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Energy and Wiener Index of Zero–Divisor Graphs

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ABSTRACT

Let R be a commutative ring and $\Gamma(R)$ be its zero–divisor graph. In this article, we study Wiener index and energy of $\Gamma(Z_n)$ where $n = pq$ or $n = p^2q$ and p, q are primes. A MATLAB code for our calculations is also presented.

Keywords: Zero–divisor graph, energy, Wiener index.

1. INTRODUCTION

The Wiener index of a graph was the first reported topological index based on graph distances, see [1]. This index is defined as the sum of all distances between vertices of the graph. We encourage the reader to see the special issue of MATCH Communications in Mathematical and in Computer Chemistry for more information on this topic [2].

In this article we discuss the eigenvalue and Wiener index of the graph zero-divisors of Z_n. Throughout this paper, all rings are assumed to be commutative with unity 1. If R is a ring, $Z(R)$ denotes its set of all zero–divisors and $Z^*(R)$ denotes its set of all nonzero zero-divisors. By the zero–divisor graph of R, denoted $\Gamma(R)$, we mean the graph whose vertices are the nonzero zero-divisors of R and for distinct r,s in $Z^*(R)$, there is an edge connecting r and s if and only if $rs = 0$. We use $M(\Gamma(R))$ to denote adjacency matrix of $\Gamma(R)$ [3].

Consider Z_n, the set of integers modulo n. For $n=p$, where p is a prime number, the graph has no edges as Z_n in this case is a field and has no nonzero zero-divisors. Let $n = p^2$ where p is a prime number; in this case $Z^*(Z_n) = \{p, 2p, \ldots, (p-1)p\}$. For every x, y in $Z^*(Z_n)$, clearly $xy = 0$. So, $|\Gamma(Z_{p^2})| = p - 1$ and distinct vertices of $\Gamma(Z_{p^2})$ are adjacent. Therefore $\Gamma(Z_{p^2}) = K_{p-1}$. Let $n=pq$, p and q are primes. In this case, we can write:

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$$Z^*(Z_n) = A \bigcup B$$
$$A = \{kp \mid k = 1,2,...,q-1\}$$
$$B = \{kq \mid k = 1,2,...,p-1\}$$

We can easily prove that for every x,y in Z_n, $xy=0$ if and only if $x \in A$, $y \in B$ or $x \in B$, $y \in A$. So $\Gamma(Z_{pq}) = K_{p-1,q-1}$. Now, suppose that $n = p^2q$. In this case, we have $Z^*(Z_n) = A \bigcup B \bigcup C \bigcup D$ where:

$$A = \{x \mid x = kp\}$$
$$B = \{x \mid x = kq\}$$
$$C = \{x \mid x = kp^2\}$$
$$D = \{x \mid x = kpq\}$$

Notice that p and q don't divide k. Since $n=p^2q$, for every $x, y \in Z_n$, $xy = 0$ if and only if $x \in A$, $y \in D$ or $x \in B$, $y \in C$ or $x \in C$, $y \in D$ or $x, y \in D$. Thus, we have:

$$M(\Gamma(Z_n)) = \begin{bmatrix} 0 & 0 & 0 & M_1 \\ 0 & 0 & M_2 & 0 \\ 0 & M_2^T & 0 & M_3 \\ M_1^T & 0 & M_3^T & M_4 - I \end{bmatrix}$$

where M_i is an all one matrix, M_i^T is its transpose and I is an identity matrix.

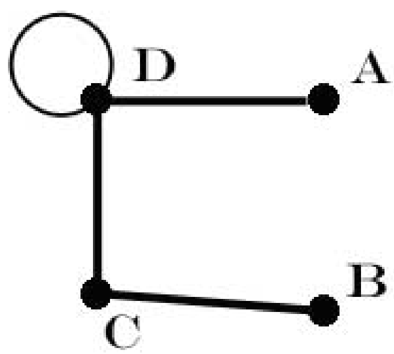

Figure 1. The General Form of $\Gamma(Z_{p^2q})$.

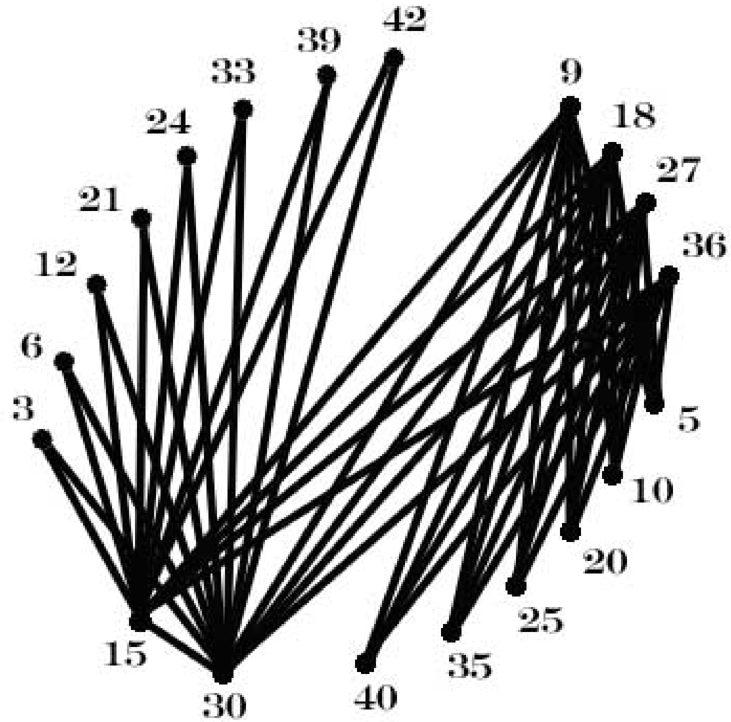

Figure 2. The 2D Perception of $\Gamma(Z_{63})$.

2. WIENER AND ENERGY OF $\Gamma(Z_n)$

In this section, we calculate energy and Wiener index of zero−devisor graph of two rings Z_{p^2} and Z_{p^2q}. In the graph theory, the energy of a graph is sum of absolute of adjacency matrix eigenvalues, see [4–10]. If we denote the length of shortest path between every pair of vertices $x, y \in \Gamma(R)$ with *d(x,y)*, then the Wiener index is

$$W(\Gamma) = \sum_{x,y \in V(\Gamma)} d(x,y)$$

Theorem 1. If p is a prime number then energy of $\Gamma(Z_{p^2})$ is $2p-4$.

Proof: Let p be a prime number. According to above we know $\Gamma(Z_{p^2}) = K_{p-1}$. Therefore:

$$M(\Gamma(Z_{p^2})) = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}_{p-1,q-1}$$

We have:

$$f(\lambda)=|\lambda I_{p-1}-M(\Gamma(Z_{p^2}))|=\begin{vmatrix}\lambda & -1 & \cdots & -1\\ -1 & \lambda & \cdots & -1\\ \vdots & \vdots & & \vdots\\ -1 & -1 & \cdots & \lambda\end{vmatrix}_{p-1}=(\lambda-1)^{p-2}(\lambda+p-2)$$

.

If $f(\lambda)=0$ then $\lambda=1,2-p$. Therefore $\sum_{i=1}^{p-1}|\lambda_i|=2p-4$. □

Theorem 2. If p and q are two prime numbers, then only non zero eigenvalues of $M(\Gamma(Z_{pq}))$ are $\pm\sqrt{(p-1)(q-1)}$ and so $E(\Gamma(Z_{pq}))=2\sqrt{(p-1)(q-1)}$.

Proof: Suppose that p and q are two prime numbers. Since $\Gamma(Z_{pq})$ is bipartite,

$$M(\Gamma(Z_{pq}))=\begin{bmatrix}0 & M_1\\ M_2 & 0\end{bmatrix}$$

where M_1 and M_2 are two all one matrices of dimensions $(p-1)\times(q-1)$ and $(q-1)\times(p-1)$, respectively. Then by an easy calculation, we have:

$$f(\lambda)=|\lambda I_{p+q-2}-M(\Gamma(Z_{pq}))|=\lambda^{p+q-4}(\lambda^2-(p-1)(q-1))$$

Thus nonzero eigenvalues are $\pm\sqrt{(p-1)(q-1)}$ and so $E(\Gamma(Z_{pq}))=2\sqrt{(p-1)(q-1)}$. □

Theorem 3. If p is a prime number then $W(\Gamma(Z_{p^2}))$ is $(p-1)(p-2)/2$.

Proof: Let p be a prime number. According above we know $\Gamma(Z_{p^2})=K_{p-1}$. Hence for every $x,y\in\Gamma(Z_{p^2})$, $d(x,y)=1$. Therefore

$$W(\Gamma(Z_{p^2}))=\sum d(x,y)=\sum_{x\neq y}1=\binom{p-1}{2}=(p-1)(q-1)/2.$$ □

Theorem 4. If p and q are two prime numbers then

$$W(\Gamma(Z_{pq}))=p^2+q^2+pq-4p-4q+5.$$

Proof: Let p and q be two prime numbers. According to the below section we know $\Gamma(Z_{pq})=K_{p-1,q-1}$. So we can write $Z^*(Z_{pq})=A\cup B$ where $A=\{kp \mid k=1,2,\dots,q-1\}$ and $B=\{kq \mid k=1,2,\dots,p-1\}$. For every $x\in A$:

$$\begin{aligned} W(x) &= \sum_{y\in B} d(x,y) + \sum_{y\in A, x\neq y} d(x,y) \\ &= \sum_{y\in B} 1 + \sum_{y\in A, x\neq y} d(x,y) \\ &= p-1+2(q-1) \end{aligned}$$

and for every $x \in B$:

$$\begin{aligned} W(x) &= \sum_{y\in A} d(x,y) + \sum_{y\in B, x\neq y} d(x,y) \\ &= \sum_{y\in A} 1 + \sum_{y\in B, x\neq y} d(x,y) \\ &= q-1+2(p-2) \end{aligned}.$$

Therefore,

$$\begin{aligned} W(\Gamma(Z_{pq})) &= \frac{1}{2}\left[\sum_{x\in A} W(x) + \sum_{x\in B} W(y)\right] \\ &= \frac{1}{2}\left[(q-1)(p-1+2(q-2)) + (p-1)(q-1+2(p-2))\right]. \\ &= p^2+q^2+pq-4p-4q+5 \end{aligned}$$

□

3. Computer Program

In this section we offer an algorithm for calculating energy and Wiener index with Matlab software. This algorithm includes several sub algorithms. It is enough to input *n*. In the first stage, we obtain M(Γ(Z_n)) and plot Γ(Z_n) by function "*graph_zero_divisor_zn2*". In the second stage, we calculate Energy and Wiener index with using "*energy*" and "*Wiener*" functions:

```
function Gz=graph_zero_divisor_zn2(p)
n=p;
M=[];
for i=1:n-1
   for j=1:n-1
      if mod(i*j,n)==0
         M=[M,i];
         break;
      end
   end
```

```
end


n=length(M);
for i=0:n-1
   axes(i+1,:)=[cos(2*pi*i/n),sin(2*pi*i/n)];
end
Gz=zeros(n);
hold on
for i=1:n
   plot(axes(i,1),axes(i,2),'*')
   if mod(M(i)^2,p)==0
      Gz(i,i)=1;
      plot(axes(i,1),axes(i,2),'rO')
   end
end
for i=1:n-1
   for j=i+1:n
      if mod(M(i)*M(j),p)==0
         Gz(i,j)=1;Gz(j,i)=1;
         plot(axes([i,j],1),axes([i,j],2));
      end
   end
end
function s=Wiener(B)
B(B==0)=inf;
A=triu(B,1)+tril(B,-1);
m=length(A);
B=zeros(m);
j=1;
while j<=m
    for i=1:m
        for k=1:m
            B(i,k)=min(A(i,k),A(i,j)+A(j,k));
        end
    end
   A=B;
   j=j+1;
end
s=sum(sum(A))/2;
function e=energy(a)
eg=eig(a);
e=sum(abs(eg));
```

In Table 1, some calculations given by our program are presented.

Table 1. The Values of $W(\Gamma(Z_n))$ and $E(\Gamma(Z_n))$ for n = 15, 24, 36 and 63.

n	$W(\Gamma(Z_n))$	$E(\Gamma(Z_n))$
15	22	5.6569
24	208	15.6546
36	520	22.6262
63	649	23.0556

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