Remarks on Distance–Balanced Graphs

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ABSTRACT

Distance-balanced graphs are introduced as graphs in which every edge uv has the following property: the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. Basic properties of these graphs are obtained. In this paper, we study the conditions under which some graph operations produce a distance-balanced graph.

Keywords: Distance-balanced graphs, graph operation.

1. INTRODUCTION

For an edge e = ab of a graph G, let $n_a^G(e)$ be the number of vertices closer to a than to b. That is, $n_a^G(e) = |\{u \in V(G) \mid d(u, a) < d(u, b)\}|$. In addition, let $n_0^G(e)$ be the number of vertices with equal distances to a and b; $n_0^G(e) = |\{u \in V(G) \mid d(u, a) = d(u, b)\}|$.

Here is our key definition. We call a graph G distance-balanced, if $n_a^G(e) = n_b^G(e)$ holds for any edge e = ab of G. These graphs were, at least implicitly, first studied by Handa [4] who considered distance-balanced partial cubes. The term itself, however, is due to Jerebic et al. [1] who studied distance-balanced graphs in the framework of various kinds of graph products. The transmission T(u) of a vertex $u \in V$ is defined as follows:

$$T(u) = \sum_{v \in V} d(u, v).$$

A graph *G* is said to be transmission-regular if all its vertices have the same transmission. As examples of transmission-regular graphs, we can cite the complete graph K_n on $n \ge 2$ vertices, the complete bipartite graph $K_{n,n}$ on $2n \ge 2$ vertices.

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Let *G* and *H* be two graphs. The corona product G o H is obtained by taking one copy of *G* and |V(G)| copies of *H*; and by joining each vertex of the i-th copy of *H* to the ith vertex of *G*, i = 1, 2, ..., |V(G)|, see [2,3]. The join G + H of graphs *G* and *H* with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G \bigcup H$ together with all the edges joining V_1 and V_2 . The symmetric difference $G \oplus H$ of two graphs *G* and *H* is the graph with vertex set $V(G) \times V(H)$ and edge set

 $E(G \oplus H) = \{ (u_1, u_2)(v_1, v_2) \mid u_1 v_1 \in E(G) \text{ or } u_2 v_2 \in E(H) \text{ but not both} \}.$

The cluster G{H} is obtained by taking one copy of *G* and |V(G)| copies of a rooted graph *H*, and by identifying the root of the ith copy of *H* with the ith vertex of *G*, i = 1,2,..., |V(G)|. The composite graph G{H} was studied by Schwenk [9]. Throughout this paper our notation is standard and taken mainly from the standard book of graph theory. We encourage the reader to consult papers [5,7,8,10–12] for background material as well as basic computational techniques.

2. MAIN RESULTS

A regular graph is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree k is called a k-regular graph or regular graph of degree k. In this section, we study the conditions under which some graph operations produce a distance-balanced graph. We begin by the following theorem which states the relationship between distance-balanced and transmission-regular graphs:

Theorem 1. A graph G is distance-balanced if and only if G is transmission-regular.

Proof. It is well–known fact that if *G* is a connected graph and $uv = e \in E(G)$, then $n_u^G(e) = n_v^G(e)$ if and only if T(u) = T(v) [6], proving the result.

Theorem 2. Let *G* and *H* be connected graphs. Then G + H is distance-balanced if and only if *G* and *H* are *r* and *k* regular graphs, respectively, and |V(G)| - r = |V(H)| - k.

Proof. Consider the following partition of E(G + H):

$$\begin{aligned} A &= \left\{ uv \in E(G+H) \mid u, v \in V(G) \right\}, \\ B &= \left\{ uv \in E(G+H) \mid u, v \in V(H) \right\}, \\ C &= \left\{ uv \in E(G+H) \mid u \in V(G) \text{ and } v \in V(H) \right\}. \end{aligned}$$

We first assume that *G* and *H* are r- and k-regular graphs respectively, and |V(G)| - r = |V(H)| - k. Let $uv = e \in A$ and $m_0^G(e) = |\{x \in V(G) \mid d(u, x) = d(v, x) = 1\}|$. Notice that

$$d_{G+H}(x, y) = \begin{cases} 0 & x = y \\ 1 & (x \in V(G) \text{ and } y \in V(H)) \text{ or } (xy \in E(H)) \text{ or } (xy \in E(G)) \\ 2 & \text{otherwise} \end{cases}$$

Thus we have $n_u^{G+H}(e) = \deg_G(u) - m_0^G(e)$ and $n_v^{G+H}(e) = \deg_G(v) - m_0^G(e)$. Since *G* is regular $\deg_G(u) = \deg_G(v)$, and thus $n_u^{G+H}(e) = n_v^{G+H}(e)$. We now assume that $uv = e \in B$. In a similar way we can see that $n_u^{G+H}(e) = n_v^{G+H}(e)$. Assume that $uv = e \in C$. Then we have $n_u^{G+H}(e) = |V(H)| - \deg_H(v)$ and $n_v^{G+H}(e) = |V(G)| - \deg_G(u)$. Therefore, $n_u^{G+H}(e) = n_v^{G+H}(e)$ and thus G + H is distance-balanced. Conversely, assume that G + H is distance-balanced. By above argument for an edge e of A, we see $n_u^{G+H}(e) = n_v^{G+H}(e)$ implies that any two adjacent vertices of *G* have the same degree. Since *G* is connected, this implies that *G* is r-regular for some *r*. In a similar way we can see that *H* is k-regular, for some *k*. For an edge $uv = e \in C$, it follows again from earlier analysis that $n_u^{G+H}(e) = |V(H)| - \deg_H(v)$ and $n_v^{G+H}(e) = |V(G)| - \deg_G(u)$. Since G + H is distance - balanced, two above equations imply that $|V(H)| - \deg_H(v) = |V(G)| - \deg_G(u)$.

Corollary. Let *G* and *H* be connected graphs. G + H is transmission-regular if and only if *G* and *H* be *r* and *k* regular respectively, such that |V(G)| - r = |V(H)| - k.

Proof. The proof follows from Theorems 1 and 2.

A graph *G* is called nontrivial if |V(G)| > 1.

Theorem 3. The corona product of two arbitrary, nontrivial and connected graphs is not distance–balanced.

Proof. Let *G* and *H* be arbitrary, nontrivial and connected graphs and H_i be the i-th copy of *H*. Assume that $uv = e \in E(GoH)$ such that $u \in V(G)$ and $v \in V(H_i)$. Thus, we have :

$$n_u^{GoH}(e) = |V(G)| (|V(H)| + 1) - \deg_{GoH}(v) \text{ and } n_v^{GoH}(e) = 1.$$

Therefore, we have n_u^{GoH} (e) $\neq n_v^{GoH}$ (e). Thus GoH is not distance-balanced.

Corollary. The corona product of two arbitrary, nontrivial and connected graphs is not transmission - regular.

Proof. The proof follows from Theorems 1 and 3.

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Let $e = (a,x)(b,y) \in E(G \oplus H)$ such that $ab \in E(G)$, and $N_{(a,x)}(e) = \{(u,v) \in V(G \oplus H) | d((u,v),(a,x)) < d((u,v),(b,y))\}$. Consider the following partition of $N_{(a,x)}(e)$:

 $\begin{array}{l} A_{(a,x)} = \ \{(u,v) \in V(G \oplus H) \mid au \in E(G) \ , \ vx \notin E(H) \ , \ ub \in E(G) \ , \ vy \in E(H) \ \}, \\ B_{(a,x)} = \ \{(u,v) \in V(G \oplus H) \mid (u,v) \neq (b,y), \ au \in E(G) \ , \ vx \notin E(H) \ , \ ub \notin E(G) \ , \ vy \notin E(H) \ \}, \\ C_{(a,x)} = \ \{(u,v) \in V(G \oplus H) \mid au \notin E(G) \ , \ vx \in E(H) \ , \ ub \in E(G) \ , \ vy \in E(H) \ \}, \\ D_{(a,x)} = \ \{(u,v) \in V(G \oplus H) \mid au \notin E(G) \ , \ vx \in E(H) \ , \ ub \notin E(G) \ , \ vy \notin E(H) \ \} \ and \\ F_{(a,x)} = \ \{(a,x)\}. \ We \ have: \end{array}$

Theorem 4. Let G and H be nontrivial and regular graphs. Then the symmetric difference $G \oplus H$ is distance-balanced.

Proof. Let $e = (a,x)(b,y) \in E(G \oplus H)$, where $ab \in E(G)$. Then $n_{(a,x)}(e) = |N_{(a,x)}(e)|$, $N_{(a,x)}(e) = A_{(a,x)} \bigcup B_{(a,x)} \bigcup C_{(a,x)} \bigcup D_{(a,x)} \bigcup F_{(a,x)}$ and $N_{(b,y)}(e) = A_{(b,y)} \bigcup B_{(b,y)} \bigcup C_{(b,y)} \bigcup D_{(b,y)} \bigcup F_{(b,y)}$. On the other hand, since *G* and *H* are regular, $|A_{(a,x)}| = |A_{(b,y)}|$, ..., $|B_{(a,x)}| = |B_{(b,y)}|$, $|C_{(a,x)}| = |C_{(b,y)}|$, $|D_{(a,x)}| = |D_{(b,y)}|$ and $|F_{(a,x)}| = |F_{(b,y)}|$. Therefore, $n_{(a,x)}(e) = n_{(b,y)}(e)$. If e = (a,x)(b,y), $xy \in E(H)$, then a similar argument shows that $n_{(a,x)}(e) = n_{(b,y)}(e)$, proving the result.

Theorem 5. The cluster of two arbitrary, nontrivial and connected graphs is not distance-balanced.

Proof. Let *G* and *H* be arbitrary, nontrivial and connected graphs and H_i be the i-th copy of *H*. Assume that $uv = e \in E(G\{H\})$ such that *u* is the root of the ith copy of *H* and $u \neq v \in V(H_i)$. Thus, we have :

$$n_{u}^{G{H}}(e) = |V(H)| (|V(G)| - 1) + n_{u}^{H}(e) \text{ and } n_{v}^{G{H}}(e) = n_{v}^{H}(e).$$

Therefore, $n_u^{G\{H\}}(e) \neq n_v^{G\{H\}}(e)$ and so $G\{H\}$ is not distance-balanced.

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