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Distance-Based Topological Indices of Tensor Product of Graphs

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ABSTRACT

Let G and H be connected graphs. The tensor product $G + H$ is a graph with vertex set $V(G+H) = V(G) \times V(H)$ and edge set $E(G + H) =\{(a , b)(x , y)|\ ax \in E(G)\ \&\ by \in E(H)\}$. The graph H is called the strongly triangular if for every vertex u and v there exists a vertex w adjacent to both of them. In this article the tensor product of $G + H$ under some distance-based topological indices are investigated, when H is a strongly triangular graph. As a special case most of results given by Hoji, Luob and Vumara in [Wiener and vertex PI indices of Kronecker products of graphs, Discrete Appl. Math., 158 (2010), 1848-1855] will be deduced.

Keywords: Tensor product, Wiener type invariant, strongly triangular graph.

1 INTRODUCTION

Throughout this article G is a simple connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A topological index is a numerical quantity related to a graph that is invariant under graph automorphisms. As usual, the distance between the vertices u and v of G is denoted by $d_G(u , v)(d(u , v)$ for short) and it is defined as the number of edges in a minimal path connecting them. A topological index related to distance function $d(–, –)$ is called a "distance–based topological index". The Wiener index $W(G)$ is the first distance–based topological index defined as the sum of all distances between vertices of G [21]. The Wiener index has noteworthy applications in chemistry and interested readers can be referred to [2, 3] and references therein for details. Hosoya [10] was the first scientist introduced the name "topological index" and reformulated the Wiener index in terms of distance function $d(– , –)$.

In graph theory such numbers is usually called graph invariant. The hyper−Wiener index was proposed by Klein et al.[17], as a generalization of the Wiener index of graphs. It is defined as $WW(G) = \frac{1}{2}W(G) + \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} d^2(u,v)$, where$d^2(u,v) = d(u,v)^2$. We encourage the reader to consult [14, 15] for mathematical properties of the hyper-Wiener index and its applications in chemistry. These definitions can be further generalized in the following natural way:$W_\lambda(G) = \sum_{u,v\in V(G)} d^\lambda(u,v)$where $d^\lambda(u,v) = d(u,v)^\lambda$and λ is a real number[6, 7].

Several particular instances of the invariant W_λ have been previously studied for instance, W_{-2} ,W_{-1} ,$\frac{1}{2}W_1 + \frac{1}{2}W_2$ *and* $\frac{1}{6}W_1 + \frac{1}{2}W_2 + \frac{1}{3}W_3$ are the so called Harary index, reciprocal Wiener index, hyper-Wiener index and Tratch–Stankevich–Zefirov index, see [7,12] and references therein for details. In the chemical literature also $W_{\frac{1}{2}}$ [23] as well as the general case W_λ were examined [5, 6, 8].

The reverse-Wiener index $\Lambda(G) = \frac{n(n-1)D}{2} - W(G)$ was proposed by Balaban et al. in 2000 [1]. It is important for a reverse problem and it is also found applications in modeling of structure–property relations. Some mathematical properties of the reverse–Wiener index may be found in [19, 20].

Up to now, many distance-based topological indices of Cartesian product of graphs are determined [4, 11, 13, 16, 22]. In this paper we consider distance-based topological indices of tensor product of graphs. The tensor product of graphs has been extensively investigated as regards graph colorings, graph recognition and decomposition, graph embedding, matching theory and stability in graphs (see, for example [18]).

Throughout this paper, $D(G)$,K_n, $K_{n\times 2}$, $srg(n, k, r, s)$ and T denote the diameter of graph, the complete graph, the cocktail party graph, the strongly regular graph with parameters k, r, s and the number of edges lying on a triangle, respectively. Our other notation is standard and taken mainly from the standard book on graph theory.

2 MAIN RESULTS

In this section, some distance–based topological indices of graphs under tensor product are computed. The exact formulas are given for the Wiener, hyper-Wiener, reverse–Wiener, Harary, reciprocal Wiener and Tracth–Stankevich–Zefirov indices of the tensor product of connected graphs. The following lemma is crucial in our calculations.

Lemma 1. $d_{G+H}\left((u,v),(u',v')\right) \geq Max\{d_G(u,u'), d_H(v,v')\}$.

Proof. It is clear that every shortest path connecting (u,v) and (u',v') are containing walk between u, u' and walk between v, v', as desired.

Theorem 2. Let G and H be connected graphs, where H be a strongly triangular graph. Then for every vertex $(u, v),(u', v')\in V(G + H)$ we have:

$$d_{G+H}\big((u,v),(u',v')\big) = \begin{cases} 2 & uu' \in E(G), vv' \notin E(H) and\, uu' is\ \ an\, edge\, of\, a\, triangle; \\ & or\, uu' \in E(G), v = v' and\, uu' is\, an\, edge\, of\, a\, triangle. \\ 3 & uu' \in E(G), vv' \notin E(H) and\, uu' is\, not\, an\, edge\, of\, a\, triangle; \\ & or\, uu' \in E(G), v = v' and\, uu' is\, not\, an\, edge\, of\, a\, triangle. \\ d_G(u,u') & Otherwise \end{cases}$$

Proof. Suppose $P : u = u_1, u_2, \ldots , u_n = u'$ is a shortest path connecting u and u' in G, w is adjacent to v, $v'\in H$ and v', w are adjacent to z. Now our main proof will consider three separate cases as follows:

Case 1: Let (u, v) and (u', v') be two arbitrary vertices of $G + H$ such that $u\neq u'$ and $v \neq v'$. We prove this case in three steps.

1) Suppose that $d_G(u, u') = 2k + 1, k > 0$. If $vv'\in E(H)$, then the path $(u_1, v),(u_2, v')$, (u_3, v), …, $(u_{n-1}, v),(u_n, v')$ is connecting $(u, v)$and (u', v') and its length is $d_G(u, u')$, and if $vv' \notin E(H)$, (u_1, v), (u_2,w), (u_3, z), (u_4, v'), (u_5, z), . . ., (u_{n-1}, z), (u_n, v') *is* a shortest path of length $d_G(u, u')$. Apply Lemma 1, we have $d_{G+H}((u, v), (u', v')) = d_G(u, u')$.

2) Suppose that $d_G(u , u') = 2k, k > 0$. Hence (u_1, v), (u_2 , w), (u_3 , v), (u_4 , w), ..., (u_{n-1}, w), (u_n , v') is connecting (u , v) and (u',v') and its length is $d_G(u , u')$. By Lemma 1, $d_{G+H}((u, v),(u', v')) = d_G(u, u')$.

3) Let $d_G(u, u') = 1$. If $vv' \in E(H)$ then (u, v), (u', v') is a path of length one. Suppose $vv'\notin E(H)$. If uu' is an edge of a triangle, since $vv' \notin E(H)$, $d_{G+H}((u, v), (u', v')) \geq 2$. Assume that u'' is adjacent to u and u'. Therefore, (u, v), (u'', w), (u', v') is a path of length 2. Thus, $d_{G+H}((u, v), (u', v'))= 2$. If uu' is not an edge of a triangle, then clearly $d_{G+H}((u, v), (u', v')) \geq 3$. On the other hand (u, v), (u', w), (u, z), (u', v') is a path of length 3 which implies that $d_{G+H}((u, v), (u', v')) = 3$.

Case 2: Let (u, v) and (u', v') be arbitrary vertices of $G + H$, such that $u\neq u'$ and $v = v'$. Suppose that vertex v_1 is adjacent to v and vertex w' is adjacent to both of them too. If $d_G(u,u') = 2k, k > 0$. Then (u_1, v), (u_2, v_1), (u_3, v), . . ., (u', v) is connecting (u, v) and (u', v) having length $d_G(u, u')$. If $d_G(u, u') = 2k + 1, k > 0$, then (u_1, v), (u_2, v_1), (u_3, w), (u_4, v), ..., (u_{n-1}, w), (u', v) is a shortest path of length $d_G(u, u')$. Thus, Lemma 1 shows that $d_{G+H}((u,v),(u', v')) = d_G(u, u')$.

Now if $d_G(u,u') = 1$ then using a similar argument as those are given in 3, we have $d_{G+H}((u,v), (u', v')) = 2$, where uu' is an edge of a triangle. On the other hand, $d_{G+H}((u,v),(u',v')) = 3$, when uu' is not an edge of a triangle.

Case 3: Suppose (u, v) and (u', v') are arbitrary vertices of $G + H$ such that $u = u'$ and $v \neq v'$. In this case (u_1, v), (u_2, w), (u_3, v') is a shortest path of length 2 between (u, v), (u, v'),which completes our proof.

Theorem 3. Let G and H be connected graphs such that H is a strongly triangular graph. Then

$$W_\lambda = H^2 W_\lambda(G) + \left((3^\lambda - 1)|E(G)| + (2^\lambda - 3^\lambda)|T|(|H|^2 - 2|E(H)|) + 2^\lambda |G| \binom{|H|}{2}\right).$$

Proof. We compute the Wiener type invariant of $(G + H)$ in a similar argument as Theorem 2. We prove the theorem in six cases as follows:

1) We assume that $u \neq u'$, $v \neq v'$ and $d_G(u, u') > 1$. Then
$$I_{A_1} = \sum d_{G+H}^\lambda \left((u,v),(u',v')\right) = 2\sum_{v,v'}\sum_{u,u'} d_G^\lambda(u,u') = 2\binom{|H|}{2}(W_\lambda(G) - |E(G)|).$$

2) Let $u \neq u'$ and $d_G(u,u') = 1$ then we have the following three cases:

i) $vv' \in E(H)$ then
$$I_{A_2} = \sum d_{G+H}^\lambda \left((u,v),(u',v')\right) = 2\sum_{v,v'}\sum_{u,u'} 1^\lambda = 2|E(H)||E(G)|.$$

ii) $vv' \notin E(H)$ and uu' is an edge of a triangle then
$$I_{A_3} = \sum d_{G+H}^\lambda \left((u,v),(u',v')\right) = 2\sum_{v,v'}\sum_{u,u'} 2^\lambda = 2^{\lambda+1}|T|\left(\binom{|H|}{2} - |E(H)|\right).$$

iii) $vv' \notin E(H)$ and uu' is not an edge of a triangle then
$$I_{A_4} = \sum d_{G+H}^\lambda \left((u,v),(u',v')\right) = 2\sum_{v,v'}\sum_{u,u'} 3^\lambda = 2.3^\lambda(|E(G)| - |T|)\left(\binom{|H|}{2} - |E(H)|\right).$$

3) Assume that $u \neq u'$, $v = v'$ and $d_G(u, u') > 1$. Then
$$I_{B_1} = \sum d_{G+H}^\lambda \left((u,v),(u',v')\right) = \sum_v \sum_{u,u'} d_G^\lambda(u,u') = |H|(W_\lambda(G) - |E(G)|).$$

4) Suppose uu' is an edge of a triangle and $v = v'$ then
$$I_{B_2} = \sum d_{G+H}^\lambda \left((u,v),(u',v')\right) = \sum_v \sum_{u,u'} 2^\lambda = 2^\lambda |T||H|.$$

5) Let uu' is not an edge of a triangle and $v = v'$ then
$$I_{B_3} = \sum d_{G+H}^\lambda \left((u,v),(u',v')\right) = \sum_v \sum_{u,u'} 3^\lambda = 3^\lambda |H|(|E(G)| - |T|).$$

6) Consider $u = u'$ and $v \neq v_0$. Then

$$I_{C_1} = \sum d^{\lambda}_{G+H}\left((u,v),(u',v')\right) = 2\sum_{v,v'}\sum_{u} 2^{\lambda} = 2^{\lambda}|G|\binom{|H|}{2}.$$

Therefore, by using the above results and adding them, proof is completed.

Corollary 4. Let G and *H* be connected graphs and *H* be strongly triangular graph. Then

a) $W(G+H) = |H|^2W(G) + (|H|^2 - 2|E(H)|)(2|E(G)| - |T|) + 2|G|\binom{|H|}{2}$,

b) $W_{-1}(G+H) = |H|^2W_{-1}(G) + (|H|^2 - 2|E(H)|)\left(\frac{1}{6}|T| - \frac{2}{3}|E(G)|\right) + \frac{1}{2}|G|\binom{|H|}{2}$,

c) $W_{-2}(G+H) = |H|^2W_{-2}(G) + (|H|^2 - 2|E(H)|)\left(\frac{5}{36}|T| - \frac{8}{9}|E(G)|\right) + \frac{1}{4}|G|\binom{|H|}{2}$,

d) $WW(G+H) = |H|^2WW(G) + (|H|^2 - 2|E(H)|)(5|E(G)| - 3|T|) + 3|G|\binom{|H|}{2}$,

e) $W_{\frac{1}{2}}(G+H) = |H|^2W_{\frac{1}{2}}(G) + (|H|^2 - 2|E(H)|)\left((\sqrt{3}-1)|E(G)| - (\sqrt{3}-\sqrt{2})|T|\right) + \sqrt{2}|G|\binom{|H|}{2}$,

f) $\frac{1}{6}W_3(G+H) + \frac{1}{2}W_2(G+H) + \frac{1}{3}W_1(G+H) = |H|^2\left(\frac{1}{6}W_3(G) + \frac{1}{2}W_2(G) + \frac{1}{3}W_1(G)\right) +$
$(|H|^2 - 2|E(H)|)(9|E(G)| - 6|T|) + 4|G|\binom{|H|}{2}$.

Proof. Apply Theorem 3.

Corollary 5. Suppose *G* and *H* are connected graphs and *H* is strongly triangular graph. Then

$$\Lambda(G+H) = \begin{cases} \frac{|G||H|(|G||H|-1)D(G)}{2} - |H|^2W(G) - (|H|^2 - 2|E(H)|)(2|E(G) - |T|) - 2|G|\binom{|H|}{2} & D(G) \geq 3 \\ \frac{3|G||H|(|G||H|-1)}{2} - |H|^2W(G) - (|H|^2 - 2|E(H)|)(2|E(G) - |T|) - 2|G|\binom{|H|}{2} & D(G) < 3, |T| \neq |E(G)|. \\ |G||H|(|G||H|-1) - |H|^2W(G) - (|H|^2 - 2|E(H)|)(2|E(G) - |T|) - 2|G|\binom{|H|}{2} & D(G) < 3, |T| = |E(G)| \end{cases}$$

Proof. Apply Theorem 2 and Corollary 4(a).

Corollary 6. Suppose *G* is a connected graph and *H= srg*(*n, k, r, s*), wherer *s >0*. Then

a) $W(G+H) = n^2W(G) + n(n-k)(2|E(G)| - |T|) + n(n-1)|G|$,

b) $WW(G+H) = n^2WW(G) + n(n-k)(5|E(G)| - 3|T|) + \frac{3}{2}n(n-1)|G|$,

c) $W_{-1}(G+H) = n^2W_{-1}(G) + n(n-k)\left(\frac{1}{6}|T| - \frac{2}{3}|E(G)|\right) + \frac{1}{4}n(n-1)|G|$,

d) $W_{-2}(G+H) = n^2W_{-2}(G) + n(n-k)\left(\frac{5}{36}|T| - \frac{8}{9}|E(G)|\right) + \frac{1}{8}n(n-1)|G|$,

e) $W_{\frac{1}{2}}(G+H) = n^2W_{\frac{1}{2}}(G) + n(n-k)\left((\sqrt{3}-1)|E(G)| - (\sqrt{3}-\sqrt{2})|T|\right) + \frac{\sqrt{2}}{2}n(n-1)|G|$.

Proof. Apply Corollary 4.

Corollary 7 (Hoji, Luob and Vumara [9]). Suppose *G* is a connected graph. Then for *n >2*,

$$W(G+K_n)=n^2W(G)+n(2|E(G)|-3|T|)+n(n-1)|G|,$$
$$WW(G+H)=n^2WW(G)+n(5|E(G)|-|T|)+\frac{3}{2}n(n-1)|G|.$$

Proof. Apply Corollary 6.

Corollary 8 (Hoji, Luob and Vumara [9]). Let *G* be a connected triangle–free graph of order at least 2. Then for $n\geq 3$,

$$W(G+K_n)=n^2W(G)+2n|E(G)|+n(n-1)|G|,$$
$$WW(G+K_n)=n^2WW(G)+5n|E(G)|+\frac{3}{2}n(n-1)|G|.$$

Proof. Apply Corollary 6.

Corollary 9. Suppose *G* is a connected graph then for $n\geq 3$

a) $W_{-1}(G+K_n)=n^2W_{-1}(G)+n\left(\frac{1}{6}|T|-\frac{2}{3}|E(G)|\right)+\frac{1}{4}n(n-1)|G|,$

b) $W_{-2}(G+K_n)=n^2W_{-2}(G)+n\left(\frac{5}{36}|T|-\frac{8}{9}|E(G)|\right)+\frac{1}{8}n(n-1)|G|,$

c) $W_{\frac{1}{2}}(G+K_n)=n^2W_{\frac{1}{2}}(G)+n\left((\sqrt{3}-1)|E(G)|-(\sqrt{3}-\sqrt{2})|T|\right)+\frac{\sqrt{2}}{2}n(n-1)|G|.$

Proof. Apply Corollary 6.

Corollary 10. Suppose *G* is a connected graph and $K_{n\times 2}$ where $n>2$ is a cocktail party graph. Then

a) $W(G+K_{n\times 2})=(2n)^2W(G)+4n(2|E(G)|-|T|)+2n(2n-1)|G|,$
b) $WW(G+K_{n\times 2})=(2n)^2WW(G)+4n(5|E(G)|-3|T|)+3n(2n-1)|G|,$
c) $W_{-1}(G+K_{n\times 2})=(2n)^2W_{-1}(G)+4n\left(\frac{1}{6}|T|-\frac{2}{3}|E(G)|\right)+\frac{1}{2}n(2n-1)|G|,$
d) $W_{-2}(G+K_{n\times 2})=(2n)^2W_{-2}(G)+4n\left(\frac{5}{36}|T|-\frac{8}{9}|E(G)|\right)+\frac{1}{4}n(2n-1)|G|,$
e) $W_{\frac{1}{2}}(G+K_{n\times 2})=(2n)^2W_{\frac{1}{2}}(G)+4n\left((\sqrt{3}-1)|E(G)|-(\sqrt{3}-\sqrt{2})|T|\right)+\sqrt{2}n(2n-1)|G|.$

Proof. Apply Corollary 6.

Corollary 11. Suppose *G* is a connected graph and $H=srg(n,k,r,s)$ is a strongly regular graph with parameters $r, s>0$ then for $n\geq 3$,

$$\Lambda(G+H)=\begin{cases}\frac{n|G|(n|G|-1)D(G)}{2}-n^2W(G)-n(n-k)(2|E(G)-|T|)-n(n-1)|G|, & D(G)\geq 3\\ \frac{3n|G|(n|G|-1)}{2}-n^2W(G)-n(n-k)(2|E(G)-|T|)-n(n-1)|G|, & D(G)<3, |T|\neq|E(G)|\\ n|G|(n|G|-1)-n^2W(G)-n(n-k)(2|E(G)-|T|)-n(n-1)|G|. & D(G)<3, |T|=|E(G)|\end{cases}$$

$$\Lambda(G+K_n)=\begin{cases}\frac{n|G|(n|G|-1)D(G)}{2}-n^2W(G)-n(2|E(G)-|T|)-n(n-1)|G|, & D(G)\geq 3\\ \frac{3n|G|(n|G|-1)}{2}-n^2W(G)-n(2|E(G)-|T|)-n(n-1)|G|, & D(G)<3, |T|\neq|E(G)|\\ n|G|(n|G|-1)-n^2W(G)-n(2|E(G)-|T|)-n(n-1)|G|. & D(G)<3, |T|=|E(G)|\end{cases}$$

$$\Lambda(G+K_{n\times 2})=\begin{cases}n|G|(2n|G|-1)D(G)-(2n)^2W(G)-4n(2|E(G)-|T|)-2n(2n-1)|G|, & D(G)\geq 3\\ 3n|G|(2n|G|-1)-(2n)^2W(G)-4n(2|E(G)-|T|)-2n(2n-1)|G|, & D(G)<3, |T|\neq|E(G)|\\ |G||H|(|G||H|-1)-|H|^2W(G)-(|H|^2-2|E(H)|)(2|E(G)-|T|)-2n(2n-1)|G|. & D(G)<3, |T|=|E(G)|\end{cases}$$

Proof. Apply Corollary 5.

Corollary 12. Suppose G is a connected graph and $H = srg(n, k, r, s)$ with parameters r, $s > 0$ then

a) $\frac{1}{6}W_3(G+H)+\frac{1}{2}W_2(G+H)+\frac{1}{3}W_1(G+H)=n^2\left(\frac{1}{6}W_3(G)+\frac{1}{2}W_2(G)+\frac{1}{3}W_1(G)\right)+n(n-k)(9|E(G)|-6|T|)+2n(n-1)|G|$,

b) $\frac{1}{6}W_3(G+K_n)+\frac{1}{2}W_2(G+K_n)+\frac{1}{3}W_1(G+K_n)=n^2\left(\frac{1}{6}W_3(G)+\frac{1}{2}W_2(G)+\frac{1}{3}W_1(G)\right)+n(9|E(G)|-6|T|)+2n(n-1)|G|$,

c) $\frac{1}{6}W_3(G+K_{n\times 2})+\frac{1}{2}W_2(G+K_{n\times 2})+\frac{1}{3}W_1(G+K_{n\times 2})=(2n)^2\left(\frac{1}{6}W_3(G)+\frac{1}{2}W_2(G)+\frac{1}{3}W_1(G)\right)+4n(9|E(G)|-6|T|)+4n(2n-1)|G|$.

Proof. Apply Corollary 4(f).

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