Iranian Journal of Mathematical Chemistry, Vol. 2, No. 2, December 2011, pp. 67 – 71 **IJMC**

Remarks on Distance–Balanced Graphs

M. TAVAKOLI AND H. YOUSEFI-AZARI•

School of Mathematics, Statistics and Computer Science, University of Tehran, Tehran, I. R. Iran

(*Received May10, 2011*)

ABSTRACT

Distance-balanced graphs are introduced as graphs in which every edge uv has the following property: the number of vertices closer to u than to v is equal to the number of vertices closer to v than to u. Basic properties of these graphs are obtained. In this paper, we study the conditions under which some graph operations produce a distance-balanced graph.

Keywords: Distance–balanced graphs, graph operation.

1. INTRODUCTION

For an edge $e = ab$ of a graph G, let $n_a^G(e)$ be the number of vertices closer to a than to b. That is, $n_a^G(e) = |\{u \in V(G) \mid d(u, a) < d(u, b)\}|$. In addition, let $n_0^G(e)$ be the number of vertices with equal distances to a and b; $n_0^G(e) = |\{u \in V(G) \mid d(u, a) = d(u, b)\}|$.

Here is our key definition. We call a graph G distance-balanced, if $n_a^G(e) = n_b^G(e)$ holds for any edge $e = ab$ of G. These graphs were, at least implicitly, first studied by Handa [4] who considered distance-balanced partial cubes. The term itself, however, is due to Jerebic et al. [1] who studied distance–balanced graphs in the framework of various kinds of graph products. The transmission $T(u)$ of a vertex $u \in V$ is defined as follows:

$$T(u) = \sum_{v \in V} d(u,v).$$

A graph G is said to be transmission-regular if all its vertices have the same transmission. As examples of transmission-regular graphs, we can cite the complete graph K_n on $n \geq 2$ vertices, the complete bipartite graph $K_{n,n}$ on $2n \geq 2$ vertices.

• Corresponding author (Email: hyousefi@ut.ac.ir).

Let *G* and *H* be two graphs. The corona product G o H is obtained by taking one copy of *G* and |V(G)| copies of *H*; and by joining each vertex of the i-th copy of *H* to the i-th vertex of *G*, i = 1, 2, …, |V(G)|, see [2,3]. The join G + H of graphs *G* and *H* with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G \bigcup H$ together with all the edges joining V_1 and V_2. The symmetric difference $G \oplus H$ of two graphs *G* and *H* is the graph with vertex set V(G) × V(H) and edge set

$$E(G \oplus H) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G) \quad \text{or} \quad u_2v_2 \in E(H) \text{ but not both}\}.$$

The cluster G{H} is obtained by taking one copy of *G* and |V(G)| copies of a rooted graph *H*, and by identifying the root of the ith copy of *H* with the i^{th} vertex of *G*, i = 1,2 ,..., |V(G)|. The composite graph G{H} was studied by Schwenk [9]. Throughout this paper our notation is standard and taken mainly from the standard book of graph theory. We encourage the reader to consult papers [5,7,8,10–12] for background material as well as basic computational techniques.

2. MAIN RESULTS

A regular graph is a graph where each vertex has the same number of neighbors. A regular graph with vertices of degree *k* is called a k-regular graph or regular graph of degree *k*. In this section, we study the conditions under which some graph operations produce a distance-balanced graph. We begin by the following theorem which states the relationship between distance-balanced and transmission-regular graphs:

Theorem 1. A graph *G* is distance-balanced if and only if *G* is transmission-regular.

Proof. It is well–known fact that if *G* is a connected graph and uv = e $\in$ E(G), then $n_u^G(e) = n_v^G(e)$ if and only if T(u) = T(v) [6], proving the result. ▼

Theorem 2. Let *G* and *H* be connected graphs. Then G + H is distance-balanced if and only if *G* and *H* are *r* and *k* regular graphs, respectively, and $|V(G)| - r = |V(H)| - k$.

Proof. Consider the following partition of E(G + H):

$$A = \{uv \in E(G+H) \mid u, v \in V(G)\},$$
$$B = \{uv \in E(G+H) \mid u, v \in V(H)\},$$
$$C = \{uv \in E(G+H) \mid u \in V(G) \text{ and } v \in V(H)\}.$$

We first assume that *G* and *H* are r– and k–regular graphs respectively, and $|V(G)| - r = |V(H)| - k$. Let uv = e $\in$ A and $m_0^G(e)$ = |{x $\in$ V(G) | d(u, x) = d(v, x) = 1}|. Notice that

$$d_{G+H}(x,y)=\begin{cases}0 & x=y\\ 1 & (x\in V(G) \text{ and } y\in V(H)) \text{ or } (xy\in E(H)) \text{or } (xy\in E(G))\\ 2 & \text{otherwise}\end{cases}$$

Thus we have n_u^{G+H}(e) = $\deg_G$(u) – $m_0^G(e)$ and n_v^{G+H}(e) = $\deg_G$(v) – $m_0^G(e)$. Since *G* is regular $\deg_G$(u) = $\deg_G$(v), and thus $n_u^{G+H}(e) = n_v^{G+H}(e)$. We now assume that uv = e$\in$B. In a similar way we can see that $n_u^{G+H}(e) = n_v^{G+H}(e)$. Assume that uv=e$\in$C. Then we have n_u^{G+H}(e) = |V(H)| – $\deg_H$(v) and n_v^{G+H}(e) = |V(G)| – $\deg_G$(u). Therefore, n_u^{G+H}(e) = n_v^{G+H}(e) and thus G + H is distance-balanced. Conversely, assume that G + H is distance-balanced. By above argument for an edge e of A, we see n_u^{G+H}(e) = n_v^{G+H}(e) implies that any two adjacent vertices of *G* have the same degree. Since *G* is connected, this implies that *G* is r–regular for some *r*. In a similar way we can see that *H* is k-regular, for some *k*. For an edge uv = e $\in$ C, it follows again from earlier analysis that n_u^{G+H}(e) = |V(H)| – $\deg_H$(v) and n_v^{G+H}(e) = |V(G)| – $\deg_G$(u). Since G + H is distance - balanced, two above equations imply that |V(H)| – $\deg_H$(v) = |V(G)| – $\deg_G$(u). ▼

Corollary. Let *G* and *H* be connected graphs. G + H is transmission-regular if and only if *G* and *H* be *r* and *k* regular respectively, such that $|V(G)| - r = |V(H)| - k$.

Proof. The proof follows from Theorems 1 and 2. ▼

A graph *G* is called nontrivial if |V(G)| > 1.

Theorem 3. The corona product of two arbitrary, nontrivial and connected graphs is not distance–balanced.

Proof. Let *G* and *H* be arbitrary, nontrivial and connected graphs and H_i be the i-th copy of *H*. Assume that uv = e $\in$ E(GoH) such that u$\in$V(G) and v$\in$V(H_i). Thus, we have :

$$n_u^{GoH}\text{(e)} = |V(G)|\,(|V(H)| + 1) - \deg_{GoH}(v) \text{ and } n_v^{GoH}\text{(e)} = 1.$$

Therefore, we have n_u^{GoH}(e) ≠ n_v^{GoH} (e). Thus *GoH* is not distance-balanced. ▼

Corollary. The corona product of two arbitrary, nontrivial and connected graphs is not transmission - regular.

Proof. The proof follows from Theorems 1 and 3. ▼

Let $e = (a,x)(b,y) \in E(G\oplus H)$ such that $ab \in E(G)$, and $N_{(a,x)}(e) = \{(u,v) \in V(G\oplus H) |\ d((u,v),(a,x)) < d((u,v),(b,y))\}$. Consider the following partition of $N_{(a,x)}(e)$:

$A_{(a,x)} = \{(u,v) \in V(G\oplus H) \mid au\in E(G) , vx\notin E(H) , ub\in E(G) , vy\in E(H) \}$,
$B_{(a,x)} = \{(u,v) \in V(G\oplus H) \mid (u,v) \neq (b,y), au\in E(G) , vx\notin E(H) , ub\notin E(G) , vy\notin E(H) \}$,
$C_{(a,x)} = \{(u,v) \in V(G\oplus H) \mid au\notin E(G) , vx\in E(H) , ub\in E(G) , vy\in E(H) \}$,
$D_{(a,x)} = \{(u,v) \in V(G\oplus H) \mid au\notin E(G) , vx\in E(H) , ub\notin E(G) , vy\notin E(H) \}$ and
$F_{(a,x)} = \{(a,x)\}$. We have:

Theorem 4. Let *G* and *H* be nontrivial and regular graphs. Then the symmetric difference $G \oplus H$ is distance-balanced.

Proof. Let $e = (a,x)(b,y) \in E(G\oplus H)$, where $ab\in E(G)$. Then $n_{(a,x)}(e) = |N_{(a,x)}(e)|$, $N_{(a,x)}(e) = A_{(a,x)} \cup B_{(a,x)} \cup C_{(a,x)} \cup D_{(a,x)} \cup F_{(a,x)}$ and $N_{(b,y)}(e) = A_{(b,y)} \cup B_{(b,y)} \cup C_{(b,y)} \cup D_{(b,y)} \cup F_{(b,y)}$. On the other hand, since *G* and *H* are regular, $|A_{(a,x)}| = |A_{(b,y)}|, \ldots, |B_{(a,x)}| = |B_{(b,y)}|, |C_{(a,x)}| = |C_{(b,y)}|, |D_{(a,x)}| = |D_{(b,y)}|$ and $|F_{(a,x)}| = |F_{(b,y)}|$. Therefore, $n_{(a,x)}(e) = n_{(b,y)}(e)$. If $e = (a,x)(b,y)$, $xy\in E(H)$, then a similar argument shows that $n_{(a,x)}(e) = n_{(b,y)}(e)$, proving the result. ▼

Theorem 5. The cluster of two arbitrary, nontrivial and connected graphs is not distance-balanced.

Proof. Let *G* and *H* be arbitrary, nontrivial and connected graphs and H_i be the i-th copy of *H*. Assume that $uv = e \in E(G\{H\})$ such that *u* is the root of the i^{th} copy of *H* and $u \neq v \in V(H_i)$. Thus, we have :

$$n_u^{G\{H\}}(e) = |V(H)|\,(\,|V(G)| - 1) + n_u^H(e) \quad \text{and} \quad n_v^{G\{H\}}(e) = n_v^H(e).$$

Therefore, $n_u^{G\{H\}}(e) \neq n_v^{G\{H\}}(e)$ and so $G\{H\}$ is not distance-balanced. ▼

REFERENCES

1. J. Jerebic, S. Klavžar and D. F. Rall, Distance–balanced graphs, *Ann. Combin.* **12** (2008) 71–79.
2. Y. N. Yeh and I. Gutman, On the sum of all distances in composite graphs , *Discrete Math.* **135** (1994) 359–365.
3. D. Stevanovic, Hosoya polynomial of composite graphs, *Discrete Math.* **235** (2001) 237−244.
4. K. Handa, Bipartite graphs with balanced (a,b)–partitions, *Ars Combin.* **51** (1999) 113–119.
5. M. H. Khalifeh, H. Yousefi−Azari and A. R. Ashrafi, The hyper−Wiener index of graph operations, *Comput. Math. Appl.* **56** (2008) 1402–1407.

6. M. Aouchiche and P. Hansen**,** On a conjecture about the Szeged index, *European J. Combin.* **31** (2010) 1662−1666.
7. M. K. Khalifeh, H. Yousefi–Azari, A. R. Ashrafi and S. G. Wagner, Some new results on distance−based graph invariants, *European J. Combin.* **30** (2009) 1149–1163.
8. A. Ilić, S. Klavžar and M. Milanović, On distance-balanced graphs, *European J. Combin.* **31** (2010) 733−737.
9. A. J. Schwenk, Computing the characteristic polynomial of a graph, in: A. Dold and B. Eckmann, eds., Graphs and Combinatorics, *Lecture Notes in Mathematics*, **406** (Springer, Berlin, 1974) 153−172.
10. M. Tavakoli and H. Yousefi−Azari**,** Computing PI and Hyper−Wiener indices of corona product of some graphs, *Iranian J. Math. Chem.* **1** (2010) 131–135.
11. A. T. Balaban, P. V. Khadikar and S. Aziz, Comparison of topological indices based on iterated 'sum' versus ' product' operations, *Iranian J. Math. Chem.* **1** (2010) 43–60.
12. P. V. Khadikar, Padamakar–Ivan Index in Nanotechnology, *Iranian J. Math. Chem*, **1** (2010) 7–42.