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Research Article

Solving Fully Interval Linear Programming Problems Using Ranking Interval Numbers

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Abstract

In most of real world problems data suffer from inaccuracy, thus considering an interval for data perturbation seems to be a good idea. There exist many papers in literature about Interval linear programming problems (ILP) dealing with non negative variables. Here the general form of an ILP is considered where all the parameters and variables are considered to be intervals. Moreover, in this study more general conditions for variables are considered, variables which are unrestricted in sign. Although this is the case in most of the real world problems, due to high complexity it has not been dealt with in literature yet. In this paper a new method is presented in order to obtain fully interval linear programming problems ($FILP_P$) using a nonlinear programming problem (NLP). Furthermore, in order to demonstrate how the proposed method works two numerical examples are illustrated.

Keywords : Interval linear system; Interval number vector; Linear programming; ranking interval.

1 Introduction

Due to inexactness in data in real world problems system of interval linear programming (ILP) problems are gained significant attentions. Thus many researchers presented models and methods for dealing with this kind of problems. In their paper, Zhou et al. [27], provided an enhanced interval programming problems. Considering this method, stated in Zhou et al. [27] about the advantage of this model, corresponding solution space is absolutely feasible while being compared to the solution of ILP. As advantage of this method is that while it is compared to ILP, corresponding degree of uncertainty is much lower than that of ILP model considering the feasible solution space. As stated in Lodwick and Jamison, [15], they proposed a method, in the computed result, automatical controlling the errors interval analysis is introduced and utilized. This method used for solving LP problems with interval coefficient. Note that it incorporates uncertainty of numbers with out any assumption about the distributions. Some examples of this method are as follows: Huang and Moore, 1993 [11],Chinneck and Ramadan, [4]; Sengupta et al., [20]. Specifically, for solving a special LP model with limited constraints, while the upper and lower bounds are at hand, a basic model provided

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by Ben-Israel and Robers [3]. This method enhanced by Rommelfanger et al. [19] and Inuiguchi and Sakawa [13] with an interval objective function while having upper and lower bounds. Under such circumstances a new LP model and a new BWS method are also provided by Huang and Moore [11] and Tong [25], respectively.
Hladik [10] presented a new approach in which an exact range for optimal value function, while considering any linear interval system, is computed. As an advantage of the presented method it can be mentioned that the constraint matrix coefficients are dependant to each other. Moreover as Hladik [10] stated, relations between the primal and dual solution sets can be characterized. In regards of inexactness inherent in real world problems, Hladik [9], considering interval data, generalized linear fractional programming problems. In this generalized approach, a new method presented for computing the range of optimal values. One important key feature of this method is that, while an ideal interior point solver method is accounted for, computation done in polynomial time. As regards of recent researches, multiobjective linear programming (MOLP) problems gained specific attentions. This subject discussed with interval data in Oliveira, C., Antunes, C.H. [17] differently. In a paper provided by Oliveira, C.,Antunes, C.H. [18], mentioned condition is taken into account merely in objective functions coefficients. Noted that a significant issue in MOLP is necessary efficiency, for all realizations of interval data, showing that a feasible point is efficient, as stated in Ida, M. [12], Hladik [8]. In 2011, Hladik [7] for checking necessary efficiency while an interval multiobjective linear programming is considered, some complexity results presented. Wu [26] presented a paper discussed weak and strong duality theorems in presence of interval data are considered. This issue is proposed in Soyster [23], [22], Thuente [24]. The provided approach in Wu [26] is based upon the concept of an inner product of closed intervals. This approach is some how relates to MOP problems. In a paper Sengupta and Pal [21] considered two interval numbers on the real line and presented a survey of the existing works for comparing and ranking such intervals. Then, they proposed two approaches for ranking. One describes a value judgment index and the other defines strict and fuzzy preference ordering in regards of a pessimistic decision maker's viewpoint. Li et al. [2] in their paper, considered linear programming problems with interval right-hand side and checked weak optimality with the necessary and sufficient conditions. Hladk [14] presented a robust optimal solutions in interval linear programming. Ashayerinasab et al. [6]2018 considered interval linear equations system and presented a new algorithm for solving an interval LP model.
In this paper a new method defined which acquires algebraic solutions of fully interval linear programming problems ($FILP_P$) using a nonlinear programming problem(NLP). Novotn et al. [16] discussed about the issue of duality gap exists in the interval linear programming. As an important issue they talked about the specifications of strongly and weakly zero duality gap in interval linear programming. They also paid attention to the important case in which there exists the real matrix of coefficients. Garajov et al. [5] in their paper discussed the transformations which are usually utilized in linear programming. They talked about imposing non-negativity of free variables, splitting equations into inequalities, and their effects on interval programs. They noted that some of the mostly used properties do not holds in the general case. Thus, in their paper they taken into account a special class of interval programs, where uncertainty exists in objective function and the right-hand-side vectors.
This paper unfolds as follows: First some preliminaries and basic definition will be briefly reviewed. Then, in section 3, main subject, ranking interval numbers and the new for finding interval optimal solution of FILP problems presented. Section 4 conclude the paper.

2 Preliminaries

In this section, some necessary backgrounds and notions of interval numbers theory and definition are briefly reviewed.

2.1 Basic definitions

Definition 2.1. *An interval number* $[x]$ *is defined as the set of real numbers such that* $[x] =$

$[\underline{x}, \overline{x}] = \{x' \in \mathbb{R} : \underline{x} \leqslant x' \leqslant \overline{x}\}$ where $\underline{x} \leqslant \overline{x}$. We denote the set of all interval numbers by $\mathbb{I}$.

Definition 2.2. *A vector $[X] = ([x_1], [x_2], \ldots, [x_n])^T$, where $[x_i] = [\underline{x_i}, \overline{x_i}]$, $1 \leqslant i \leqslant n$ are called an interval number vectors. In this case, we denote $[X] \in \mathbb{I}^n$.*

Definition 2.3. *Let $[x] = [\underline{x}, \overline{x}]$ and $[y] = [\underline{y}, \overline{y}]$ be two interval numbers, then*

$$[\underline{x}, \overline{x}] \oplus [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$

$$[\underline{x}, \overline{x}] \ominus [\underline{y}, \overline{y}] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}],$$

$$[\underline{x}, \overline{x}] \otimes [\underline{y}, \overline{y}] = [\underline{c}, \overline{c}]$$

$$\begin{cases} \underline{c} = \min\{\underline{x}.\underline{y},\ \overline{x}.\underline{y},\ \underline{x}.\overline{y},\ \overline{x}.\overline{y}\} \\ \overline{c} = \max\{\underline{x}.\underline{y},\ \overline{x}.\underline{y},\ \underline{x}.\overline{y},\ \overline{x}.\overline{y}\}, \end{cases}$$

$$[\underline{x}, \overline{x}] \oslash [\underline{y}, \overline{y}] = [\underline{x}, \overline{x}] \otimes [1/\overline{y}, 1/\underline{y}], \overline{y},\ \underline{y} \neq 0,$$

$$k.[\underline{x}, \overline{x}] = \begin{cases} [k\underline{x}, k\overline{x}] & k \geqslant 0 \\ [k\overline{x}, k\underline{x}] & k < 0 \end{cases} \quad k \in \mathbb{R}.$$

Definition 2.4. *The width of an interval number $[x]$ is defined as follows:*

$$W([x]) = \overline{x} - \underline{x}.$$

2.2 Nonlinear Partition Interval

Definition 2.5. *[2] Consider the* **nonlinear partition interval** *of an interval number $[x] = [\underline{x}, \overline{x}]$ that $\underline{x} \geqslant 0$ or $\overline{x} \leqslant 0$ as follows:*

$$\begin{cases} [x] = [\underline{x}, \overline{x}] = [x_1, x_1'] \oplus [x_2', x_2] \\ x_1 \leqslant x_1' \leqslant 0 \leqslant x_2' \leqslant x_2, \\ \lambda_1.x_1 + \lambda_2.x_2 \ = \ 0, \\ \lambda_1 + \lambda_2 \ = \ 1, \quad \lambda_1, \lambda_2 \in \{0, 1\}. \end{cases} \tag{2.1}$$

Example 2.1. *Consider the following interval numbers*

$$[x] = [-6, -2], \qquad [y] = [3, 7].$$

Using Eq. (2.1), the following nonlinear nartition interval can be presented:

$$\begin{cases} [-6, -2] & = & [-6, -2] & \oplus & [0, 0], \\ [3, 7] & = & [0, 0] & \oplus & [3, 7]. \end{cases} \tag{2.2}$$

3 Main subject

As mentioned above this paper mainly deals with solving fully interval linear systems. In doing so a new method for ranking interval numbers will be proposed, then in accordance with that, Standard Interval Splitting (SIS) will also be defined. As regards, it is possible to solve fully interval linear systems. One great feature of the proposed method is that it helps solving fully interval linear problems. For checking validity of the proposed method some examples are provided.

3.1 New approach for ranking of interval numbers

In this subsection a new index of interval number has been defined then this index use for ranking interval numbers. The proposed index has some novel features in comparison to those existing methods for ranking interval numbers that will be explained in following.

Definition 3.1. *For an arbitrary interval number $[x] = [\underline{x}, \overline{x}]$, we define the ranking function index $Ind() : \mathbb{I} \to \mathbb{R}$ of the interval number $[X]$ as*

$$Ind([x]) = \sum_{i=0}^{n} w_i x_i, \tag{3.3}$$

where:

$$\begin{cases} x_i = i\left(\frac{\overline{x}-\underline{x}}{n}\right) + \underline{x} \quad i = 0, 1, \ldots, n, \\ \sum_{i=0}^{n} w_i = 1 \\ 0 < w_1 < w_2 < \ldots < w_n \leqslant 1, \end{cases} \tag{3.4}$$

In the above definition, (3.1), $w_i(i = 0, 1, \ldots, n)$ are the operator weights for determining $Ind([x])$. The proposed approach allows the operator weights with different values to be determined in terms of decision maker.

As explained above the $Ind([x])$ synthetically reflects the information on interval number $[x]$. Noted that the larger the magnitude of n is, the more number points of interval for determining $Ind([x])$ will be defined. Meaning of this magnitude is visual and natural. The resulted scalar value is then used to rank the interval numbers. In accordance to what has been proposed above, the larger $Ind([x])$ is, the larger interval number will be.

Definition 3.2. *For any two interval numbers $[x]$ and $[y]$, ranking of $[x]$ and $[y]$ by $Ind([x])$ on $\mathbb{I}$, with fixed operator weights $w_i (i = 0, 1, \ldots, n)$, will be defined as follows:*

1) $Ind([x]) > Ind([y])$ *if and only if* $[x] \succ [y]$.

2) $Ind([x]) < Ind([y])$ *if and only if* $[x] \prec [y]$.

3) $Ind([x]) = Ind([y])$ *if and only if* $[x] \backsim [y]$.

Then the order $\succcurlyeq$ and $\preccurlyeq$ will be formulated as $[x] \succcurlyeq [y]$ if and only if $[x] \succ [y]$ or $[x] \backsim [y]$, and $[x] \preccurlyeq [y]$ if and only if $[x] \prec [y]$ or $[x] \backsim [y]$. Obviously, $[x] \preccurlyeq [y]$ if and only if $Ind([x]) \leqslant Ind([y])$.

Using ranking function $Ind()$, we define an equivalence class for interval number $[x]$ as following:

$$\begin{aligned} &Class([x]) = \{[y] | \textstyle\sum_{i=0}^n w_i(x_i - y_i) = 0\} \\ &= \{[y] | (\underline{x} - \underline{y}) \textstyle\sum_{i=0}^n i w_i \\ &\quad + (\overline{x} - \overline{y}) \textstyle\sum_{i=0}^n (n-i) w_i = 0\}. \end{aligned} \tag{3.5}$$

Obvious, if $[t] \in Class([x])$ then $[t] \backsim [x]$.

Theorem 3.1. *For any $[x], [y], [z] \in \mathbb{I}$ and for the fixed operator weights, $w_i (i = 0, 1, \ldots, n)$, the index $Ind(.)$ defined in Definition 3.1, satisfies in the following properties:*

1. *If $[x]$ is crisp numbers, i.e., $[x] = [k, k]$ then $Ind([x]) = k$.*
2. *$Ind([x]) \in [x]$.*
3. *$Ind(k[x]) = k.Ind([x]) \quad k \in \mathbb{R}^+$.*
4. *$Ind([x] \oplus [y]) = Ind([x]) + Ind([y])$*
5. *If $\underline{x} \geqslant 0$ then $Ind([x]) \geqslant 0$ (if $\overline{x} \leqslant 0$ then $Ind([x]) \leqslant 0$)*

Proof. **1:** Let $[x]$ be crisp numbers, i.e., $[x] = [k, k]$ then $x_i = k \quad i = 0, 1, \ldots, n$. Hence

$$Ind([x]) = \sum_{i=0}^n w_i x_i = \sum_{i=0}^n w_i k = k \sum_{i=0}^n w_i = k$$

2: Using Definition 3.2, we have:

$$x_i = i\left(\frac{\overline{x} - \underline{x}}{n}\right) + \underline{x} = \frac{i}{n}\overline{x} + \frac{n-i}{n}\underline{x}$$

$$\Rightarrow \overline{x} \geqslant x_i \geqslant \underline{x}, \quad (i = 0, 1, \ldots, n)$$

$$\Rightarrow \sum_{i=0}^n w_i \overline{x} \geqslant \sum_{i=0}^n w_i x_i \geqslant \sum_{i=0}^n w_i \underline{x}$$

$$\Rightarrow \overline{x} \geqslant Ind([x]) \geqslant \underline{x} \Rightarrow Ind([x]) \in [x]$$

3: suppose $k \geqslant 0$. Then $k[x] = [k\underline{x}, k\overline{x}]$ Hence, we have:

$$Ind(k[x]) = \sum_{i=0}^n w_i \left(i \left(\frac{k\overline{x} - k\underline{x}}{n} \right) + k\underline{x} \right)$$

$$= \sum_{i=0}^n w_i k x_i = k \sum_{i=0}^n w_i x_i = k.Ind([x])$$

4: Using Definition 2.3, $[x] \oplus [y] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$. Hence, we have:

$$\begin{aligned} &Ind([x] \oplus [y]) = \\ &\textstyle\sum_{i=0}^n w_i \left(i \left(\frac{\overline{x} + \overline{y} - \underline{x} - \underline{y}}{n} \right) + \underline{x} + \underline{y} \right) = \\ &\textstyle\sum_{i=0}^n w_i ((i(\frac{\overline{x} - \underline{x}}{n}) + \underline{x}) + (i(\frac{\overline{y} - \underline{y}}{n}) + \underline{y})) \\ &= \textstyle\sum_{i=0}^n w_i \left(\left(i \left(\frac{\overline{x} - \underline{x}}{n} \right) + \underline{x} \right) \right) + \\ &\textstyle\sum_{i=0}^n w_i \left(\left(i \left(\frac{\overline{y} - \underline{y}}{n} \right) + \underline{y} \right) \right) \\ &= Ind([x]) + Ind([y]) \end{aligned}$$

5: Let $\underline{x} \geqslant 0$. Hence,

$$x_i = \frac{i}{n}\overline{x} + \frac{n-i}{n}\underline{x} \geqslant 0 \Rightarrow$$

$$Ind([x]) = \sum_{i=0}^n w_i x_i \geqslant 0 \quad (i = 0, 1, \ldots, n)$$

□

Theorem 3.2. *The ordering approaches defined formerly have the reasonable properties as following:*

1. *$[x] \preccurlyeq [x]$.*
2. *$[x] \preccurlyeq [y]$ and $[y] \preccurlyeq [x]$ if and only if $[x] \backsim [y]$.*
3. *$[x] \preccurlyeq [y]$ and $[y] \preccurlyeq [z]$ if and only if $[x] \preccurlyeq [z]$.*
4. *For any $[z] \in \mathbb{I}$ if $[x] \preccurlyeq [y]$ then $[x] \oplus [z] \preccurlyeq [y] \oplus [z]$.*
5. *If $[x] \preccurlyeq [y]$ and $\underline{z} \geqslant 0$ and $\underline{x}, \underline{y} \geqslant 0 \quad (\overline{x}, \overline{y} \leqslant 0)$ then $[x] \otimes [z] \preccurlyeq [y] \otimes [z]$.*

6. *If* $\underline{x} \geqslant \overline{y}$ *then* $[x] \succcurlyeq [y]$ *(If* $\overline{x} \leqslant \underline{y}$ *then* $[x] \preccurlyeq [y]$*).*

7. *If* $\overline{x} = \overline{y}$ *(or* $\overline{x} > \overline{y}$*) and* $\underline{x} \geqslant \underline{y}$ *then* $[x] \succcurlyeq [y]$

8. *If* $\underline{x} = \underline{y}$ *(or* $\underline{x} > \underline{y}$*) and* $\overline{x} \geqslant \overline{y}$ *then* $[x] \succcurlyeq [y]$

Proof. **1:** Using Definition 3.2, the proof is obvious.
2: Using Definition 3.2, we have:

$$\begin{cases} [x] \preccurlyeq [y] \Leftrightarrow Ind([x]) \leqslant Ind([y]) \\ [y] \preccurlyeq [x] \Leftrightarrow Ind([y]) \leqslant Ind([x]) \end{cases}$$
$$\Leftrightarrow Ind([y]) = Ind([x]) \Leftrightarrow [x] \backsim [y]$$

3: Using Definition 3.2, we have:

$$\begin{cases} [x] \preccurlyeq [y] \Leftrightarrow Ind([x]) \leqslant Ind([y]) \\ [y] \preccurlyeq [z] \Leftrightarrow Ind([y]) \leqslant Ind([z]) \end{cases}$$
$$\Leftrightarrow Ind([x]) \leqslant Ind([z]) \Leftrightarrow [x] \preccurlyeq [z]$$

4: Let $[z] \in \mathbb{I}$ and $[x] \preccurlyeq [y]$. Using Theorem 3.1, we have:

$$[x] \preccurlyeq [y] \quad \Leftrightarrow Ind([x]) \leqslant Ind([y])$$
$$\Leftrightarrow Ind([x]) + Ind([z]) \leqslant Ind([y]) + Ind([z])$$
$$\Leftrightarrow Ind([x] + [z]) \leqslant Ind([y] + [z])$$
$$\Leftrightarrow [x] \oplus [z] \preccurlyeq [y] \oplus [z]$$

5: Let $[x] \preccurlyeq [y]$ and $\underline{z} \geqslant 0$, we consider two following cases.

Case A**:** Firstly, suppose $\underline{x}, \underline{y} \geqslant 0$.
By contradiction assume that:

$[x] \otimes [z] \succ [y] \otimes [z] \iff Ind([x] \otimes [z]) = \sum_{i=0}^{n} w_i \left(\frac{i}{n}\overline{x}.\overline{z} + \frac{n-i}{n}\underline{x}.\underline{z}\right) > Ind([y] \otimes [z]) = \sum_{i=0}^{n} w_i \left(\frac{i}{n}\overline{y}.\overline{z} + \frac{n-i}{n}\underline{y}.\underline{z}\right) \implies \sum_{i=0}^{n} w_i(\overline{z}\frac{i}{n}(\overline{x} - \overline{y}) + \underline{z}\frac{n-i}{n}(\underline{x} - \underline{y})) > 0$
$\implies \sum_{i=0}^{n} w_i(\overline{z}\frac{i}{n}(\overline{x} - \overline{y}) + \overline{z}\frac{n-i}{n}(\underline{x} - \underline{y})) > 0$
$\implies \sum_{i=0}^{n} w_i \left(\frac{i}{n}(\overline{x} - \overline{y}) + \frac{n-i}{n}(\underline{x} - \underline{y})\right) > 0$
$\sum_{i=0}^{n} w_i \left(\frac{i}{n}\overline{x} + \frac{n-i}{n}\underline{x}\right) > \sum_{i=0}^{n} w_i(\frac{i}{n}\overline{y} + \frac{n-i}{n}\underline{y})$
$\iff Ind([x]) \succ Ind([y])$.□

Therefore it is concluded that it is incoherence, and the proof is down.
Case B: Now, suppose $\overline{x}, \overline{y} \leqslant 0$. The proof of this case is similar to that of Case A.

6: Let $\underline{x} \geqslant \overline{y}$ then: $\underline{x} \geqslant \underline{y}$ and $\overline{x} \geqslant \overline{y}$,

$\Rightarrow x_i = \frac{i}{n}\overline{x} + \frac{n-i}{n}\underline{x} \geqslant y_i = \frac{i}{n}\overline{y} + \frac{n-i}{n}\underline{y}$
$\Rightarrow w_i.x_i \geqslant w_i.y_i \quad i = 0, \ldots, n$
$\Rightarrow \sum_{i=0}^{n} w_i.x_i \geqslant \sum_{i=0}^{n} w_i.y_i$
$\Rightarrow Ind([x]) \geqslant Ind([y]) \quad \Rightarrow [x] \succcurlyeq [y]$

7: Let $\overline{x} = \overline{y}$ (or $\overline{x} > \overline{y}$) and $\underline{x} \geqslant \underline{y}$. Hence, we have

$$\Rightarrow x_i = \tfrac{i}{n}\overline{x} + \tfrac{n-i}{n}\underline{x} \geqslant y_i = \tfrac{i}{n}\overline{y} + \tfrac{n-i}{n}\underline{y}$$
$$\Rightarrow w_i.x_i \geqslant w_i.y_i \quad i = 0, \ldots, n$$
$$\Rightarrow \textstyle\sum_{i=0}^{n} w_i.x_i \geqslant \sum_{i=0}^{n} w_i.y_i$$
$$\Rightarrow Ind([x]) \geqslant Ind([y]) \quad \Rightarrow [x] \succcurlyeq [y]$$

8: Now suppose $\overline{x} \geqslant \overline{y}$ and $\underline{x} = \underline{y}$ (or $\overline{x} > \overline{y}$.) The proof of this case is similar to that of Case 7. □

Example 3.1. *Consider the following interval* $[X] = [2, 6]$*. Let* $n = 8$ *therefore, using Definition.(3.1) choice points on interval are as following:*
$x_0 = 2, x_1 = 2.5, x_2 = 3, x_3 = 3.5, x_4 = 4, x_5 = 4.5, x_6 = 5, x_7 = 5.5, x_8 = 6,$ *and, for* $n = 16$ *we have:*
$x_0 = 2, x_1 = 2.25, x_2 = 2.5, x_3 = 2.75, x_4 = 3, x_5 = 3.25, x_6 = x_7 = 3.75, x_8 = 4, x_9 = 4.25, x_{10} = 4.5, x_{11} = 4.75, x_{12} = 5, x_{13} = 5.25, x_{14} = 5.5, x_{15} = 5.75, x_{16} = 6,$ *let operator weights for* $n = 8$ *be as follows:*
$w_0 = \frac{1}{100}, w_1 = \frac{2}{100}, w_2 = \frac{3}{100}, w_3 = \frac{5}{100}, w_4 = \frac{8}{100}, w_5 = \frac{12}{100}, w_6 = \frac{17}{100}, w_7 = \frac{23}{100}, w_8 = \frac{29}{100},$ *likewise for* $n = 16$*:*
$w_0 = \frac{1}{1000}, w_1 = \frac{2}{1000}, w_2 = \frac{3}{1000}, w_3 = \frac{4}{1000}, w_4 = \frac{5}{1000}, w_5 = \frac{6}{1000}, w_6 = \frac{8}{1000}, w_7 = \frac{11}{1000}, w_8 = \frac{15}{1000}, w_9 = \frac{20}{1000}, w_{10} = \frac{35}{1000}, w_{11} = \frac{50}{1000}, w_{12} = \frac{80}{1000}, w_{13} = \frac{110}{1000}, w_{14} = \frac{150}{1000}, w_{15} = \frac{200}{1000}, w_{16} = \frac{300}{1000}$*. Hence, considering* $n = 8$*:*
$Ind([x]) = \sum_{i=0}^{8} w_i.x_i =$
$\frac{1}{100}2 + \frac{2}{100}2.5 + \frac{3}{100}3 + \frac{5}{100}3.5 + \frac{8}{100}4 + \frac{12}{100}4.5 + \frac{17}{100}.5 + \frac{23}{100}5.5 + \frac{29}{100}.6 = 5.5,$ *and for* $n = 16$ *as similar, the proposed index is:*

$$Ind([x]) = \sum_{i=0}^{16} w_i.x_i = 5.4213.$$

Example 3.2. *Consider the following example in which for delineating the ranking function index*

Ind(), the operator weights are set as follows:

$$\begin{array}{l} For \quad 10 \quad points: \\ w_i = \frac{1+2*i}{100} (i = 0, 1, \ldots, 9). \\ For \quad 50 \quad points: \\ w_i = \frac{1+2*i}{2500} (i = 0, 1, \ldots, 49). \\ For \quad 100 \quad points: \\ w_i = \frac{1+2*i}{10000} (i = 0, 1, \ldots, 99). \\ For \quad 100 \quad points: \\ w_i' = \frac{1+5*i}{24850} (i = 0, 1, \ldots, 99). \end{array} \tag{3.6}$$

3.2 Fully interval linear programming problem

In this section a fully interval LP problem will be introduced. Also, a method for solving such problems will be thoroughly explained and discussed.
Fully interval linear programming (FILP) problems with m equality constraints and n interval variables will be formulated as follows:

$$\begin{array}{ll} Max \ \ (Min) & [Z] = [C] \otimes [X] \\ subject \ \ to & [A] \otimes [X] \preccurlyeq (\succcurlyeq) [b] \end{array} \tag{3.7}$$

where $[A] = ([a_{ij}])_{m \times n}$ is an interval number matrix and $[C] = ([c_1], [c_2], \ldots, [c_n])^T$,$[X] = ([x_1], [x_2], \ldots, [x_n])^T$ and $[b] = ([b_1], [b_2], \ldots, [b_n])^T$ are interval number vectors.

Definition 3.3. *The interval optimal solution of FILP problem 3.7 will be a interval number $[X]$ if it satisfies the following characteristics:*

(i) $Ind([A] \otimes [X]) \leqslant (\geqslant) \ \ Ind([b])$

(ii) *If there exist any interval number vector $[X']$ such that*

$$Ind([A] \otimes [X']) \leqslant (\geqslant) \ \ Ind([b]),$$

then
$Ind([C] \otimes [X]) \geqslant Ind([C] \otimes [X'])$ *(in case of maximization problem)*
and
$Ind([C] \otimes [X]) \leqslant Ind([C] \otimes [X'])$ *(in case of minimization problem).*

Remark 3.1. *Let $[X]$ be an interval optimal solution of FILP problem 3.7. If there exist an interval number vector $[Y]$ such that,*

(i) $Ind([A] \otimes [Y]) \leqslant (\geqslant) \ \ Ind([b])$

(ii) $Ind([C] \otimes [X]) = Ind([C] \otimes [Y])$.

then $[Y]$ is said to be a an alternative interval optimal solution of 3.7.

Lemma 3.1. *For any interval numbers, $[x] = [\underline{x}, \overline{x}]$ and $[a] = [\underline{a}, \overline{a}]$ if ($\overline{x} \leq 0$ or $\underline{x} \geq 0$) then we have*

$$[a] \otimes [x] = \begin{cases} [\overline{a}.x_1, \underline{a}.x_1] \oplus [\underline{a}.x_2, \overline{a}.x_2], & \underline{a} \leqslant 0 \leqslant \overline{a}, \\ [\overline{a}.x_1, \underline{a}.x_1'] \oplus [\underline{a}.x_2', \overline{a}.x_2], & \underline{a} \geqslant 0, \\ [\overline{a}.x_1', \underline{a}.x_1] \oplus [\underline{a}.x_2, \overline{a}.x_2'], & \overline{a} \leqslant 0. \end{cases} \tag{3.8}$$

where:

$$\begin{cases} [x] = [\underline{x}, \overline{x}] = [x_1, x_1'] \oplus [x_2', x_2] \\ x_1 \leqslant x_1' \leqslant 0 \leqslant x_2' \leqslant x_2, \\ \lambda_1.x_1 + \lambda_2.x_2 \ = \ 0 \\ \lambda_1 + \lambda_2 \ = \ 1, \quad \lambda_1, \lambda_2 \in \{0, 1\} \end{cases} \tag{3.9}$$

Proof. Let the assumptions be categorized into three cases:
Case 1)
let $\underline{a} \leq 0 \leq \bar{a}$ then we have the following two subcases:
1) $\underline{x} \geq 0$ in this case according to the definition (2.3) we have
$[a] \otimes [x] = [\underline{a}\overline{x}, \overline{ax}]$
In regards of the nonlinear partition interval (2.5) we have $[x]$ as following:
$[x] = [\underline{x}, \overline{x}] = [0, 0] \oplus [x_2', x_2] = [x_2', x_2]$
on the other hand we have:
$[\overline{a}.x_1, \underline{a}.x_1] \oplus [\underline{a}.x_2, \overline{a}.x_2] = [0, 0] \oplus [\underline{a}\overline{x}, \overline{ax}] = [\underline{a}\overline{x}, \overline{ax}]$
Thus we have:
$[a] \otimes [x] = [\overline{a}.x_1, \underline{a}.x_1] \oplus [\underline{a}.x_2, \overline{a}.x_2]$
2) $\bar{x} \leq 0$ In this case according to definition (2.3) We have

$$[a] \otimes [x] = [\overline{a}\underline{x}, \underline{ax}]$$

Then, in regards of the nonlinear partition interval (2.5) $[x]$ we have

$$[x] = [\bar{x}, \underline{x}] = [x_1, x_1'] \oplus [0, 0] = [x_1, x_1']$$

on the other hand we have:

$$[\overline{a}.x_1, \underline{a}.x_1] \oplus [\underline{a}.x_2, \overline{a}.x_2] = [\overline{a}\underline{x}, \underline{ax}] \oplus [0, 0] = [\overline{a}\underline{x}, \underline{ax}]$$

Table 1: Comparative results of ranking interval numbers using ind (.) function.

interval	Ind(.) n=9 w	Ind(.) n=49 w	Ind(.) n=99 w	Ind(.) n=99 w'
$a=[4,10]$	8.1000	8.0200	8.0100	8.0161
$b=[4.5,10]$	8.2583	8.1850	8.1758	8.1814
$c=[5,10]$	8.4167	8.3500	8.3417	8.3467
Results	$a \prec b \prec c$	$a \prec b \prec c$	$a \prec b \prec c$	$a \prec b \prec c$
$a=[5,18]$	13.8833	13.7100	13.6883	13.7015
$b=[5,18.5]$	14.2250	14.0450	14.0225	14.0362
$c=[5,18.75]$	14.3958	14.2125	14.1896	14.2036
Results	$a \prec b \prec c$	$a \prec b \prec c$	$a \prec b \prec c$	$a \prec b \prec c$
$a=[2,3]$	2.6833	2.6700	2.6683	2.6693
$b=[2.2,2.8]$	2.6100	2.6020	2.6010	2.6016
$c=[2.4,2.6]$	2.5367	2.5340	2.5337	2.5339
Results	$a \succ b \succ c$	$a \succ b \succ c$	$a \succ b \succ c$	$a \succ b \succ c$
$a=[30,50]$	43.6667	43.4000	43.3667	43.3870
$b=[37,46]$	43.1500	43.0300	43.0150	43.0241
$c=[38,47]$	44.1500	44.0300	44.0150	44.0241
Results	$b \prec a \prec c$	$b \prec a \prec c$	$b \prec a \prec c$	$b \prec a \prec c$
$a=[50,100]$	84.1667	83.5000	83.4167	83.4675
$b=[52,98]$	83.4333	82.8200	82.7433	82.7901
$c=[60,97]$	85.2833	84.7900	84.7283	84.7659
Results	$b \prec a \prec c$	$b \prec a \prec c$	$b \prec a \prec c$	$b \prec a \prec c$
$a=[2000.55,3000.55]$	2683.8833	2670.5500	2668.8833	2669.8994
$b=[2000.56,3000.55]$	2683.8865	2670.5533	2668.8867	2669.9027
$c=[2000.55,3000.56]$	2683.8902	2670.5567	2668.8900	2669.9061
Results	$a \prec b \prec c$	$a \prec b \prec c$	$a \prec b \prec c$	$a \prec b \prec c$
$a=[-16,-10]$	−11.9000	−11.9800	−11.9900	−11.9839
$b=[-15.5,-10]$	−11.7417	−11.8150	−11.8242	−11.8186
$c=[-15,-10]$	−11.5833	−11.6500	−11.6583	−11.6533
Results	$a \prec b \prec c$	$a \prec b \prec c$	$a \prec b \prec c$	$a \prec b \prec c$
$a=[-18,-4]$	−8.4333	−8.6200	−8.6433	−8.6291
$b=[-18,-4.5]$	−8.7750	−8.9550	−8.9775	−8.9638
$c=[-18,-5]$	−9.1167	−9.2900	−9.3117	−9.2985
Results	$a \succ b \succ c$	$a \succ b \succ c$	$a \succ b \succ c$	$a \succ b \succ c$

Thus we have:

$$[a] \otimes [x] = [\overline{a}.x_1, \underline{a}.x_1] \oplus [\underline{a}.x_2, \overline{a}.x_2]$$

Case 2) Let $\underline{a} \geq 0$ then we consider the two following subcases.
1) $\underline{x} \geq 0$ 2) $\overline{x} \leq 0$
Case 3) Let $\bar{a} \leq 0$ then we have the following two subcases.
1) $\underline{x} \geq 0$ 2) $\bar{x} \leq 0$. □

3.3 Proposed method to find the fuzzy optimal solution of FILP problems

Now, a new method is proposed to find the fuzzy optimal solution of following type of FILP problems:

$$\begin{array}{l} Max(Min)\ \ [z]=\sum_{j=1}^{n}([\underline{c_j},\overline{c_j}]\otimes[\underline{x_j},\overline{x_j}]) \\ subject\ \ to: \\ \sum_{j=1}^{n}\left([\underline{a_{ij}},\overline{a_{ij}}]\otimes[\underline{x_j},\overline{x_j}]\right) \\ \preccurlyeq(\succcurlyeq)[\underline{b_i},\overline{b_i}]\quad i=1,\ldots,m. \end{array} \tag{3.10}$$

Let the interval number vector $[X]=([x_1],[x_2],\ldots,[x_n])^T$ be a feasible solution of FILP problem (3.10) that $0\notin(\underline{x_j},\overline{x_j})$ $j=1,\ldots,n$ then:

$$\begin{array}{l} [z]=\sum_{j=1}^{n}\left([\underline{c_j},\overline{c_j}]\otimes[\underline{x_j},\overline{x_j}]\right) \\ \sum_{j=1}^{n}\left([\underline{a_{ij}},\overline{a_{ij}}]\otimes[\underline{x_j},\overline{x_j}]\right) \\ \preccurlyeq(\succcurlyeq)[\underline{b_i},\overline{b_i}]\quad i=1,\ldots,m, \end{array} \tag{3.11}$$

So, we consider two separate cases for the elements of matrix A and C as following:

$$\begin{array}{l} \mathbb{J}_i^-=\{j\mid \overline{a_{ij}}\leqslant 0\overline{a_{ij}}\neq 0\} \\ \mathbb{G}_=^-\{j\mid \overline{c_j}\leqslant 0\overline{c_j}\neq 0\}, \\ \mathbb{J}_i^+=\{j\mid \underline{a_{ij}}\geqslant 0\overline{a_{ij}}\neq 0\} \\ \mathbb{G}_=^+\{j\mid \underline{c_j}\geqslant 0\overline{c_j}\neq 0\}, \\ \mathbb{J}_i^0=\{j\mid \underline{a_{ij}}\leqslant 0\leqslant\overline{a_{ij}}\} \\ \mathbb{G}_=^0\{j\mid \underline{c_j}\leqslant 0\leqslant\overline{c_j}\}, \\ i=1,\ldots,m \end{array} \tag{3.12}$$

Using the above mentioned separation, (3.11) is written as follows;

$$\begin{array}{l} \sum_{\mathbb{J}_i^-}([\underline{a_{ij}},\overline{a_{ij}}]\otimes[\underline{x_j},\overline{x_j}])+ \\ \sum_{\mathbb{J}_i^+}([\underline{a_{ij}},\overline{a_{ij}}]\otimes[\underline{x_j},\overline{x_j}])+ \\ \sum_{\mathbb{J}_i^0}([\underline{a_{ij}},\overline{a_{ij}}]\otimes[\underline{x_j},\overline{x_j}]) \\ \preccurlyeq(\succcurlyeq)[\underline{b_i},\overline{b_i}]\quad i=1,\ldots,n, \\ [z]=\sum_{\mathbb{G}_i^-}([\underline{c_j},\overline{c_j}]\otimes[\underline{x_j},\overline{x_j}])+ \\ \sum_{\mathbb{G}_i^+}([\underline{c_j},\overline{c_j}]\otimes[\underline{x_j},\overline{x_j}])+ \\ \sum_{\mathbb{G}_i^0}([\underline{c_j},\overline{c_j}]\otimes[\underline{x_j},\overline{x_j}]) \\ i=1,\ldots,n, \end{array} \tag{3.13}$$

Now, using Lemma 3.1, system (3.13) is written as follows:

$$\begin{array}{l} [z]= \\ \sum_{\mathbb{G}^-}([\overline{c_j}x'_{j1}+\underline{c_j}x_{j2},\underline{c_j}x_{j1}+\overline{c_j}x'_{j2}])+ \\ \sum_{\mathbb{G}^+}([\overline{c_j}x_{j1}+\underline{c_j}x'_{j2},\underline{c_j}x'_{j1}+\overline{c_j}x_{j2}])+ \\ \sum_{\mathbb{G}^0}([\overline{c_j}x_{j1}+\underline{c_j}x_{j2}\ ,\ \underline{c_j}x_{j1}+\overline{c_j}x_{j2}]) \\ \\ \sum_{\mathbb{J}^-}([\overline{a_{ij}}x'_{j1}+\underline{a_{ij}}x_{j2},\underline{a_{ij}}x_{j1}+\overline{a_{ij}}x'_{j2}])+ \\ \sum_{\mathbb{J}^+}([\overline{a_{ij}}x_{j1}+\underline{a_{ij}}x'_{j2},\underline{a_{ij}}x'_{j1}+\overline{a_{ij}}x_{j2}]) \\ \sum_{\mathbb{J}^0}([\overline{a_{ij}}x_{j1}+\underline{a_{ij}}x_{j2},\underline{a_{ij}}x_{j1}+\overline{a_{ij}}x_{j2}]) \\ \preccurlyeq(\succcurlyeq)[\underline{b_i},\overline{b_i}],\qquad i=1,2,\ldots,n, \\ \\ [\underline{x_j},\overline{x_j}]=[x_{j1},x'_{j1}]\oplus[x'_{j2},x_{j2}] \\ x_{j1}\leqslant x'_{j1}\leqslant 0\leqslant x'_{j2}\leqslant x_{j2}, \\ \lambda_{j1}.x_{j1}+\lambda_{j2}.x_{j2}=0 \\ \lambda_{j1}+\lambda_{j2}=1,\quad \lambda_{j1},\lambda_{j2}\in\{0,1\}, \\ j=1,\ldots,n. \end{array} \tag{3.14}$$

Now we set:

$$\begin{array}{l} \sum_{\mathbb{G}^-}\left(\overline{c_j}x'_{j1}+\underline{c_j}x_{j2}\right)+\sum_{\mathbb{G}^+}\left(\overline{c_j}x_{j1}+\underline{c_j}x'_{j2}\right) \\ +\sum_{\mathbb{G}^0}\left(\overline{c_j}x_{j1}+\underline{c_j}x_{j2}\right)=:\underline{CX}, \\ \sum_{\mathbb{G}^-}\left(\underline{c_j}x_{j1}+\overline{c_j}x'_{j2}\right)+\sum_{\mathbb{G}^+}\left(\underline{c_j}x'_{j1}+\overline{c_j}x_{j2}\right) \\ +\sum_{\mathbb{G}^0}\left(\underline{c_j}\cdot x_{j1}+\overline{c_j}\cdot x_{j2}\right)=:\overline{CX}, \\ \\ \sum_{\mathbb{J}^-}\left(\overline{a_{ij}}x'_{j1}+\underline{a_{ij}}x_{j2}\right)+\sum_{\mathbb{J}^+}\left(\overline{a_{ij}}x_{j1}+\underline{a_{ij}}x'_{j2}\right) \\ +\sum_{\mathbb{J}^0}\left(\overline{a_{ij}}x_{j1}+\underline{a_{ij}}x_{j2}\right)=:\underline{(AX)}_i \\ \sum_{\mathbb{J}^-}\left(\underline{a_{ij}}x_{j1}+\overline{a_{ij}}x'_{j2}\right)+\sum_{\mathbb{J}^+}\left(\underline{a_{ij}}x'_{j1}+\overline{a_{ij}}x_{j2}\right) \\ +\sum_{\mathbb{J}^0}\left(\underline{a_{ij}}x_{j1}+\overline{a_{ij}}x_{j2}\right)=:\overline{(AX)}_i, \\ i=1,2,\ldots,m. \end{array} \tag{3.15}$$

Then using (3.15) model (3.10) can be rewritten as follows:

$$\begin{array}{l} Max(Min)[z]=[\underline{CX},\overline{CX}] \\ subjectto: \\ [\underline{(AX)}_i,\overline{(AX)}_i]\preccurlyeq(\succcurlyeq)[\underline{b_i},\overline{b_i}]i=1,\ldots,m \\ x_{j1}\leqslant x'_{j1}\leqslant 0\leqslant x'_{j2}\leqslant x_{j2}, \\ \lambda_{j1}.x_{j1}+\lambda_{j2}.x_{j2}=\ 0 \end{array} \tag{3.16}$$

$$\lambda_{j1}+\lambda_{j2} \;=\; 1, \quad \lambda_{j1},\lambda_{j2}\in\{0,1\},$$
$$j=1,\ldots,n.$$

Now, using the operator weights w_i $i = 0,1,\ldots,n$, the ranking function index $Ind(.)$, as presented in section (2) can be delineated. Then, applying the ranking function index $Ind(.)$ it is possible to obtain the interval optimal solution of FILP problem 3.16 as follows:

$$\begin{array}{l}
Max(Min)\quad Ind([z]) = Ind([\underline{CX},\overline{CX}])\\
subjectto:\\
Ind([\underline{(AX)}_i,\overline{(AX)}_i]) \leqslant (\geqslant) Ind([\underline{b_i},\overline{b_i}])\\
i=1,\ldots,m,\\
x_{j1}\leqslant x'_{j1}\leqslant 0\leqslant x'_{j2}\leqslant x_{j2},\\
\lambda_{j1}.x_{j1}+\lambda_{j2}.x_{j2} \;=\; 0\\
\lambda_{j1}+\lambda_{j2} \;=\; 1, \quad \lambda_{j1},\lambda_{j2}\in\{0,1\},\\
j=1,\ldots,n
\end{array} \tag{3.17}$$

The above NLP 3.17 can be easily solved by using GAMS or LINGO software package.

Theorem 3.3. *Let* $X^* = ([\underline{x}_1,\overline{x}_1],[\underline{x}_2,\overline{x}_2],\ldots,[\underline{x}_n,\overline{x}_n])$ *be an interval optimal solution of FILP problem (3.10) then* $X^*_{Ind} = (x_{11},x_{12},x'_{11},x'_{12},x_{21},x_{22},x'_{21},x'_{22},\ldots,x_{n1},x_{n2},x'_{n1},x'_{n2})$ *is an optimal solution of (3.17) where* $[\underline{x}_j,\overline{x}_j] = [x_{j1},x'_{j1}]\oplus[x'_{j2},x_{j2}]$ $(j=1,\ldots,n)$.

Proof. Using Definitions (3.3) and Eqs.(3.11)-(3.15), the proof is obvious. □

In addition, it is possible to explain the new proposed method by following algorithm:

3.3.1 Algorithm

step 1 Consider the following FILP problem:

$$\begin{array}{l}
Max(Min)[Z]=[C]\otimes[X]\\
subjectto\\
[A]\otimes[X]\preccurlyeq(\succcurlyeq)[b]
\end{array} \tag{3.18}$$

step 2 Now using Lemma 3.1,model (3.18) can be written as model (3.16).

step 3 Using the operator weights $w_i(i = 0,1,\ldots,n)$ it is possible to delineate the ranking function index $Ind()$ as in section (2).

step 4 Apply the ranking function index $Ind(.)$ to obtain the interval optimal solution of FILP problem (3.16) as model (3.17).

step 5 Solving model (3.17) $x_{j1},x_{j2},x'_{j1},x'_{j2}, j = 1,2,\ldots,n$ and $[\underline{x_j},\overline{x_j}]$ are found which is the **interval optimal solution** of FILP problem.

Example 3.3. *Consider the following FILP problem:*

$$\begin{array}{l}
Min[z] = [-145,-28][x_1]+[31,186][x_2]\\
\qquad +[-44,127][x_3]+[-62,281][x_4],\\
subject\ \ to:\\
[46,137][x_1]+[35,152][x_2]+[72,183][x_3]\\
\qquad +[115,218][x_4]\preccurlyeq\ [-1400,1500],\\
[-75,48][x_1]+[-121,162][x_2]+\\
\quad [-183,192][x_3]+[82,253][x_4]\\
\preccurlyeq[-26000,20000],\\
[-152,-39][x_1]+[28,179][x_2]+\\
\quad [-53,138][x_3]=[100,2100],\\
\quad [x_4]\succcurlyeq[0,0],\\
\quad [x_1],[x_2],[x_3]\quad isfree.
\end{array} \tag{3.19}$$

Now using Lemma 3.1, system (3.19) can be written as follows:

$$\begin{array}{l}
Min\ \ [z]=[-145x_{12}-28x'_{11}+186x_{21}\\
\quad +31x'_{22}+127x_{31}-44x_{32}-62x_{42}\\
\quad -145x_{11}-28x'_{12}+186x_{22}+31x'_{21}\\
\quad +127x_{32}-44x_{31}+281x_{42}]\\
subjectto:\\
\quad [137x_{11}+46x'_{12}+152x_{21}+35x'_{22}\\
\quad +183x_{31}+72x'_{32}+115x'_{42}\ ,\ 137x_{12}\\
\quad +46x'_{11}+152x_{22}+35x'_{21}+183x_{32}\\
\quad +72x'_{31}+218x_{42}]\preccurlyeq[-1400,1500],\\
\quad [48x_{11}-75x_{12}+162x_{21}-121x_{22}\\
\quad +192x_{31}-183x_{32}+82x'_{42}\ ,\ 48x_{12}\\
\quad -75x_{11}+162x_{22}-121x_{21}+192x_{32}\\
\quad -183x_{31}+253x_{42}]\preccurlyeq[-26000,20000],\\
\quad [-152x_{12}-39x'_{11}+179x_{21}+28x'_{22}\\
\quad +138x_{31}-153x_{32}\ ,\ -152x_{11}-39x'_{12}\\
\quad +179x_{22}+28x'_{21}+138x_{32}-153x_{31}]
\end{array} \tag{3.20}$$

$$= [100, 2100]$$
$$x_{j1} \leqslant x'_{j1} \leqslant 0 \leqslant x'_{j2} \leqslant x_{j2}, x_{41} \geqslant 0$$
$$\lambda_{j1}.x_{j1} + \lambda_{j2}.x_{j2} = 0$$
$$\lambda_{j1} + \lambda_{j2} = 1, \quad \lambda_{j1}, \lambda_{j2} \in \{0, 1\},$$
$$j = 1, \ldots, 3$$

Now for delineating the ranking function index $Ind()$, set the operator weights $w_i = \frac{1+12*i}{550}(i = 0, 1, \ldots, 9)$. Obviously $\sum_{i=0}^{9} w_i = 1$, $0 < w_1 < w_2 < \ldots < w_9 \leqslant 1$. Hence,
$Ind([a, b]) = \sum_{i=0}^{9} w_i \left(\frac{i}{9}b + \frac{9-i}{9}a\right) = 0.33 * a + 0.67 * b$.
Apply the ranking function index $Ind([a, b]) = 0.3 * a + 0.7 * b$ to obtain the interval optimal solution of FILP problem 3.19 as following:

$$\begin{aligned}
&Min \quad Z' = 0.3(-145x_{12} - 28x'_{11} + \\
&186x_{21} + 31x'_{22} + 127x_{31} - 44x_{32} - 62x_{42}) \\
&+0.7(-145x_{11} - 28x'_{12} + 186x_{22} + 31x'_{21} \\
&+127x_{32} - 44x_{31} + 281x_{42}) \\
&subject \ \ to: \\
&0.3(137x_{11} + 46x'_{12} + 152x_{21} + 35x'_{22} \\
&+183x_{31} + 72x'_{32} + 115x'_{42}) + \\
&0.7(137x_{12} + 46x'_{11} + 152x_{22} + 35x'_{21} \\
&+183x_{32} + 72x'_{31} + 218x_{42}) \leqslant 6300, \\
&0.3(48x_{11} - 75x_{12} + 162x_{21} - 121x_{22} \\
&+192x_{31} - 183x_{32} + 82x'_{42}) + \\
&0.7(48x_{12} - 75x_{11} + 162x_{22} - 121x_{21} \\
&+192x_{32} - 183x_{31} + 253x_{42}) \leqslant 6200, \\
&0.3(-152x_{12} - 39\alpha_1 + 179x_{21} + 28\beta_2 \\
&+138x_{31} - 153x_{32}) + \\
&0.7(-152x_{11} - 39\beta_1 + 179x_{22} + 28\alpha_2 \\
&+138x_{32} - 153x_{31}) = -1250, \\
&x_{j1} \leqslant x'_{j1} \leqslant 0 \leqslant x'_{j2} \leqslant x_{j2}, \qquad x_{41} \geqslant 0, \\
&\lambda_j.x_{j2} + \lambda_j.x_{j1} - x_{j1} = 0, \\
&\lambda_j \in \{0, 1\} \quad j = 1, \ldots, 4.
\end{aligned} \tag{3.21}$$

Thus, the obtained results are as follows:

$$, \quad \begin{array}{ll}
x_{11} = 0, & x_{12} = 89.12946 \\
x'_{11} = 0, & x'_{12} = 0, \\
x_{21} = 0, & x_{22} = 11.45378, \\
x'_{21} = 0, & x'_{22} = 11.45378, \\
x_{31} = -61.38407, & x_{32} = 0, \\
x'_{31} = 0, & x'_{32} = 0, \\
x_{41} = 0, & x_{42} = 0, \\
x'_{41} = 0, & x'_{42} = 0.
\end{array} \tag{3.22}$$

Hence, the following results are obtained as:

$$[x_1^*] = [0, 89.12946],$$
$$[x_2^*] = [11.45378, 11.45378],$$
$$[x_3^*] = [-61.38407, 0], \qquad [x_4^*] = [0, 0].$$

And $z^* = -5962.190$

4 Conclusion

This article introduce a ranking function for ordering interval numbers and explain some features of this function. Moreover, a new method for solving FILS in general case presented while using standard splitting provided by Allahviranloo et al. In this method positive and negative optimal interval solutions obtained which are algebraically verified in feasible region in regards of introduced ranking function. In doing so, a FILS model transferred into a NLP which can be solved by related softwares. Great advantages of this model are its simplicity and that it can be used in any kind of FILS models. Also, further improvements can be done for more simplifying the proposed model.

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