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Multiplication-Like Modules

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Abstract

In this paper we introduce the concept of multiplication-like modules and we obtain some related results. We show that an R-module M is multiplication-like if and only if for each ideal I of R, $I = (IM :_R M)$. We prove that any multiplication-like module is faithful and r-multiplication. So we get that any flat and multiplication-like module is faithfully flat.

Keywords : Modules; Free modules, Flat modules, Multiplication modules; Multiplication-like modules.

1 Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary. Let M be an R-module. For a submodule N of M, let $(N :_R M)$ denote the set of all elements r in R such that $rM \subseteq N$. The annihilator of M, denoted by $Ann_R(M)$, is $(0 :_R M)$. A proper submodule N of M is called prime (*primary*) if $rx \in N$, for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r \in (N :_R M)$ ($r^n \in (N :_R M)$, for some $n \in \mathbb{N}$). We denote the set of prime submodules of M by Spec(M). For a submodule N of M, $V(N)$ denotes $\{P \in Spec(M) | N \subseteq P\}$, and $rad(N) = \bigcap V(N)$, is called the radical of N and was introduced in [9], [10] and [11]. A proper submodule N of M is said to be primary-like if $rm \in N$, for $r \in R$ and $m \in M$, implies that either $m \in rad(N)$ or $r \in (N :_R M)$ (see [7]).

It is said that M is a multiplication module, if for each submodule N of M, there is an ideal I of R, such that $N = IM$. Equivalently, M is a multiplication module if and only if for each submodule N of M, we have $N = (N :_R M)M$ [5] and [6].

In [3] the notion of a comultiplication module was introduced as a dual of the concept of a multiplication module. An R-module M is called comultiplication, if for every submodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$. For example, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is a comultiplication module since all of its proper submodules are of the form $(0 :_M P^i\mathbb{Z})$ for $i = 0, 1, ...$ It is clear that M is comultiplication if and only if for every submodule N of M, we have $N = (0 :_M (0 :_R N))$. An R-module M is said to be strong comultiplication, if for every submodule N of M there is exactly one ideal I of R with $N = (0 :_M I)$ (*see*[4]).

M is said to be an r-multiplication module, when $IM \neq M$ for every proper ideal I of R (see [12]).

A non-zero submodule N of M is said to be

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second, if for each $a \in R$, the homomorphism $N \xrightarrow{a} N$ is either surjective or zero [15]. An R-module M is said to be distributive, if the lattice of its submodule is distributive, i.e. $(X+Y) \cap Z = (X \cap Z) + (Y \cap Z)$, for any of its submodules X, Y and Z. A non-zero module M over a ring R is said to be prime, if the annihilator of M is the same as the annihilator of N for every non-zero submodule N of M (see [2]).

In this article, we introduce multiplication-like module and obtain some basic results and characterizations.

2 Multiplication-Like Modules

Definition 2.1. *An R-module M is said multiplication-like, if for any ideal I of R, there exists a submodule N of M such that $I = (N :_R M)$.*

Example 2.1. *(i) Every vector space is multiplication-like.*
(ii) $R[X]$ is a multiplication-like R-module.
(iii) $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module are not multiplication-like.

It is clear that every free module is multiplication-like; but $M = \mathbb{Z} \oplus \mathbb{Z}_2$ is a multiplication-like $\mathbb{Z}$-module, which is not free.

Lemma 2.1. *An R-module M is multiplication-like if and only if for each ideal I of R, $I = (IM :_R M)$.*

Proof. The sufficiency is clear. Conversely, suppose that M is a multiplication-like. Then there exists a submodule N of M such that $I = (N :_R M)$. So we have $IM \subseteq N$. Hence $I \subseteq (IM :_R M) \subseteq (N :_R M) = I$. This implies that $I = (IM :_R M)$ as desired. □

Proposition 2.1. *Let M be an R-module. Then M is multiplication-like if and only if for every ideal I of R, there exist submodules N_i of M ($i \in J$), such that $I = \sum_{i \in J}(N_i :_R M) = (\sum_{i \in J} N_i :_R M)$.*

Proof. Let M be multiplication-like and let I be an ideal of R. Then $I = (IM :_R M)$. On the other hand, $I = \sum_{r_i \in I} Rr_i$ and for each $r_i \in I$, $Rr_i = (r_iM :_R M)$. So we have
$I = \sum_{r_i \in I} Rr_i = \sum_{r_i \in I}(r_iM :_R M) = (\sum_{r_i \in I} r_iM :_R M)$.

Hence the proof is completed. □

Theorem 2.1. *Let M be an R-module. Then the following statements are equivalent.*

(i) M is multiplication-like.

(ii) For every ideal I of R and each submodule N of M with $I \subset (N :_R M)$, there exists a submodule L of M such that $L \subset N$ and $I = (L :_R M)$.

(iii) For every ideal I of R and each submodule N of M with $I \subset (N :_R M)$, there exists a submodule L of M such that $L \subset N$ and $I \subseteq (L :_R M)$.

Proof. (i)$\Longrightarrow$ (ii) Let $I \subset (N :_R M)$. Since M is multiplication-like, $I = (IM :_R M)$. Put $L = IM \cap N$. Since $I = (IM :_R M) \subset (N :_R M)$, hence $L \subset N$ and we have
$(L :_R M) = (IM \cap N :_R M) = (IM :_R M) \cap (N :_R M) = I$.

$(ii) \Longrightarrow (iii)$ It is obvious.

$(iii) \Longrightarrow (i)$ Let I be an ideal of R and put
$H = \{L : L \textit{ is a submodule of } M \textit{ and } I \subseteq (L :_R M)\}$.

Clearly H is a non-empty set, so by Zorn's Lemma, H has a minimal member like K and so $I \subseteq (K :_R M)$. Assume that $I \neq (K :_R M)$. Then by part (iii), there exists a submodule U of M with $U \subset K$ and $I \subseteq (U :_R M)$. But this is a contradiction by the choice of K. Thus we have $I = (K :_R M)$. This shows that M is multiplication-like. □

Example 2.2. *Let $M = \mathbb{Z}_6$ and $R = \mathbb{Z}_6$. Then M is multiplication-like but, $\bar{2}\mathbb{Z}_6$ and $\bar{3}\mathbb{Z}_6$ are not multiplication-like modules.*

Let M be a torsion-free R-module. Clearly, every non-zero cyclic submodule of M is a multiplication-like R-module. But, if every non-zero cyclic submodule of an R-module M is multiplication-like, then M is not necessarly multiplication-like. As the following example Shows:

Example 2.3. *Let $M = \mathbb{Q}$ and $R = \mathbb{Z}$. Then every non-zero cyclic submodule of M is free and so multiplication-like; but $\mathbb{Q}$ is not a multiplication-like R-module.*

Theorem 2.2. *Let R be a comultiplication ring and M be a faithful R-module. Then M is a multiplication-like R-module.*

Proof. Assume that I is a proper ideal of R and $rM \subseteq IM$, for $r \in R$. Then $rAnn_R(I)M = 0$. Since M is faithful and R is a comultiplication ring, we have $r \in I$. Thus M is a multiplication-like module. □

It is straightforward to prove that R is a comultiplication ring if and only if $(I :_R J) = (Ann_R(J) :_R Ann_R(I))$, for each ideals I and J of R. So by Theorem 2.2, we have:

Corollary 2.1. *Let R be a ring such that for every ideal I and J of R, $(I :_R J) = (Ann_R(J) :_R Ann_R(I))$. Then every faithful R-module is multiplication-like module.*

By Example 3.8 [3] and Theorem 2.2, we obtain the following corollary.

Corollary 2.2. *Let R be a semi-simple ring. Then every faithful R-module is multiplication-like.*

Corollary 2.3. *Let M be a strong comultiplication module which has a maximal submodule over a reduced ring R (recall that a reduced ring is one with no nilpotents). Then M is a multiplication-like R-module.*

Proof. As M is strong comultiplicatin, then $Ann_R(M) = 0$. Now it follows easily from Corollary 4.5 [12] and Corollary 2.2. □

By Proposition 4.3 [12] and Theorem 2.2, we get the following corollary.

Corollary 2.4. *Let M be a non-zero multiplication and strong comultiplication R-module. Then M is a multiplication-like R-module.*

Clearly, if M' is a multiplication-like R-module and $\rho : M \longrightarrow M'$ is an R-epimorphism, then M is a multiplication-like module.

Also, let M be an R-module and N be a submodule of M. If $\frac{M}{N}$ is a multiplication-like R-module, then we can conclude that M is a multiplication-like R-module. But, the converse is not true in general, as the following example shows:

Example 2.4. *$\mathbb{Z}$ as $\mathbb{Z}$-module is a multiplication-like R-module, but for submodule $n\mathbb{Z}$, $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is not a multiplication-like $\mathbb{Z}$-module.*

Lemma 2.2. *Let M be a multiplication-like R-module.*

(i) If for submodule N of M, $N \subseteq IM$ for each non-zero ideal I of R and $\frac{M}{N}$ is faithful, then $\frac{M}{N}$ is a multiplication-like R-module.

(ii) If M' is a faithful R-module, $\rho : M \longrightarrow M'$ is an epimorphism and for any non-zero ideal I of R, $ker(\rho) \subseteq IM$, then M' is a multiplication-like R-module.

Proof. We have $I(\frac{M}{N}) = \frac{IM}{N}$. Hence $\frac{M}{N}$ is a multiplication-like R-module.

(ii) This is clear by part (i). □

Corollary 2.5. *Let N be a faithful second submodule of a multiplication-like R-module M. Then for every non-zero ideal I of R, there is a submodule L of $\frac{M}{N}$ such that $I = (L :_R \frac{M}{N})$.*

Proof. Since N is second and faithful, we have that $IN = N$, for each non-zero ideal I of R. So $N \subseteq IM$. By Lemma 2.2 (i), the proof is complete. □

Proposition 2.2. *Let M be a multiplication-like R-module and I be an ideal of R. Then $\frac{M}{IM}$ is a multiplication-like $\frac{R}{I}$-module.*

Proof. It is enough to prove that for each ideal J of R containing I, $(\frac{J}{I}(\frac{M}{IM}) :_{\frac{R}{I}} \frac{M}{IM}) \subseteq \frac{J}{I}$. Since M is multiplication-like, we have $J = (JM :_R M)$. If $(r + I) \in (\frac{J}{I}(\frac{M}{IM}) :_{\frac{R}{I}} \frac{M}{IM})$, then for every $x \in M$, $(r + I)(x + IM) \in \frac{J}{I}(\frac{M}{IM}) = \frac{JM}{IM}$.

This implies that $rx \in JM$. So we have that $r \in J$. It follows $r + I \in \frac{J}{I}$. □

Corollary 2.6. *Let M be a multiplication-like R-module. Then for any ideal I of R such that $I \subseteq Ann_R(M)$, M is a multiplication-like $\frac{R}{I}$-module.*

Remark 2.1. The converse of previous corollary is not true in general. For example, $\mathbb{Z}_n$ is a multiplication-like $\mathbb{Z}_n$-module, while $\mathbb{Z}_n$ as $\mathbb{Z}$-module is not multiplication-like.

The following proposition shows the behavior of modules that are multiplication-like module over localizations.

Proposition 2.3. *Let M be an R-module and S be a multiplicative closed subset of R.*

(i) If M is a finitely generated multiplication-like R-module, then M_S is a multiplication-like R_S-module.

(ii) If M_S is a multiplication-like R_S-module and for any ideal I of R and any $r \notin I$, $S \cap (I :_R r) = \emptyset$, then M is a multiplication-like R-module.

Proof. (i) Since M is a multiplication-like module, $I = (IM :_R M)$ for any ideal I of R. So we have $I_S = (I_S M_S :_{R_S} M_S)$, as M is finitely generated.

(ii) Let I be an ideal of R and $r \in (IM :_R M)$. So $\frac{r}{1} M_S \subseteq I_S M_S$. Since M_S is a multiplication-like R_S-module, $\frac{r}{1} \in I_S$. So there exists $u \in S$ such that $ur \in I$. If $r \notin I$, then $u \in S \cap (I :_R r)$ which is a contradiction. Hence $r \in I$. □

We now give an example to show that in Proposition 2.3 (ii), the condition is necessary.

Example 2.5. *Let $M = \mathbb{Q}$, $R = \mathbb{Z}$ and $S = \mathbb{Z} - \{0\}$. Then M_S is a vector space on field $R_S = \mathbb{Q}$. So M_S is a multiplication-like R_S-module; but M is not a multiplication-like R-module.*

Corollary 2.7. *Let (R, m) be a local ring and M be a finitely generated R-module. Then M is a multiplication-like R-module if and only if M_m is a multiplication-like R_m-module.*

Proposition 2.4. *Let M and N be R-modules and $M \otimes_R N$ be a multiplication-like module. Then M and N are multiplication-like R-modules.*

Proof. Let I be an ideal of R and $r \in (IM :_R M)$. Then $rM \otimes_R N \subseteq IM \otimes_R N$. This implies that $r(M \otimes_R N) \subseteq I(M \otimes_R N)$, so that $r \in (I(M \otimes_R N) :_R M \otimes_R N) = I$. Hence M and similarly N are multiplication-like R-modules. □

It is clear that, if M is a multiplication-like R-module and N is a free R-module, then the converse of above proposition is true.

Proposition 2.5. *Let M_1 and M_2 be two R-modules which M_1 or M_2 is multiplication-like R-module. Then $M_1 \oplus M_2$ is a multiplication-like R-module.*

Proof. Let I be an ideal of R such that $r(M_1 \oplus M_2) \subseteq I(M_1 \oplus M_2)$ and M_1 be a multiplication-like R-module. Then $I = (IM_1 :_R M_1)$ which implies that $r \in I$. Therefore, $M_1 \oplus M_2$ is a multiplication-like R-module. □

The converse of above lemma is not true in general.

Example 2.6. *Consider $M = \mathbb{Z}_6 = (\bar{2}) \oplus (\bar{3})$ and $R = \mathbb{Z}_6$. Then M is a multiplication-like R-module. But it is easy to see that $N = \bar{2}\mathbb{Z}_6$ and $L = \bar{3}\mathbb{Z}_6$ are not multiplication-like module.*

Corollary 2.8. *Let M_i $(i \in I)$ be R-modules such that for some i, M_i is a multiplication-like R-module. Then $\oplus_{i \in I} M_i$ is a multiplication-like R-module.*

Lemma 2.3. *Let R be a ring and M be an R-module such that $I \neq (IM :_R M)$, for some ideal I. Then there exists an ideal K and $r \notin K$ such that $I \subseteq K$ and $(K :_R r)$ is maximal ideal of ring R.*

Proof. By hypothesis there exists an element r in R such that $r \in (IM :_R M)$ but $r \notin I$. Let S denote the collection of ideals L of R such that $I \subseteq L$ but $r \notin L$. Clearly S is non-empty and so by Zorn's Lemma, S contains a maximal member like K.
Thus $I \subseteq K$ and $r \notin K$. Let s be an element of R such that $s \notin (K :_R r)$. It follows that K is a

proper subset of $K+Rsr$ and hence $K+Rsr \notin S$. Thus $r \in K + Rsr$. Therefore, there exists $b \in R$ and $u \in K$ such that $r = u+bsr$ and so $(1-bs)r \in K$. It follows that $(K :_R r)$ is a maximal ideal of R. □

Theorem 2.3. *Let R be a ring. Then the following statements are equivalent for R-module M.*

(i) M is a multiplication-like R-module.

(ii) $I = (IM :_R M)$, for every ideal I of R.

(iii) Given ideals I, J of R, $IM \subseteq JM$ implies that $I \subseteq J$.

(iv) Given any ideal I of R and $r \in R$, $rM \subseteq IM$ implies that $r \in I$.

(v) Given any ideal I of R and $r \in R$, $rM \subseteq IM$ implies that $(I :_R r)$ is not a maximal ideal.

Proof. $(i) \iff (ii)$ By Lemma 2.1.

$(ii) \implies (iii)$ Let $IM \subseteq JM$. Then $(IM :_R M) \subseteq (JM :_R M)$. By (ii), $I \subseteq J$.

$(iii) \implies (ii)$ We know that always $IM = (IM :_R M)M$. By (iii), $I = (IM :_R M)$.

$(iii) \iff (iv)$ It is obvious.

$(iv) \implies (v)$ Let $rM \subseteq IM$. By (iv), $r \in I$, and hence $(I :_R r) = R$. Therefore, $(I :_R r)$ is not a maximal.

$(v) \implies (iv)$ Let $rM \subseteq IM$ such that $r \notin I$. By Lemma 2.3, there exists an ideal K of R such that $I \subseteq K$, $r \notin K$ and $(K :_R r)$ is maximal ideal. But this is a contradiction. □

3 Properties of Multiplication-Like Modules

In this section we shall show that multiplication-like modules have some interesting properties.

Theorem 3.1. *Let M be a multiplication-like R-module. Then*

(i) M is a faithful module.

(ii) M is an r-multiplication module.

(iii) The set of all prime submodules of an R-module M is non-empty ($Spec_R(M) \neq \emptyset$).

(iv) For every ideal I of R, $Ann_R(I) = Ann_R(IM)$.

(v) $Z(R) = \{a \in R : \exists$ non − zero submodule N s.t $(N :_R M) \neq 0,\ a(N :_R M) = 0\}$ (here $Z(R)$ denotes the set of zero divisor of R).

Proof. (i) By Lemma 2.1, $0 = (0M :_R M) = Ann_R(M)$.

(ii) If there exists a proper ideal I of R such that $IM = M$, then $I = (IM :_R M) = R$. This is a contradiction and the proof is completed.

(iii) Let $m \in Max(R)$. By part (ii), $m = (mM :_R M)$. This shows that mM is a prime submodule of M.

(iv) It is enough to prove that $Ann_R(IM) \subseteq Ann_R(I)$. Now let $r \in Ann_R(IM)$, then $rIM = 0$. Now by using part (i), we have $Ir = 0$.

(v) Let $a \in Z(R)$. Then there exists $0 \neq b \in R$ such that $ab = 0$. It implies that $a(bM :_R M) = 0$, because M ia multiplication-like module. The converse is clear. □

The following examples show that Converse parts of the previous theorem do not hold in general.

Example 3.1. *Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$. It is clear that M satisfies in parts (i), (iii), (iv) and (v), but M is not a multiplication-like R-module.*

Example 3.2. *Let $R = \mathbb{Z}$ and $M = \bigoplus_{p \in max(R)} \mathbb{Z}_p$. Clearly M is an r-multiplication and $4\mathbb{Z} \neq (4\mathbb{Z}M :_R M) = 2\mathbb{Z}$. Therefore M is not a multiplication-like module.*

Lemma 3.1. *Let M be an R-module. Then M is multiplication-like and second module if and only if M is a vector space.*

Proof. It is sufficient to show that R is a field. For each non-unit such as $r \in R$, $rM \neq M$, as M is multiplication-like module. So $r = 0$, beacause M is second and faithful. The set of non-units is zero ideal. Therefore R is a field and M is a vector space. □

Lemma 3.2. *Let M be an r-multiplication module which every proper submodule of it is multiplication-like R-module. Then M is a multiplication-like R-module.*

Proof. Let I be an ideal of R. By Lemma 2.1, $I = (I^2M :_R IM)$. Let $rM \subseteq IM$. It follows that $IrM \subseteq I^2M$. So we have $r \in I$. Therefore, M is a multiplication-like R-module. □

Corollary 3.1. *Let M be a finitely generated R-module that every submodule of it is multiplication-like R-module. Then M is a multiplication-like module.*

Example 2.2, show that if R-module M is multiplication-like module, then every non-zero submodule of M need not necessarily be multiplication-like. By Theorem 3.1 (ii) and Proposition 2.11.24 [13], we get the following lemma.

Lemma 3.3. *Let M be a flat and multiplication-like R-module. Then M is a faithfully flat.*

If M is a multiplication (comultiplication) module, then it is not concluded that M is a multiplication-like and conversely.

Example 3.3.
(i) $\mathbb{Z}_n$ as $\mathbb{Z}$-module is a multiplication module, but it is not multiplication-like.

(ii) $\mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$-module is multiplication-like, but is not multiplication.

(iii) $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module is a comultiplication, but is not multiplication-like.

(iv) $\mathbb{Z}$ as a $\mathbb{Z}$-module is multiplication-like module, but is not a comultiplicatiom module, by Example 3.9 [3].

Remark 3.1. By Example 2.2, we can see that if R-module M is multiplication-like, then every submodule of M is not r-multiplication.

Proposition 3.1. *Let R be a Noetherian domain which is not a field and M be a multiplication-like R-module. Then every non-zero maximal submodule of M, is r-multiplication.*

Proof. Suppose that N is a non-zero maximal submodule of M. If N is not an r-multiplication, then there exists a proper ideal I of R such that $IN = N$.
Since N is a maximal submodule and M is multiplication-like, we must have $N = IM$ and $I = I^2 = (N :_R M)$. Hence there exists $a \in I$ such that $(1 - a)I = 0$. Since R is domain, we have $I = R$ or $I = 0$, which is a contradiction. □

Proposition 3.2. *Let R be a local Noetherian ring that is not a field and M be a multiplication-like R-module. Then every non-zero maximal submodule of M is r-multiplication.*

Proof. Suppose that N is a non-zero maximal submodule of M. If N is not an r-multiplication, then there exists a proper ideal I of R such that $IN = N$. Since N is a maximal submodule and M is multiplication-like, we have $N = IM$ and $I = I^2 = (N :_R M)$. By Nakayama lemma, $I = 0$, which is a contradiction. Hence N is an r-multiplication. □

Lemma 3.4. *Let M be a multiplication R-module. Then M is a multiplication-like if and only if M is finitely generated and faithful.*

Proof. Let M be a multiplication-like R-module. By Theorem 3.1, M is faithful and for each proper ideal I of R, $IM \neq M$. It follows that M is finitely generated. Conversely, let I be a proper ideal of R. Note that $IM = (IM :_R M)M$. Since M is multiplication, faithful and finitely generated, $I = (IM :_R M)$. Therefore, M is multiplication-like. □

Lemma 3.5. *Let M be a faithful multiplication R-module. Then M is an r-multiplication if and only if M is a multiplication-like.*

Proof. Let M be a multiplication-like R-module. By Theorem 3.1, M is r-multiplication. Conversely, let I be an ideal of R. Note that $IM =$

$(IM :_R M)M$. Since M is faithful multiplication and r-multiplication, so M is finitely generated. Now by Lemma 3.4, M is multiplication-like. □

Corollary 3.2. *If M is a multiplication and multiplication-like R-module, then $|Spec_R(M)|= |Spec(R)|$.*

Corollary 3.3. *Let M be a multiplication and multiplication-like R-module. Then for every I ideal of R, there exists an unique R-submodule K of M such that $I = (K :_R M)$.*

Corollary 3.4. *Let M be a multiplication-like R-module. Then M is multiplication if and only if for every I of R, there exists an unique submodule N of M such that $I = (N :_R M)$.*

Lemma 3.6. *Assume that M is a comultiplication and multiplication-like R-module. Then M is a strong comultiplication.*

Proof. Suppose N be a submodule of M. If there exist ideals I and J such that $N = (0 :_M I)$ and $N = (0 :_M J)$, then $IM = JM$, by Proposition 4.1 [12]. Now by Theorem 2.3, $I = J$. □

Proposition 3.3. *If M is a comultiplication and multiplication-like R-module, then for every submodule N of M, there exists an ideal I of R such that $(N :_R M) = Ann_R(I)$.*

Proof. Let N be a submodule of M. Since M is a comultiplication R-module, there exists an ideal I of R such that $N = (0 :_M I)$ and hence
$(N :_R M) = ((0 :_M I) :_R M) = Ann_R(IM) = Ann_R(I)$. □

Lemma 3.7. *Let M be a multiplication-like R-module. Then for every ideal I and J of R*

(i) $(IJM :_R M) = (IM :_R M)(JM :_R M)$.

(ii) $(IM + JM :_R M) = (IM :_R M) + (JM :_R M)$.

(iii) $((I\cap J)M :_R M) = (IM :_R M)\cap(JM :_R M)$.

Proof. This follows from Lemma 2.1. □

Remark 3.2. Lemma 3.7 shows properties which hold for multiplication-like modules (for ideals of ring), but part (ii) is not valid in general for submodules of module.

Consider $M = \mathbb{Z}[X]\oplus\mathbb{Z}[X]$ as $R = \mathbb{Z}[X]$-module. Then $((X) \oplus \mathbb{Z}[X] :_R M) + (\mathbb{Z}[X] \oplus (X) :_R M) \subset ((X) \oplus \mathbb{Z}[X] + \mathbb{Z}[X] \oplus (X) :_R M) = R$.

Proposition 3.4. *Let M be a Noetherian multiplication-like R-module. Then R is Noetherian.*

Proof. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be an ascending chain of ideals of R. It follows that $I_1M \subseteq I_2M \subseteq I_3M \subseteq \ldots$ is an ascending chain of submodules of M. So there exists a positive integer k such that $I_kM = I_{k+1}M = \ldots$, and hence $I_k = I_{k+1} = \ldots$, as M is multiplication-like. □

Proposition 3.5. *Let M be an Artinian multiplication-like R-module. Then R is Artinian.*

The following example shows that if M is multiplication-like over a Noetherian (Artinian) ring, then it is not necessarily to be a Noetherin (Artinian) module.

Example 3.4. *Let V be a vector space over a field F. It follows that V is multiplication-like and F is Artinian and Noetherian. But if V has an infinite dimension, then V is not Artinian and Noetherian.*

Proposition 3.6. *Let M be a faithful module over a Noetherian ring R such that for every primary ideal q of R, $q = (qM :_R M)$. Then M is multiplication-like.*

Proof. Let I be an ideal of R and let $I = \bigcap_{i=1}^n q_i$ be a reduced primary decomposition of I in R, where q_i are primary. It follows that
$I \subseteq (IM :_R M) = ((\bigcap_{i=1}^n q_i)M :_R M) \subseteq \bigcap_{i=1}^n(q_iM :_R M) = \bigcap_{i=1}^n q_i = I$. □

Lemma 3.8. *If R-module M is a multiplication-like R-module and each submodule of M has a reduced primary decomposition, then every ideal of R has a reduced primary decomposition.*

Proof. Let I be an ideal of R. Since M is multiplication-like it follows that $I = (IM :_R M)$. By hypothesis, $IM = \bigcap_{i=1}^n q_i$, when q_i are P_i-primary. Hence
$I = (IM :_R M) = (\bigcap_{i=1}^n q_i :_R M) = \bigcap_{i=1}^n(q_i :_R$

M).
It follows that I has reduced primary decomposition in R. □

Recall that an integral domain R is a valution ring if and only if the ideals of R are totally ordered by inclusion.

Lemma 3.9. *Let M be a multiplication-like R-module and R be an integral domain.*
Then for any submodules N, L of M, $(N :_R M) \subseteq (L :_R M)$ or $(L :_R M) \subseteq (N :_R M)$ if and only if R is valuation ring.

Proof. Obvious □

Proposition 3.7. *If for some $P \in Max(R)$, PM is a multiplication-like R-module, then M is a multiplication-like R-module.*

Proof. If $PM = M$, then the proof is complete. Now assume that $PM \neq M$ and let I be any ideal of R and $r \in (IM :_R M)$.
It implies that $rPM \subseteq PIM$. Hence $r \in I$. □

Remark 3.3. Example 2.6 shows that the converse of Proposition 3.7 is not true, in general.

Anderson and Fuller [1] called the submodule N a pure submodule, if $IN = N \cap IM$ for every ideal I of R.

Proposition 3.8. *Let N be a pure submodule of an R-module M. If N is multiplication-like, then M is a multiplication-like module.*

Proof. Let I be an ideal of R. Then $I = (IN :_R N)$. Assume that $rM \subseteq IM$. Since N is pure, we have $rN \subseteq IN$, and hence $r \in I$. Therefore, M is multiplication-like. □

Recall that a ring R is discrete valuation ring (DVR) if and only if it is valuation and Noetherian ring. If R is a DVR, then every non-zero ideal I of R is uniquely of the type $I = m^n$ (for some $n \in \mathbb{N}$), where m is the unique maximal ideal R.

Lemma 3.10. *Let M be a faithful finitely generated module over discrete valuation ring R. Then M is a multiplication-like.*

Proof. Let I be an ideal of R and m be the unique maximal ideal. Then there exists $n \in \mathbb{N}$ such that $I = m^n$. We have $m^n \subseteq (m^nM :_R M) \subseteq m^{n-1}$. Hence $m^n = (m^nM :_R M)$ or $(m^nM :_R M) = m^{n-1}$. If $(m^nM :_R M) = m^{n-1}$, then $m^nM = m^{n-1}M$. Hence by Nakayama lemma, $m = 0$ which is a contradiction. So $(m^nM :_R M) = m^n$. □

A Dedekind domain (D.d) is a Noetherian integrally closed domain in which every non-zero primes ideal is maximal.

Corollary 3.5. *Let M be a faithful finitely generated module over D.d R. Then for every non-zero prime ideal P of R, M_P is multiplication-like R_P-module.*

Proposition 3.9. *Let M be a faithful finitely generated R-module. Then for every radical ideal like I, $I = (IM :_R M)$.*

Proof. Let I be a radical ideal of R. Then $I = \sqrt{I} = \bigcap_{P \in V(I)} P$. For each $P \in V(I)$, $(PM :_R M) = P$, as M is a faithful finitely generated module. Thus

$I \subseteq (IM :_R M) = ((\bigcap_{P \in V(I)} P)M :_R M) \subseteq \bigcap_{P \in V(I)} (PM :_R M) = \bigcap_{P \in V(I)} P = I$. □

Lemma 3.11. *Let N be an R-submodule of M. If N is a multiplication-like such that for every ideal I of R, IN is primary-like submodule and $rad(IN) \subset N$, then M is a multiplication-like module.*

Proof. Let I be an ideal of R. Since N is a multiplication-like, $I = (IN :_R N)$. We show that $IM \subseteq IN$. It follows to show that $I \subseteq (IN :_R M)$. Let $r \in I$. Since $rad_R(IN) \subset N$, we can find an element $n \in N - rad_R(IN)$. Then $rn \in IN$. Hence $r \in (IN :_R M)$, as IN is primary-like. Therefore, M is multiplication-like. □

Lemma 3.12. *Let M be a distributive multiplication-like R-module and for any two submodule N and L of M, $(N :_R M) + (L :_R M) = (N + L :_R M)$. Then R is a distributive ring.*

Proof. Let A, B and C be ideals of R. Since M is multiplication-like, there exist submodules N, K and L of M such that $A = (N :_R M), B = (K :_R M)$ and $C = (L :_R M)$. Then

$(A + B) \cap C = ((N :_R M) + (K :_R M)) \cap (L :_R M) = (N + K :_R M) \cap (L :_R M)$
$= ((N + K) \cap L :_R M) = ((N \cap L) + (K \cap L) :_R M) = (N \cap L :_R M) + (K \cap L :_R M) = (N :_R M \cap (L :_R M) + (K :_R M) \cap (L :_R M) = A \cap C + B \cap C.$ □

The following example shows that in above theorem, the conditions, M is distributive and for any two submodule N and L of M, $(N :_R M) + (L :_R M) = (N + L :_R M)$ can not be omitted.

Example 3.5. *Let $M = \mathbb{Z}[X] \oplus \mathbb{Z}[X]$, $R = \mathbb{Z}[X]$, $N = (X) \oplus \mathbb{Z}[X]$ and $L = \mathbb{Z}[X] \oplus (X)$. It is clear that $((X) \oplus \mathbb{Z}[X] :_R M) + (\mathbb{Z}[X] \oplus (X) :_R M) \subset ((X) \oplus \mathbb{Z}[X] + \mathbb{Z}[X] \oplus (X) :_R M) = R$. Also R is not distributive, by Theorem 6.6 [8] and M is not distributive module, by [14].*

Proposition 3.10. *Let M be a multiplication-like R-module. If I is an ideal of R such that IM is a second submodule of M, then I is a second ideal of R.*

Proof. Let $\psi_a : I \longrightarrow I$ be the non-zero homomorphism defined by $r \longmapsto ar$. Thus $aIM \neq 0$, because M is faithful module. It follows that $aIM = IM$, since IM is a second submodule. Since M is multiplication-like

$$aI = (aIM :_R M) = (IM :_R M) = I.$$

□

Corollary 3.6. *Let M be a multiplication-like R-module. If I is an ideal of R such that IM is a second submodule of M, then for each non-zero $r \in R$, $r \in Z(R)$ or $I = Ir$.*

Lemma 3.13. *Let M be a multiplication-like and prime R-module. Then for any non-zero ideal I of R, $Ann_R(I) = 0$.*

Proof. Let I be any ideal of R. By Theorem 3.1 (i) and (iv), $Ann_R(I) = 0$. □

Corollary 3.7. *Let M be a multiplication-like and prime R-module. Then $Z(R) = 0$.*

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