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Eccentric and Total Eccentric Connectivity Indices of Caterpillars

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ABSTRACT

Consider a simple connected graph G with the vertex set $V(G)$. The eccentric and total connectivity indices of G are defined as $\xi^c(G) = \sum_{v\in V(G)} ec_G(v)deg_G(v)$ and $EC(G) = \sum_{v\in V(G)} ec_G(v)$, respectively. Here, $deg_G(v)$ denotes the degree of a vertex v and $ec_G(v)$ is its eccentricity. In this paper, these two indices are calculated for a given arbitrary caterpillar.

Keywords: Eccentric connectivity index, total eccentricity index, caterpillar.

1 INTRODUCTION

If G is a simple connected graph, $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. $|V(G)| = n(G)$ is called the order of G. Moreover, the distance between u and v in G, $d_G(u, v)$, is the length of the shortest $u - v$ path in G. The eccentricity of a vertex $v \in V(G)$, $ec_G(v)$, is the maximum distance between v and any other vertex in G. We define the eccentric connectivity index of G as $\sum_{v\in V(G)} ec_G(v)deg_G(v)$ and total eccentricity of G as $\sum_{v\in V(G)} ec_G(v)$.

By a caterpillar we mean a tree such that by deleting all leaves we obtain a path which is called the spine of the caterpillar. The vertices of the spine are called the main vertices and the vertices with degree one are called the secondary vertices. We denote a caterpillar on m main vertices by T_m.

Throughout this paper, graphs have been assumed to be simple and connected, unless stated otherwise. A path on n vertices is denoted by P_n. Moreover, if no ambiguity is possible, the subscript G may be omitted.

2. MAIN RESULTS

In this section, we calculate the eccentric and total connectivity indices of an arbitrary caterpillar on m main vertices, T_m. By definition, it is clear that for determining a caterpillar, the number of main vertices and the number of leaves of each main vertex (secondary vertex) should be known. Let us denote the i −th main vertex (from the left side) by u_i and the j-th secondary vertex of u_i by v_{ij}. We also define a map f on the main vertices of T_m as follows:

$$f:\{u_1,u_2,\dots,u_m\}\longrightarrow N\cup\{0\}$$
$$f(u_1)=\deg(u_1)-1$$
$$f(u_m)=\deg(u_m)-1$$
$$f(u_i)=\deg(u_i)-2 \qquad (i=2,3,\dots,m-1)$$

In a caterpillar, the eccentricity of each main vertex is equal to the eccentricity of ist spine added by one and the eccentricity of each secondary vertex is number one plus the eccentricity of the corresponding main vertex. If we denote the set of main vertices by V_1 and the set of secondary vertices by V_2, then :

$$\xi^c(T_{2k})=\sum_{v\epsilon V_1(T_{2k})} ec(v)\deg(v)+\sum_{v\epsilon V_2(T_{2k})} ec(v)\deg(v) \qquad (m=2k)$$

We calculate each part separately:

$$\sum_{v\epsilon V_1(T_{2k})} ec(v)\deg(v)=$$
$$(f_1+1)(2k)+(f_2+2)(2k-1)+\cdots+(f_k+2)(2k-k+1)$$
$$+(f_{k+1}+2)(2k-k+1)+\cdots+(f_{2k-1}+2)(2k-1)+(f_{2k}+1)(2k)$$
$$=(2k)(f_1+f_2+\cdots+f_{2k})+(-f_2+\cdots+(-k+1)f_k+(-k+1)f_{k+1}+$$
$$\dots+(-1)f_{2k-1})+2(2k+(4k-2)+\cdots+(4k-2k+2))$$
$$=2k(\sum_{i=1}^{2k} f_i)+\sum_{i=1}^{k}(-i+1)(f_i+f_{2k-i+1})+6k^2-2k$$

and,

$$\sum_{v\epsilon V_2(T_{2k})} ec(v)\deg(v)=\sum_{i=1}^{2k}(ec(u_i)+1)f_i=\sum_{i=1}^{2k} ec(u_i)f_i+\sum_{i=1}^{2k} f_i$$
$$=(2k)\left(\sum_{i=1}^{2k} f_i\right)+\sum_{i=1}^{k}(-i+1)f_i+\sum_{i=1}^{k}(-i+1)f_{2k-i+1}+\sum_{i=1}^{2k} f_i$$

$$= (2k+1)\sum_{i=1}^{2k} f_i + \sum_{i=1}^{k}(-i+1)(f_i + f_{2k-i+1})$$

As a consequence,

$$\xi^c\big(T_{2k(=m)}\big) = (4k+1)\textstyle\sum_{i=1}^{2k} f_i + 2\sum_{i=1}^{k}(-i+1)(f_i + f_{2k-i+1}) + 6k^2 - 2k$$

$$= (4k+1)\sum_{i=1}^{2k} \deg(u_i) + 2\sum_{i=2}^{k}(-i+1)(\deg(u_i) + \deg(u_{2k-i+1}))$$

$$-2(3k^2 + k - 1)$$

$$= (2m+1)\sum_{i=1}^{m} \deg(u_i) + 2\sum_{i=2}^{m/2}(-i+1)(\deg(u_i) + \deg(u_{m-i+1}))$$

$$-\,{}^{3}\!/_{2}\, m^2 - m + 2$$

$$= 2m\sum_{i=1}^{m} \deg(u_i) + \sum_{i=1}^{m} \deg(u_i) + 2\sum_{i=2}^{m/2}(-i)(\deg(u_i) + \deg(u_{m-i+1})$$

$$+ 2\sum_{i=2}^{m/2}(\deg(u_i) + \deg(u_{m-i+1})) - {}^{3}\!/_{2}\, m^2 - m + 2.$$

Since $\sum_{i=1}^{m} \deg(u_i) = m + n - 2$ $(n = |V_{T_{2k}}|)$,

$$\xi^c\big(T_{m(=2k)}\big) = 2\sum_{i=2}^{m/2}(-i)(\deg(u_i) + \deg(u_{m-i+1})) - 2\deg(u_1) - 2\deg(u_m)$$

$$+ {}^{1}\!/_{2}\, m^2 + 2(n-1)m + 3n - 4$$

By using the same method, we can obtain the following results:

$$\sum\nolimits_{v\epsilon V_1(T_{2k+1})} ec(v)\deg(v)$$

$$= (2k)\left(\sum_{i=1}^{2k+1} f_i + 4k\right) + f_1 + f_{2k+1} + 2 + (-k+1)(f_{k+1} + 2)$$

$$+ \sum_{i=3}^{k}(-i+2)(f_i + f_{2k-i+2} + 4)$$

and,

$$\sum_{v\epsilon V_2(T_{2k+1})} ec(v)\deg(v)$$

$$= (2k+1)\sum_{i=1}^{2k+1} f_i + (-k+1)f_{k+1} + \sum_{i=1}^{k}(-i+2)(f_i + f_{2k-i+2}).$$

Thus,

$$\xi^c(T_{2k+1(=m)}) = (4k+1)\sum_{i=1}^{2k+1} f_i + 2(f_1 + f_{2k+1}) + (-2k+2)f_{k+1}$$

$$+ 2\sum_{i=3}^{k}(-i+2)(f_i + f_{2k-i+2}) + 6k^2 + 4k$$

$$= (4k+1)\sum_{i=1}^{2k+1} \deg(u_i) + 2\sum_{i=3}^{k}(-i+2)(\deg(u_i) + \deg(u_{2k-i+2}))$$

$$+ 2\deg(u_1) + 2\deg(u_{2k+1}) + (2-2k)\deg(u_{k+1}) - 6k^2 - 8k$$

$$= (2m-1)\sum_{i=1}^{m} \deg(u_i) + 2\sum_{i=3}^{(m-1)/2}(-i+2)(\deg(u_i) + \deg(u_{m-i+1}))$$

$$+ 2\deg(u_1) + 2\deg(u_m) + (3-m)\deg\left(u_{(m+1)/2}\right) - {}^3/_2\, m^2 - m + {}^5/_2$$

If $n = |V_{T_{2k+1}}|$ then

$$\xi^c(T_{m(=2k+1)}) = (2m-1)(m-2+n) + 2\sum_{i=3}^{(m-1)/2}(-i)(\deg(u_i) + \deg(u_{m-i+1})) + 4(m+n$$

$$-2-\deg(u_1) - \deg(u_2) - \deg\left(u_{(m+1)/2}\right) - \deg(u_{m-1}) - \deg(u_m))$$

$$+ 2\deg(u_1) + 2\deg(u_m) + (3-m)\deg\left(u_{(m+1)/2}\right) - {}^3/_2\, m^2 - m + {}^5/_2$$

$$= 2\sum_{i=3}^{(m-1)/2}(-i)(\deg(u_i) + \deg(u_{m-i+1})) - 2\deg(u_1) - 4\deg(u_2)$$

$$-(m+1)\deg\left(u_{(m+1)/2}\right) - 4\deg(u_{m-1}) - 2\deg(u_m) + {}^1/_2\, m^2$$

$$+ 2(n-1)m + 3n - {}^7/_2$$

Now, we can conclude one of our main results as follows:

Theorem 2. 1. Let T_m be an arbitrary caterpillar on m main vertices. If $|V_{T_m}| = n$ and u_i denotes the i −th main vertex from the left side, then

$$\xi^c(T_m) = \begin{cases} 2\sum_{i=2}^{m/2}(-i)(\deg(u_i) + \deg(u_{m-i+1})) - 2\deg(u_1) - 2\deg(u_m) & m = 2k \\ + 1/2\, m^2 + 2(n-1)m + 3n - 4 & \\ 2\sum_{i=3}^{(m-1)/2}(-i)(\deg(u_i) + \deg(u_{m-i+1})) - 2\deg(u_1) - 4\deg(u_2) & m = 2k+1 \\ -(m+1)\deg\left(u_{(m+1)/2}\right) - 4\deg(u_{m-1}) - 2\deg(u_m) + 1/2\, m^2 & \\ +2(n-1)m + 3n - 7/2 & \end{cases}.$$

Theorem 2. 2. By the previous theorem's conditions, we have:

$$EC(T_m) = \begin{cases} \sum_{i=2}^{m/2}(-i)(\deg(u_i) + \deg(u_{m-i+1})) - \deg(u_1) - \deg(u_m) & m = 2k \\ + 1/4\, m^2 + (n - 1/2)m + 2(n-1) & \\ \sum_{i=3}^{(m-1)/2}(-i)(\deg(u_i) + \deg(u_{m-i+1})) - \deg(u_1) - 2\deg(u_2) & \\ -\frac{m+1}{2}\deg\left(u_{(m+1)/2}\right) - 2\deg(u_{m-1}) - \deg(u_m) + 1/4\, m^2 & \\ +(n - 1/2)m + 2n - 7/4 & m = 2k+1 \end{cases}.$$

Proof. By using the methods described for calculation of $\xi^c(T_n)$, we have:

$$EC(T_{2k}) = (2k+1)\sum_{i=1}^{2k} f_i + \sum_{i=2}^{k}(-i+1)(f_i + f_{2k-i+1}) + 3k^2 + k$$

$$= (2k+1)\sum_{i=1}^{2k} \deg(u_i) + \sum_{i=2}^{k}(-i+1)(\deg(u_i) + \deg(u_{2k-i+1}) - 3k^2 - k + 2$$

$$= \sum_{i=2}^{m/2}(-i)(\deg(u_i) + \deg(u_{m-i+1})) - \deg(u_1) - \deg(u_m) + {}^1/_4\, m^2$$

$$+ \left(n - {}^1/_2\right)m + 2(n-1)$$

The second part is similar to argument given for the first one and so omitted.

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