

Chain Hexagonal Cacti: Extremal With Respect To The Eccentric Connectivity Index

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ABSTRACT

In this paper we present explicit formulas for the eccentric connectivity index of three classes of chain hexagonal cacti. Further, it is shown that the extremal chain hexagonal cacti with respect to the eccentric connectivity index belong to one of the considered types. Some open problems and possible directions of further research are mentioned in the concluding section.

Keywords: Chain hexagonal cactus, eccentric connectivity index.

1. INTRODUCTION

The eccentric connectivity index is a graph invariant that attracted a lot of attention of researchers working in the area of QSAR/QSPR. It has been found useful in modeling various physico-chemical properties of several classes of chemical compounds [12, 20, 22]. After some initial delay, it also became a subject of a number of mathematical papers [1, 4, 6, 7, 17, 24, 25]. Here we continue with investigation of its behavior on a class of graphs of some relevance in statistical mechanics, chemistry, and theory of networks.

The central objects of this note, the cactus graphs, were introduced in the scientific literature some sixty years ago under the name of **Husimi trees**. They appeared in papers of Husimi [16] and Riddell [19] concerned with cluster integrals in the theory of condensation

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in statistical mechanics[23]. Besides statistical mechanics, where they serve as simplified models of real lattices [18, 21], the Husimi trees were also found useful in the theory of electrical and communication networks [28] and in chemistry [15, 26].

From the mathematical point of view, the Husimi trees were first studied in a series of papers by Harary, Uhlenbeck and Norman, concerned with their enumerative properties [13, 14]. Later they become known as cactus graphs, and under that name attracted some attention when it was found out that some facility allocation problems that are NP-hard for general graphs can be solved in polynomial time for the cactus graphs [2, 27]. Also, they spawned a number of generalizations, such as block-cactus graphs [3, 29].

In this paper we study certain uniform and regular classes of cactus graphs. We present explicit formulas for the eccentric connectivity index of such cacti and show that the extremal values of this quantity are achieved on two of the considered classes.

The paper is organized as follows. In Section 2 we formally introduce the classes of graphs relevant for our investigation. Section 3 is concerned with the explicit formulas for three types of chain hexagonal cacti. In Section 4 we establish the extremal values and find the extremal chains with respect to the eccentric connectivity index. The paper is concluded by indicating some possible directions for future research.

2. DEFINITIONS AND PRELIMINARY RESULTS

All graphs considered in this paper will be finite, simple and connected. For a graph G , we denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. For two vertices u and v of $V(G)$ we define their **distance** $d(u, v)$ as the length of a shortest path between u and v in G . For a given vertex u of $V(G)$ its **eccentricity** $\varepsilon(u)$ is the largest distance between u and any other vertex v of G . Hence, $\varepsilon(u) = \max_{v \in V(G)} d(u, v)$. The **eccentric connectivity index** $\xi(G)$ of a graph G is defined as

$$\xi(G) = \sum d(u) \varepsilon(u),$$

where $d(u)$ denotes the degree of vertex u , i. e., the number of its neighbors in G .

A **cactus graph** is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same size m , the cactus is m -uniform.

A **hexagonal cactus** is a 6-uniform cactus, i.e., a cactus in which every block is a hexagon. A vertex shared by two or more hexagons is called a **cut-vertex**. If no hexagon of a hexagonal cactus G has more than two cut-vertices, and each cut-vertex is shared by exactly two hexagons, we say that G is a **chain hexagonal cactus**. Hence, in a chain hexagonal cactus there are no branching hexagons. The number of hexagons in G is called the **length** of the chain. An example of a chain hexagonal cactus is shown in Fig. 1.

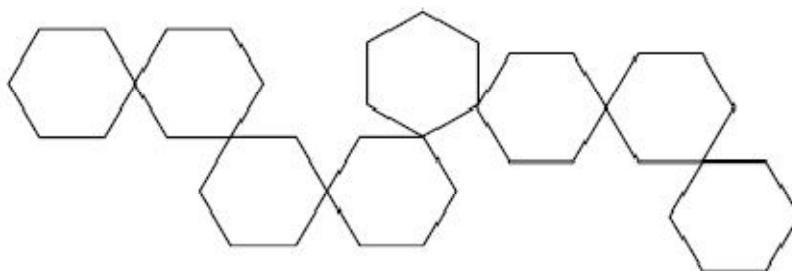


Figure 1. A Chain Hexagonal Cactus of Length 8.

Obviously, a chain hexagonal cactus of length n has $5n + 1$ vertices and $6n$ edges. Furthermore, any chain hexagonal cactus of length greater than one has exactly two **terminal** hexagons, i.e., two hexagons with only one cut-vertex. Any remaining hexagons, if present, are called **internal** hexagons. We denote the set of all chain hexagonal cacti of length n by $(CHC)_n$. It is obvious from the definition that all chain hexagonal cacti are planar, and that all their bounded faces are hexagons. A class of graphs with somewhat similar properties (planar graphs whose all bounded faces are cycles of length 6) known as **benzenoid graphs**, has been studied for a long time as the mathematical model of a wide and important class of chemical compounds called benzenoid hydrocarbons. As a result, many terms of chemical origin became well established in the theory of benzenoid graphs. We adopt some of those terms and use them as a mean of concise description of three types of chain hexagonal cacti.

Let us consider a hexagon C_6 . Two vertices u and v of C_6 are said to be in **ortho-position** if they are neighbors in C_6 . If the distance between u and v is 2, they are in **meta-position**. Finally, if the distance between u and v is 3, we say that they are in **para-position**. The ortho-, meta-, and para-position of two vertices in C_6 are shown in Fig. 2.

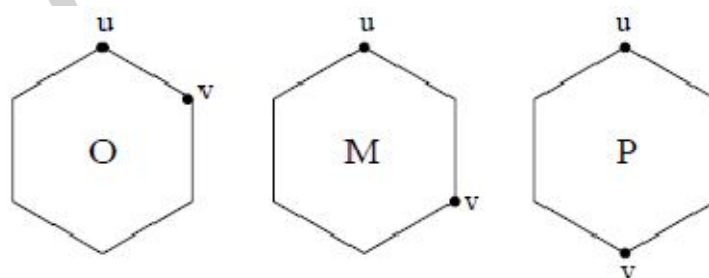


Figure 2. Ortho-, Meta-, and Para-Position of Two Vertices in C_6

An internal hexagon in a chain hexagonal cactus is called **ortho-hexagon**, **meta-hexagon**, or **para-hexagon** if its cut-vertices are in ortho-, meta-, and para-position, respectively. If all internal hexagons of a hexagonal chain cactus are of the same type, we say that the chain is regular. Obviously, three given types of internal hexagons give rise to three classes of regular chain cacti. A chain hexagonal cactus $(CHC)_n$ is an **ortho-chain** if all its internal hexagons are ortho-hexagons. The **meta-chain** and **para-chain** are defined in a completely analogous manner. The ortho-chain of length n is denoted by $(OC)_n$ and the meta-chain is denoted by $(MC)_n$. The para-chain of length n will be denoted by $(PC)_n$.

We conclude this section by noting that the number of chain hexagonal cacti grows exponentially with the number of hexagons.

Theorem 2.1. There are $\frac{1}{2} \left(3^{n-2} + 3^{\lfloor \frac{n-1}{2} \rfloor} \right)$ different chain hexagonal cacti of length n .

The result follows by counting words of length $n - 2$ in a ternary alphabet and eliminating palindromes. We refer the reader to [5] for the proof.

3. ECCENTRIC CONNECTIVITY INDEX OF REGULAR CHAINS

The unbranched nature of chain cacti allows for a natural ordering of their hexagons. It is clear that the eccentricity of vertices of a given hexagon mostly depends on the position of the hexagon in the chain and then on the way it is connected to its neighbors. The dependence is particularly simple (in fact linear) for regular chains. This fact allows us to derive explicit formulas for the eccentric connectivity index of three classes of regular chains. Throughout this section we assume $n \geq 2$.

3.1. Ortho-chain.

Theorem 3.1. The eccentric connectivity index of an ortho-chain of length n is given by

$$\xi((OC)_n) = \begin{cases} 9n^2 + 36n & 2 \mid n \\ 9n^2 + 36n + 3 & 2 \nmid n \end{cases}$$

Proof. We start by labeling vertices of $(OC)_n$ in the way shown in Fig. 3. The vertices of degree 4 are labeled by positive integers from 1 to $n - 1$, while the vertices of degree 2 are labeled by a pair of integers, the first one indicating the hexagon they belong to, and the second one denoting their place within that hexagon. There are two cases

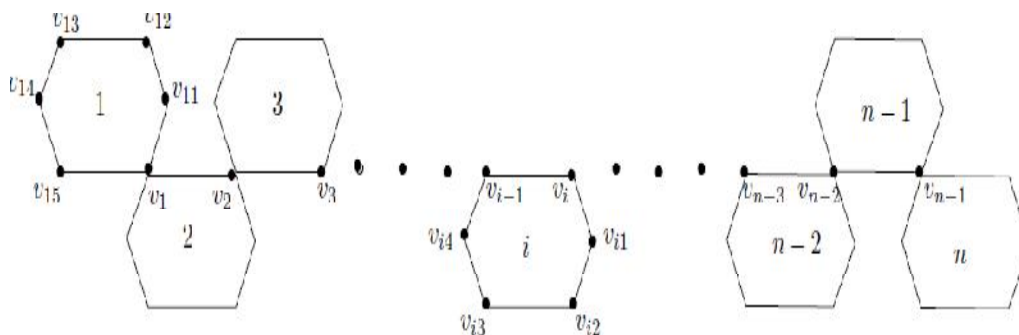


Figure 3. The Labeling of Vertices of $(OC)_n$.

Case 1: Let n be even, $n = 2k$. Because of the symmetry, it is enough to calculate the eccentricities of vertices of first k hexagons. Their eccentricities are obtained as follows:

- i. For any $i, 1 \leq i \leq k, \varepsilon(v_i) = n + 2 - i$.
- ii. For any $i, 2 \leq i \leq k, \varepsilon(v_{i1}) = n + 3 - i, \varepsilon(v_{i2}) = \varepsilon(v_{i4}) = n + 4 - i$ and $\varepsilon(v_{i3}) = n + 5 + i$.
- iii. $\varepsilon(v_{11}) = \varepsilon(v_{15}) = n + 2, \varepsilon(v_{12}) = \varepsilon(v_{14}) = n + 3$ and $\varepsilon(v_{13}) = n + 4$.

The total contribution of vertices of degree 2 of the i -th hexagon is now equal to $2(4n + 16 - 4i)$. By adding all contributions, doubling the result, and subtracting the contribution of the middle vertex that was included twice, we obtain

$$\begin{aligned} \xi((OC)_n) &= \sum_{v \in V((OC)_n)} d(v)\varepsilon(v) \\ &= 2 \sum_{i=1}^{n/2} 4(n + 2 - i) - 4(n + 2 - k) \\ &\quad + 2 \sum_{i=2}^{n/2} 2(4n + 16 - 4i) + 4(5n + 14) \\ &= 9n^2 + 36n. \end{aligned}$$

Case 2: Let n be odd, $n = 2k + 1$. The eccentricities of vertices in the first $k = (n - 1)/2$ hexagons are given by the same formulas as in the even case. For the middle hexagon we have $\varepsilon(v_{(k+1)1}) = \varepsilon(v_{(k+1)4}) = n + 3 - k$ and $\varepsilon(v_{(k+1)2}) = \varepsilon(v_{(k+1)3}) = n + 4 - k$. The total contribution of the vertices of degree 2 in the middle hexagon is now given by $2(2n + 7 - 2k) = 2(n + 8)$. The other contributions remain the same as in the previous case. Therefore

$$\begin{aligned} \xi((OC)_n) &= \sum_{v \in V((OC)_n)} d(v)\varepsilon(v) \\ &= 2 \sum_{i=1}^{(n-1)/2} 4(n+2-i) \\ &+ 2 \sum_{i=2}^{(n-1)/2} 2(4n+16-4i) + 4(5n+14) + 4(n+8) \\ &= 9n^2 + 36n + 3. \end{aligned}$$

□

3.2. Meta-Chain

Theorem 3.2. Let $(MC)_n$ be a meta-chain with n hexagons. Then the eccentric connectivity index of $(MC)_n$ for $n > 3$ is given by

$$\xi((MC)_n) = \begin{cases} 18n^2 + 18n & 2 | n \\ 18n^2 + 18n + 2 & 2 \nmid n \end{cases}$$

and $\xi((MC)_3) = 208$.

Proof. The vertices of degree 4 and the vertices of degree 2 of first and i -th hexagons of MC_n are labeled in the way shown in Fig. 4.

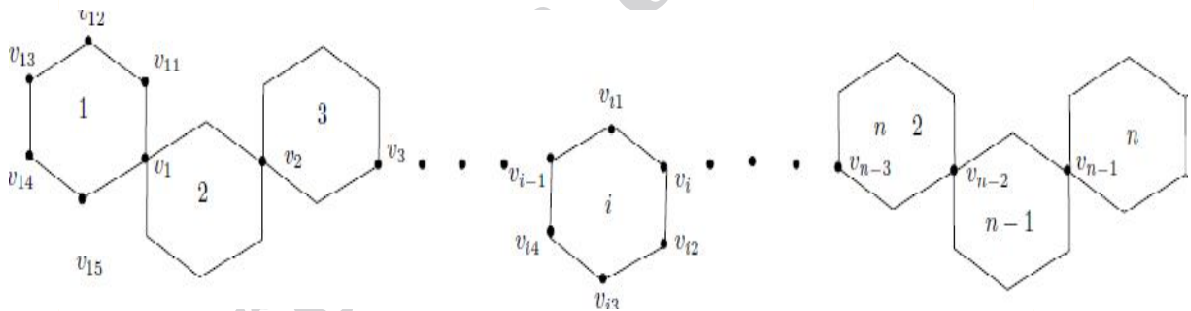


Figure 4. The Labeled Vertices of $(MC)_n$.

Again we consider two cases.

Case1: Let n be even, $n = 2k$. It is enough to calculate the eccentricities of vertices of the first k hexagons.

- i. For any $i, 1 \leq i \leq k, \varepsilon(v_i) = 2(n-i) + 1,$
- ii. For any $i, 2 \leq i \leq k, \varepsilon(v_{i1}) = \varepsilon(v_{i2}) = 2(n-i) + 2, \varepsilon(v_{i3}) = 2(n-i) + 3$ and $\varepsilon(v_{i4}) = 2(n-i) + 4.$

$$\text{iii. } \varepsilon(v_{11}) = \varepsilon(v_{15}) = 2n, \varepsilon(v_{12}) = \varepsilon(v_{14}) = 2n + 1 \text{ and } \varepsilon(v_{13}) = 2n + 2.$$

Then the total contribution of vertices of degree 2 of the i -th hexagon is equal to $2(8(n-i)+11)$. By the above calculation we have

$$\begin{aligned} \xi((MC)_n) &= \sum_{v \in V((MC)_n)} d(v)\varepsilon(v) \\ &= 2 \sum_{i=1}^{n/2} 4(2(n-i)+1) - 4(2(n-k)+1) \\ &\quad + 2 \sum_{i=2}^{n/2} 2(8(n-i)+11) + 2.2(10n+4) \\ &= 18n^2 + 18n. \end{aligned}$$

Case 2: Let n be odd, $n = 2k + 1$. The eccentricities of vertices of the first k hexagons remain the same, while for the middle hexagon we have $\varepsilon(v_{(k+1)1}) = 2(n-k)$, $\varepsilon(v_{(k+1)2}) = \varepsilon(v_{(k+1)4}) = 2(n-k) + 2$ and $\varepsilon(v_{(k+1)3}) = 2(n-k) + 1$. Therefore

$$\begin{aligned} \xi((MC)_n) &= \sum_{v \in V((MC)_n)} d(v)\varepsilon(v) \\ &= 2 \sum_{i=1}^{(n-1)/2} 4(2(n-i)+1) \\ &\quad + 2 \sum_{i=2}^{(n-1)/2} 2(8(n-i)+11) + 2 \times 2(10n+4) + 2(4n+9) \\ &= 18n^2 + 18n + 2. \end{aligned}$$

□

3.3. Para-chain.

Theorem 3.3. The eccentric connectivity index of a para-chain on n vertices is given by

$$\xi((PC)_n) = \begin{cases} 27n^2 & 2 \mid n \\ 27n^2 + 1 & 2 \nmid n \end{cases}$$

Proof. We label the vertices of $(PC)_n$ in the way shown in Figure 5.

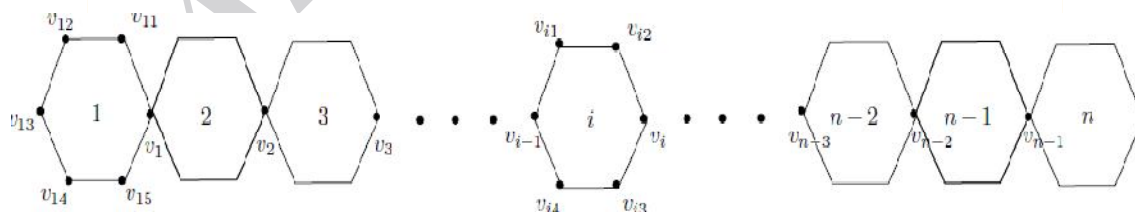


Figure 5. The Labeled Vertices of $(PC)_n$.

Case 1: Let n be even, $n = 2k$. It is enough to calculate the eccentricities of vertices of the first k hexagons.

- i. For any i , $1 \leq i \leq k$, $\varepsilon(v_i) = 3(n-1)$.
- ii. For any i , $2 \leq i \leq k$, $\varepsilon(v_{i1}) = \varepsilon(v_{i4}) = 3(n-i) + 2$ and $\varepsilon(v_{i2}) = \varepsilon(v_{i3}) = 3(n-i) + 1$.
- iii. $\varepsilon(v_{11}) = \varepsilon(v_{15}) = 3n - 2$, $\varepsilon(v_{12}) = \varepsilon(v_{14}) = 3n - 1$ and $\varepsilon(v_{13}) = 3n$.

By the above calculation we have

$$\begin{aligned}\xi((PC)_n) &= \sum_{v \in V((PC)_n)} d(v)\varepsilon(v) \\ &= 2 \sum_{i=1}^{n/2} 12(n-i) - 12(n-k) \\ &\quad + 2 \sum_{i=2}^{n/2} 2(12(n-i) + 6) + 4(15n - 6) \\ &= 27n^2.\end{aligned}$$

Case 2: Let n be odd, $n = 2k + 1$. The contributions of the non-middle hexagons remain the same, while for the middle hexagon we have $\varepsilon(v_{(k+1)1}) = \varepsilon(v_{(k+1)4}) = 3(n-k)$ and $\varepsilon(v_{(k+1)2}) = \varepsilon(v_{(k+1)3}) = 3(n-k) - 1$.

Therefore

$$\begin{aligned}\xi((PC)_n) &= \sum_{v \in V((PC)_n)} d(v)\varepsilon(v) \\ &= 2 \sum_{i=1}^{(n-1)/2} 4.3(n-i) \\ &\quad + 2 \sum_{i=2}^{(n-1)/2} 2(12(n-i) + 6) + 4(15n - 6) + 2(12(n-k) - 4) \\ &= 27n^2 + 1.\end{aligned}$$

□

As expected, all three formulas give the same value of 108 for $n = 2$. It is clear from the leading coefficients that for long regular chains the para-chain has the largest and the ortho-chain the smallest eccentric connectivity index. In the next section we show that those chains remain extremal also when we drop the condition of regularity.

4. EXTREMAL CHAIN HEXAGONAL CACTI

In what follows, we prove a general theorem for obtaining extremal chain hexagonal cacti with respect to the eccentric connectivity index.

Theorem 4.1. Let H_1 and H_2 be connected disjoint graphs, such that $u \in V(H_1)$ and $v \in V(H_2)$. The graphs G_1 , G_2 and G_3 are the graphs obtained by identifying the vertices u and v with para-, meta- and ortho-position of vertices in C_6 , respectively (see Fig. 6). Then

$$\xi(G_3) \leq \xi(G_2) \leq \xi(G_1).$$

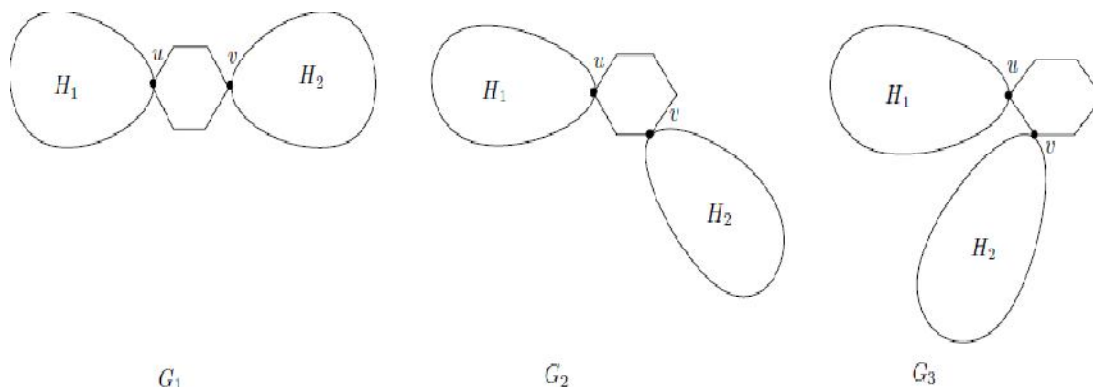


Figure 6. Graphs G_1 , G_2 and G_3 .

Proof. There is nothing to prove if H_1 or H_2 is trivial, i.e., if $\varepsilon_{H_1}(u) = 0$ or $\varepsilon_{H_2}(v) = 0$. Hence we may assume that $d(u)$ and $d(v)$ are at least equal to 3. It is clear that for any $w \in V(H_1) \cup V(H_2)$ its degree remains the same in all three cases, $d_{G_1}(w) = d_{G_2}(w) = d(w)$. We start by proving the right inequality. We break the argument into 3 steps. In the first step we look at the vertices of $V(H_1) \cup V(H_2)$ and compare their contributions to $\xi(G_1)$ and $\xi(G_2)$. In the second step we compare the contributions of vertices u and v . Finally, in the third step, we look at the vertices of degree 2 in the hexagon C_6 connecting H_1 and H_2 .
Step 1: For any $w, w' \in V(H_1)$ ($w, w' \in V(H_2)$), we have $d_{G_1}(w, w') = d_{G_2}(w, w')$. Also, for any $w \in V(H_1)$ and $w' \in V(H_2)$, we have

$$\begin{aligned} d_{G_1}(w, w') &= d_{G_1}(w, u) + d_{G_1}(u, v) + d_{G_1}(v, w') \\ &= d_{G_2}(w, u) + 3 + d_{G_1}(v, w') \\ &> d_{G_2}(w, u) + 2 + d_{G_2}(v, w') \\ &= d_{G_2}(w, u) + d_{G_2}(u, v) + d_{G_2}(v, w') \\ &= d_{G_2}(w, w'). \end{aligned}$$

By the above argument, we conclude that $\varepsilon_{G_1}(w) \geq \varepsilon_{G_2}(w)$ for any $w \in V(H_1) \cup V(H_2)$. In particular, we can say that if the eccentricity of $w \in V(H_1)$ ($w \in V(H_2)$) in G_1 is attained at a vertex of H_1 (H_2), then $\varepsilon_{G_1}(w) = \varepsilon_{G_2}(w)$. If the eccentricity of $w \in V(H_1)$ ($w \in V(H_2)$) in G_1 is attained at a vertex of H_2 (H_1), then $\varepsilon_{G_1}(w) = \varepsilon_{G_2}(w) + 1$.

Step 2: Let x be a vertex of G_1 . We denote by x^* a vertex of G_1 such that $\varepsilon_{G_1}(x) = d(x, x^*)$. By Step 1 it follows that for every $x \in V(H_1) \cup V(H_2)$ its eccentricity in G_2 is attained on

the same vertex x^* as in G_1 . Now we look at u and v . We claim that $u^* \in V(H_2)$ or $v^* \in V(H_1)$. Let us suppose otherwise, i.e., $u^* \in V(H_1)$ and $v^* \in V(H_2)$. Without loss of generality we can assume that $d_{G_1}(u, u^*) \leq d_{G_1}(v, v^*)$. Then we have

$$\begin{aligned} \varepsilon_{G_1}(u) &= d_{G_1}(u, u^*) \geq d_{G_1}(u, v^*) = d_{G_1}(u, v) + d_{G_1}(v, v^*) \\ &= 3 + d_{G_1}(v, v^*) \\ &\geq 3 + d_{G_1}(u, u^*) \\ &= 3 + \varepsilon_{G_1}(u). \end{aligned}$$

This is a contradiction. Hence, by Step 1, $d(u)\varepsilon_{G_1}(u) \geq d(u)(\varepsilon_{G_2}(u) + 1)$ or $d(v)\varepsilon_{G_1}(v) \geq d(v)(\varepsilon_{G_2}(v) + 1)$. Then $d(u)\varepsilon_{G_1}(u) + d(v)\varepsilon_{G_1}(v) \geq d(u)\varepsilon_{G_2}(v) + 3$. Hence, the total contribution of u and v to $\xi(G_1)$ exceeds their total contribution to $\xi(G_2)$.

Step 3: Let $\varepsilon_{H_1}(u) = m$ and let it be attained at vertex $u_* \in V(H_1)$. Similarly, let $\varepsilon_{H_2}(v) = l$ and it is attained on $v_* \in V(H_2)$. Then $d_{H_1}(u, u_*) = m$ and $d_{H_2}(v, v_*) = l$. Without loss of generality we can assume $m \leq l$. We consider two cases.

Case 1: $m = l$. Label the vertices of C_6 in G_1 and G_2 as shown in Fig. 7. Then

$$\begin{aligned} \varepsilon_{G_1}(u_1) &= \varepsilon_{G_1}(u_2) = \varepsilon_{G_1}(u_4) = \varepsilon_{G_1}(u_5) = l + 2 \text{ and } \varepsilon_{G_1}(u_3) = \varepsilon_{G_1}(u_6) = l + 3. \\ \varepsilon_{G_2}(v_1) &= \varepsilon_{G_2}(v_3) = l + 3, \varepsilon_{G_2}(v_2) = l + 2, \varepsilon_{G_2}(v_5) = l + 1 \text{ and } \varepsilon_{G_1}(v_4) = \varepsilon_{G_1}(v_6) = l + 2. \end{aligned}$$

Since the degrees of u and v in G_1 and G_2 are at least 3, then $d(u) + d(v) \geq 6$ and by using the eccentricity of vertices of C_6 in G_1 and G_2 , we have the following inequality:

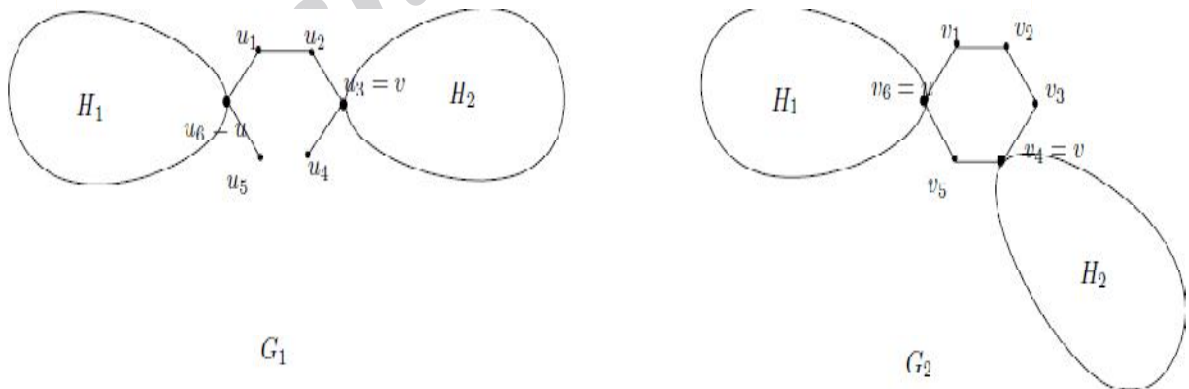


Figure 7. Labeling Vertices of C_6 in G_1 and G_2 .

$$\begin{aligned}\sum_{i=1}^6 d_{G_1}(u_i)\varepsilon_{G_1}(u_i) &= 2(4l+8) + d(u)(l+3) + d(v)(l+3) \\ &> 2(4l+9) + d(u)(l+2) + d(v)(l+2) \\ &= \sum_{i=6}^6 d_{G_2}(v_i)\varepsilon_{G_2}(v_i)\end{aligned}$$

Therefore by Step 1, we conclude that $\xi(G_2) < \xi(G_1)$.

Case 2: $m < 1$. From Fig. 7 and Step 2 we obtain the following minimum values of eccentricity of vertices of degree 2 of C_6 in G_1 and the maximum values of eccentricity of vertices of degree 2 of C_6 in G_2 .

$$\varepsilon_{G_1}(u_1) = \varepsilon_{G_1}(u_5) \geq l+2 \text{ and } \varepsilon_{G_1}(u_2) = \varepsilon_{G_1}(u_4) \geq l+1,$$

$$\varepsilon_{G_2}(v_1) \leq l+3, \varepsilon_{G_2}(v_2) \leq l+2, \varepsilon_{G_2}(v_3) = \varepsilon_{G_1}(v_5) \leq l+1. \text{ Hence}$$

$$\begin{aligned}\sum_{i=1}^6 d_{G_1}(u_i)\varepsilon_{G_1}(u_i) &\geq 2(4l+6) + d(u)\varepsilon_{G_1}(u) + d(v)\varepsilon_{G_1}(v) \\ &\geq 2(4l+6) + 3 + d(u)\varepsilon_{G_2}(u) + d(v)\varepsilon_{G_2}(v) \\ &> 2(4l+7) + d(v)\varepsilon_{G_2}(v) \\ &\geq \sum_{i=1}^6 d_{G_2}(v_i)\varepsilon_{G_2}(v_i),\end{aligned}$$

and by Step 1, we conclude that $\xi(G_2) < \xi(G_1)$.

By similar reasoning we can prove that $\xi(G_3) < \xi(G_2)$. We omit the details. Since the inequalities in Step 3 are strict, we have the following result.

Corollary 4.2. Let $(CHC)_n$ be a chain hexagonal cactus of length n . Then $\xi((OC)_n) \leq \xi((CHC)_n) \leq \xi((PC)_n)$, with the right (left) equality if and only if $(CHC)_n = (PC)_n$ ($(CHC)_n = (OC)_n$).

5. CONCLUDING REMARKS

In this section we present some results concerning the eccentric connectivity index of a family of hexagonal cacti considered in [8] and [5]. Then we show that such graphs can be also viewed as a special case of chains. That observation points toward a more general setting that encompasses in a natural way both the graphs considered here and in a series of works by Farrell [9, 10, 11].

A **star hexagonal cactus** $(SHC)_n$ is obtained by taking n copies of C_6 and splicing them all together in a single vertex u in a way shown in Fig. 8.

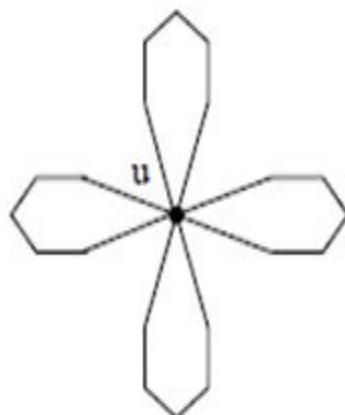


Figure 8. A Star Hexagonal Cactus.

Theorem 5.1.

$$\xi((SHC)_n) = 54n$$

The star hexagonal cacti were treated separately here since the results follow by a direct computation, much easier than for the chain cacti of the previous two sections. However, star hexagonal cacti fit neatly into the class of chains by allowing the two cut vertices of all internal hexagons to coincide. Hence, a star hexagonal cactus is a “chain” hexagonal cactus whose cut-vertices are separated by a path of length 0. Now we can abandon our chemical nomenclature and index the chains by an integer parameter equal to the distance between the cut-vertices. By doing so, we obtain a uniform notation for all hexagonal cacti considered here: $C_n(6,0) = (SHC)_n$, $C_n(6,1) = (OC)_n$, $C_n(6,2) = (MC)_n$, and $C_n(6,3) = (PC)_n$.

By using the above notation we can present our results by a single formula.

Theorem 5.2.

$$\xi(C_n(6,k)) = 9kn^2 + 18(3-k)n + \frac{1-(-1)^n}{2}(4-k)(1-d_{o,k}).$$

Here $d_{o,k} = 1$ if $k=0$ and 0 otherwise.

The general setting referred to at the beginning of this section now consists of considering the chain cacti $C_n(m, k)$ made of n copies of m -gons whose cut-vertices are at the distance k . Here we assume that $k \leq \lfloor \frac{m}{2} \rfloor$. It would be interesting to derive results for general (m, k) -chains analogous to the ones presented here. Also, it would be interesting to

find explicit formulas and extremal values and graphs for several generalizations of the eccentric connectivity index, such as, e.g., the augmented eccentric connectivity index.

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REFERENCES

1. A. R. Ashrafi, T. Došlić and M. Saheli, The eccentric connectivity index of $TUC_4C_8(R)$ nanotubes, *MATCH Commun. Math. Comput. Chem.* **65** (2011), 221–230.
2. B. Ben-Moshe, B. Bhattacharya and S. Quiaosheng, Efficient algorithms for the weighted 2-center problem in a cactus graph, *Lect. Notes Comput. Sci.* **3827** (2005), 693–703.
3. G. Chang, C. Chen and Y. Chen, Vertex and tree arboricities of graphs, *J. Comb. Optim.* **8** (2004), 295–306.
4. T. Došlić, A. Graovac and O. Ori, Eccentric connectivity index of hexagonal belts and chains, *MATCH Commun. Math. Comput. Chem.* **65** (2011), 745–752.
5. T. Došlić and F. Måløy, Chain hexagonal cacti: Matchings and independent sets, *Discrete Math.* **310** (2010), 1176–1190.
6. T. Došlić and M. Saheli, Eccentric connectivity index of benzenoid graphs, *Novel Molecular Structure Descriptors – Theory and Applications II* (I. Gutman and B. Furtula, ed.), Univ. Kragujevac, Kragujevac, 2010, pp. 169–182.
7. T. Došlić, M. Saheli and D. Vukičević, Eccentric connectivity index: extremal graphs and values, *Iranian J. Math. Chem.* **1** (2) (2010), 45–56.
8. E. J. Farrell, Matchings in hexagonal cacti, *Internat. J. Math. & Math. Sci.* **10** (1987), 321–338.
9. E. J. Farrell, Matchings in rectangular cacti, *J. Math. Sci. (Calcutta)* **9** (1998), 163–183.
10. E. J. Farrell, Matchings in pentagonal cacti, *J. Math. Sci. (Calcutta)* **11** (2000), 109–126.
11. E. J. Farrell, Matchings in triangular cacti, *J. Math. Sci. (Calcutta)* **11** (2000), 85–98.
12. S. Gupta, M. Singh, and A.K. Madan, Application of graph theory: Relationship of eccentric connectivity index and Wiener's index with anti-inflammatory activity, *J. Math. Anal. Appl.* **266** (2002), 259–268.

13. F. Harary and R. Z. Norman, The dissimilarity characteristic of Husimi trees, *Annals Math.* **58** (1953), 134–141.
14. F. Harary and G. E. Uhlenbeck, On the number of Husimi trees I, *Proc. Nat. Acad. Sci. USA* **39** (1953), 315–322.
15. H. Hosoya and K. Balasubramanian, Exact dimer statistics and characteristic polynomials of cacti lattices, *Theor. Chem. Accounts* **76** (1989), 315–329.
16. K. Husimi, Note on Mayer's theory of cluster integrals, *J. Chem. Phys.* **18** (1950), 682–684.
17. A. Ilić and I. Gutman, Eccentric connectivity index of chemical trees, *MATCH Commun. Math. Comput. Chem.* **65** (2011), 745–752.
18. J. L. Monroe, The bilayer Ising model and a generalized Husimi tree approximation, *Physica A* **335** (2004), 563–576.
19. R. J. Riddell, *Contributions to the theory of condensation*, Ph.D. thesis, Univ. of Michigan, Ann Arbor, 1951.
20. S. Sardana and A. K. Madan, Application of graph theory: Relationship of molecular connectivity index, Wiener's index and eccentric connectivity index with diuretic activity, *MATCH Commun. Math. Comput. Chem.* **43** (2001), 85–98.
21. P. Serra, J. F. Stilck, W. L. Cavalcanti and K. D. Machado, Polymers with attractive interaction on the Husimi lattice, *J. Phys. A Mat. Gen.* **37** (2004), 8811–8821.
22. V. Sharma, R. Goswami and A. K. Madan, Eccentric connectivity index: A novel highly discriminating topological descriptor for structure-property and structure-activity studies, *J. Chem. Inf. Comput. Sci.* **37** (1997), 273–282.
23. G. E. Uhlenbeck and G. W. Ford, *Lectures in Statistical Mechanics*, AMS, Providence RI, 1956.
24. Z. Yarahmadi, Eccentric connectivity and augmented eccentric connectivity indices of N-branched phenylacetylenes nanostar dendrimers, *Iranian J. Math. Chem.*, **1** (2) (2010), 105–110.
25. B. Zhou and Z. Du, On eccentric connectivity index, *MATCH Commun. Math. Comput. Chem.* **63** (2010), 181–198.
26. B. Zmazek and J. Žerovnik, Computing the weighted Wiener and Szeged number on weighted cactus graphs in linear time, *Croat. Chem. Acta* **76** (2003), 137–143.
27. B. Zmazek and J. Žerovnik, The obnoxious center problem on weighted cactus graphs, *Discrete Appl. Math.* **136** (2004), 377–386.
28. B. Zmazek and J. Žerovnik, Estimating the traffic on weighted cactus networks in linear time, Ninth International Conference on Information Visualization (IV'05), London, 2005, pp. 536–541.
29. V. E. Zverovich, The ratio of the irredundance number and the domination number for block-cactus graphs, *J. Graph Theory* **29** (1998), 139–149.