$$+ \sum_{i=1}^{n} (d(v_i/G) + 1)(d_2(v_i/G) + n - 1)$$
$$= 5LM_3(G) + 2(n-1)\varepsilon + 8\varepsilon + 2n^2$$
$$- n + D_2(G).$$

□

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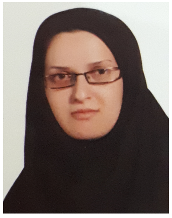

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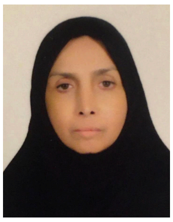

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