Some New Results On the Hosoya Polynomial of Graph Operations

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(*Received 25 May, 2009*)

ABSTRACT

The Wiener index is a graph invariant that has found extensive application in chemistry. In addition to that a generating function, which was called the Wiener polynomial, who's derivate is a q-analog of the Wiener index was defined. In an article, Sagan, Yeh and Zhang in [The Wiener Polynomial of a graph, *Int. J. Quantun Chem.*, **60** (1996), 959−969] attained what graph operations do to the Wiener polynomial. By considering all the results that Sagan et al. admitted for Wiener polynomial on graph operations for each two connected and nontrivial graphs, in this article we focus on deriving Wiener polynomial of graph operations, Join, Cartesian product, Composition, Disjunction and Symmetric difference on n graphs and Wiener indices of them.

Keywords: Topological dimensionality, Sierpinski fractals, asymptotic Wiener index.

1 INTRODUCTION

Let G be a connected graph with vertex and edge set, $V(G)$ and $E(G)$, respectively. The distance between the vertices u and v of G is denoted by $d(u,v)$ and defined as the number of edges in a minimal path connecting the vertices u and v. The Wiener index of G is defined as the summation of all distances over all unordered pairs $\{u,v\}$ of vertices of G.

The Wiener index W is the first topological index to be used in chemistry [15]. Usage of topological indices in chemistry began in 1947, when chemist Harold Wiener used the Wiener index to determine the paraffin boiling point [3]. For more information or results on the Wiener index, its polynomial version, the chemical meaning and its history, we encourage the interested readers to consult the special issues of MATCH Communication in Mathematics and in Computer Chemistry [3], Discrete Applied Mathematics [4] and survey article [2]. For the polynomial aspect of the Wiener and other topological indices, we refer to [1,6−14]. Our notation is standard and taken mainly from the book of Imrich and Klavzar [5].

2 DEFINITIONS

In this section the concepts used throughout the paper are presented. The Wiener polynomial of G is defined as $W(G;q) = \sum_{\{u,v\}\subseteq V(G)} q^{d(u,v)}$, where q is a parameter. It is easy to see that the derivative of $W(G;q)$ is a q-analog of $W(G)$.

The join G_1+G_2 of graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is the graph with vertex set $V(G_1+G_2)=V_1\bigcup V_2$ and edge set $E(G_1+G_2)=E_1\cup E_2\cup\{uv: u\in V_1, v\in V_2\}$. For the other operations; Cartesian product, composition, disjunction and symmetric difference the vertex set is $V_1\times V_2$. The Cartesian product $G_1\times G_2$ has edge set $\{(u_1,u_2)(v_1,v_2) : (u_1v_1 \in E_1 \; and \; u_2 = v_2) \; or \; (u_2v_2 \in E_2 \; and \; u_1 = v_1)\}$, the composition $G_1\circ G_2$ has the edge set $\{(u_1,u_2)(v_1,v_2) : (u_1v_1 \in E_1) or (u_2v_2 \in E_2 \; and \; u_1 = v_1)\}$, the edge set of disjunction $G_1\vee G_2$ is $\{(u_1,u_2)(v_1,v_2) : (u_1v_1 \in E_1) or (u_2v_2 \in E_2) or \; both\}$ and the edge set for the symmetric difference $G_1\oplus G_2$ is $\{(u_1,u_2)(v_1,v_2) : u_1v_1 \in E_1 \; or \; u_2v_2 \in E_2 \; but \; not \; both\}$, see [5] for details. The ordered Wiener polynomial of G is denoted by $\overline{W}(G;q) = \sum_{(u,v)\subseteq V(G)} q^{d(u,v)}$, where the sum is over all ordered pairs (u,v) of vertices, including those vertices that $u=v$. Thus

$$\overline{W}(G;q) = 2W(G;q) + |V(G)| \qquad (1)$$

Throughout this paper, we only consider connected graphs and let for graphs G_i, $1 \le i \le n$, $|V(G_i)| = n_i$ and $|E(G_i)| = k_i$. It will be convenient to have a variable for the non-edges in G_i, so let $\bar{k}_i = \frac{n_i(n_i - 1)}{2} - k_i$. Also $\prod_{i\in\phi}|A_i| = 1$, where A_i is a set.

3 MAIN RESULTS

In this section the Hosoya polynomials of some graph operations are computed.

Lemma 1.

1) If G_1 and G_2 be connected graphs then G_1+G_2 is connected.
2) The join is associative.
3) $|E(G_1+G_2)| = k_1 + k_2 + n_1n_2$
4) Let $G_1, G_2, ..., G_m$ be a graphs then

$$|E(G_1+G_2+\ldots+G_m)| = \sum_{i=1}^{m} k_i + \sum_{i=2}^{m} n_i \sum_{j=1}^{i-1} n_j \quad .$$

Proof. The proof is straightforward and so omitted. □

Theorem 1. Let $G_1, G_2, \ldots, G_m$ be connected graphs. Then we have

$$W(G_1 + G_2 + \ldots + G_m; q) = \left(\Sigma_{i=1}^{m} k_i + \Sigma_{i=2}^{m} (n_i) \Sigma_{j=1}^{i-1} n_j \right) q + \sum_{i=1}^{m} \overline{k}_i \; q^2$$

Proof. Since distance for every distinct pair of vertices in G_1+G_2 is 1 or 2 by Lemma 1 the proof is clear. □

In the following lemma, some well-known properties of Cartesian product are introduced.

Lemma 2. Suppose G_1 and G_2 are graphs with $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, $|E(G_1)| = k_1$ and $|E(G_2)| = k_2$. Then the following are holds:

1) $G_1 \times G_2$ is connected graphs if and only if G_1 and G_2 are connected.

2) The Cartesian product is associative and commutative.

3) $|E(G_1 \times G_2)| = k_1 n_2 + k_2 n_1$,

4) Suppose G_1 and G_2 are connected and nontrivial (not equal to K_1).Then

$$\overline{W}(G_1 \times G_2; q) = \overline{W}(G_1; q) . \overline{W}(G_2; q) \tag{2}$$

Proof. The proof for parts 1 and 3 are trivial and for parts 2 and 4 see [7] and [1], respectively. □

Theorem 2. Let $G_1, G_2, \ldots, G_m$ be connected graphs then we have

$$W(G_1 \times G_2 \times \ldots \times G_m; q) = \frac{1}{2}\left[\prod_{i=1}^{m} [2W(G_i; q) + n_i] - \prod_{i=1}^{m} n_i \right]$$

Proof. By using Lemma 2 part 4 and utilize relation (1) we have;

$$\begin{aligned} W(G_1 \times G_2 \times \ldots \times G_m; q) &= \left(\overline{W}(G_1 \times G_2 \times \ldots \times G_m; q) - |V(G_1 \times G_2 \times \ldots \times G_m)| \right) / 2 \\ &= \frac{1}{2}\left[\prod_{i=1}^{m} [2W(G_i; q) + n_i] - \prod_{i=1}^{m} n_i \right] \end{aligned}$$ □

Lemma 3. Let G_1 and G_2 be connected graphs then we have:

1) $|E(G_1 \circ G_2)| = k_1 n_2^2 + k_2 n_1$

2) $W(G_1 \circ G_2; q) = n_1 \left(k_2 q + \overline{k_2} q^2 \right) + n_2^2 W(G_1; q)$

Proof. The proof of part 1 is clear. To prove part 2, we apply Lemma 2 of [10]. We have:

$$d_{G_1\circ G_2}((u_1,u_2),(v_1,v_2))=\begin{cases} d_{G_1}(u_1,u_2) & u_1\neq v_1\\ 0 & u_1=v_1\ \&\ u_2=v_2\\ 1 & u_1=v_1\ \&\ u_2v_2\in \mathrm{E}(G_2)\\ 2 & u_1=v_1\ \&\ u_2v_2\notin \mathrm{E}(G_2)\end{cases}$$ □

Theorem 3. Let $G_1, G_2, \ldots, G_m$ be connected graphs then we have

$$W(G_1\circ G_2\circ\ldots\circ G_m;q)=\left(\prod_{i=1}^{m-1}n_i\right)\left(k_mq+\overline{k}_mq^2\right)+\left(\prod_{i=2}^{m}n_i^2\right)W(G_1;q)$$
$$+\sum_{l=2}^{m-1}\left[\left(\prod_{i=1}^{m-1}n_i\right)\left(\prod_{j=m-l+2}^{m}n_j^2\right)\left(k_{m-l+1}q+\overline{k_{m-l+1}}q^2\right)\right].\quad \mathit{for}\ m\geq 3$$

Proof. The proof is by induction. The case $m=2$ is a consequence of Lemma 3. Suppose the result is valid for *m* graphs and we will prove its validity for *m+1* graph. Let $G=G_1\circ G_2\circ\ldots\circ G_m$. Then by Lemma 3

$$W(G\circ G_{m+1};q)=\left(\prod_{i=1}^{m}n_i\right)\left(k_{m+1}q+\overline{k_{m+1}}q^2\right)+n_{m+1}^2.W(G;q)$$
$$=\left(\prod_{i=1}^{m}n_i\right).\left(k_{m+1}q+\overline{k_{m+1}}q^2\right)+n_{m+1}^2.\left[\left(\prod_{i=1}^{m-1}n_i\right)\left(k_mq+\overline{k_m}q^2\right)+\sum_{l=2}^{m-1}\left(\prod_{i=1}^{m-1}n_i\right)\left(\prod_{j=m-l+2}^{m}n_j^2\right)\left(k_{m-l+1}q+\overline{k_{m-l+1}}q^2\right)\right.$$
$$\left.+\left(\prod_{i=2}^{m}n_i^2\right).W(G_1;q)\right]=\left(\prod_{i=1}^{m}n_i\right)\left(k_{m+1}q+\overline{k_{m+1}}q^2\right)+n_{m+1}^2\left(\prod_{i=1}^{m-1}n_i\right)\left(k_mq+\overline{k_m}q^2\right)$$
$$+n_{m+1}^2\sum_{l=2}^{m-1}\left[\left(\prod_{i=1}^{m-1}n_i\right)\left(\prod_{j=m-l+2}^{m}n_j^2\right)\left(k_{m-l+1}q+\overline{k_{m-l+1}}q^2\right)\right]+\left(\prod_{i=2}^{m+1}n_i^2\right).W(G_1;q)$$
$$=\left(\prod_{i=1}^{m}n_i\right)\left(k_{m+1}q+\overline{k_{m+1}}q^2\right)+\left(\prod_{i=2}^{m+1}n_i^2\right).W(G_1;q)$$
$$+\sum_{l=2}^{m}\left[\left(\prod_{i=1}^{m}n_i\right)\left(\prod_{j=m-l+3}^{m+1}n_j^2\right)\left(k_{m-l+2}q+\overline{k_{m-l+2}}q^2\right)\right].$$ □

Lemma 4. Let $\mathrm{G}_1,\mathrm{G}_2,\ldots,\mathrm{G}_\mathrm{m}$ be graphs, then we have

1) If G_1 and G_2 are connected then $\mathrm{G}_1\vee\mathrm{G}_2$ and $G_1\oplus G_2$ are connected.
2) Let $G=G_1\oplus G_2\oplus\ldots\oplus G_m$ then we have $|E(G)|=\sum_{\phi\neq A\subseteq M}(-4)^{|A|-1}\prod_{i\in A}k_i\prod_{i\in M-A}n_i^2$

3) Let $G = G_1 \vee G_2 \vee \ldots \vee G_m$ then we have $\left|E(G)\right| = \sum_{\phi \neq A \subseteq M} (-2)^{|A|-1} \prod_{i \in A} k_i \prod_{i \in M-A} n_i^2$ where $M = \{1,2,\ldots,m\}$.

Proof. The proof of part 1 is clear. We prove part 2 by induction on *m*. For *m=2* one can see $\left|E(G_1 \oplus G_2)\right| = k_1 n_2^2 + k_2 n_1^2 - 4k_1 k_2$. We now assume the result is valid for *m and* $H = G \oplus G_{m+1}$. So

$$\left|E(H)\right| = \left|E(G)\right| n_{m+1}^2 + k_{m+1}\left|V(G)\right|^2 - 4\left|E(G)\right| k_{m+1} \tag{2}$$

On the other hand we know $P(M \cup \{m+1\}) = P(M) \cup \{\{m+1\} \cup A \mid A \subseteq M\}$ (3)

where $P(M)$ is the power set of M. Clearly, $\phi = P(M) \cap \{\{m+1\} \cup A \mid A \subseteq M\}$ and so

$$\left|E(H)\right| = \sum_{\phi \neq A \subseteq M} (-4)^{|A|-1} \prod_{i \in A} k_i \prod_{i \in (M \cup \{m+1\})-A} n_i^2 + k_{m+1} \prod_{i \in M} n_i^2 - 4\, k_{m+1} \sum_{\phi \neq A \subseteq M} (-4)^{|A|-1} \prod_{i \in A} k_i \prod_{i \in M-A} n_i^2 =$$

$$= \sum_{\phi \neq B \subseteq M \cup \{m+1\}} (-4)^{|B|-1} \prod_{i \in B} k_i \prod_{i \in M \cup \{m+1\}-B} n_i^2$$

The proof of part 3 is similar to the proof of part 2. □

Theorem 4. Let $G_1, G_2, \ldots, G_m$ be connected graphs then

$$W(G_1 \vee G_2 \vee \ldots \vee G_m; q) = \left[\sum_{\phi \neq A \subseteq M} (-2)^{|A|-1} \prod_{i \in A} k_i \prod_{i \in M-A} n_i^2\right] q + \left[\binom{\prod_{i=1}^{m} n_i}{2} - \sum_{\phi \neq A \subseteq M} (-2)^{|A|-1} \prod_{i \in A} k_i \prod_{i \in M-A} n_i^2\right] q^2.$$

and

$$W(G_1 \oplus G_2 \oplus \ldots \oplus G_m; q) = \left(\sum_{\phi \neq A \subseteq M} (-4)^{|A|-1} \prod_{i \in A} k_i \prod_{i \in M-A} n_i^2\right) q + \left(\binom{\prod_{i=1}^{m} n_i}{2} - \left(\sum_{\phi \neq A \subseteq M} (-4)^{|A|-1} \prod_{i \in A} k_i \prod_{i \in M-A} n_i^2\right)\right) q^2$$

Proof. Since distance between distinct vertices of graphs $G_1 \oplus G_2$ and $G_1 \vee G_2$ is 1 or 2, $W(G;q) = \left|E(G)\right| q + \left|E(\overline{G})\right| q^2$. We now apply Lemma 4 to complete the proof. □

We conclude this paper by computing the Wiener index of the operations on *m* graphs. We mentioned that the derivative of $W(G;q)$ is q-analog of $W(G)$. By Theorem [1,1.5], $W'(G;1) = W(G)$ and we have:

$$W(G_1+G_2+\ldots+G_m)=\sum_{i=1}^{m}k_i+\sum_{i=2}^{m}\left(n_i\sum_{j=1}^{m}n_j\right)+2\sum_{i=1}^{m}k_i$$

$$W(G_1\times G_2\times\ldots\times G_m)=\sum_{i=1}^{m}\left(W(G_i)\prod_{\substack{j=1\\ j\neq i}}^{m}n_j^2\right)$$

$$W(G_1\circ G_2\circ\ldots\circ G_m)=\left(\prod_{i=1}^{m-1}n_i\right)\left(k_m+2\overline{k_m}\right)+\sum_{l=2}^{m-1}\left[\left(\prod_{i=1}^{m-1}n_i\right)\left(\prod_{j=m-l+2}^{m}n_j^2\right)\left(k_{m-l+1}+2\overline{k_{m-l+1}}\right)\right]$$
$$+\left(\prod_{i=2}^{m}n_j^2\right).W(G_1) \qquad \text{for } m\geq 3$$

$$W(G_1\vee G_2\vee\ldots\vee G_m)=2\binom{\prod_{i=1}^{m}n_i}{2}-\sum_{\phi\neq A\subseteq M}(-2)^{|A|-1}\prod_{i\in A}k_i\prod_{i\in A^c}n_i^2 \qquad \textit{where } M=\{1,2,\ldots,m\}$$

$$W(G_1\oplus G_2\oplus\ldots\oplus G_m)=2\binom{\prod_{i=1}^{m}n_i}{2}-\sum_{\phi\neq A\subseteq M}(-4)^{|A|-1}\prod_{i\in A}k_i\prod_{i\in A^c}n_i^2 \qquad \textit{where } M=\{1,2,\ldots,m\}.$$

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