

## A Note on Hyper-Zagreb Index of Graph Operations

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**ABSTRACT** In this paper, the hyper-Zagreb index of the Cartesian product, composition and corona product of graphs are computed. These results correct some errors in G. H. Shirdel et al. [*Iranian J. Math. Chem.* **4** (2) (2013) 213–220].

**KEYWORDS** Hyper-Zagreb index • Zagreb index • graph operation.

### 1. INTRODUCTION

Throughout this paper, we consider only simple connected graphs. Let  $G$  be such a graph with vertex set  $V(G)$  and edge set  $E(G)$ . The degree of a vertex  $w \in V(G)$  is the number of vertices adjacent to  $w$  and is denoted by  $d_G(w)$ . We refer to [11] for unexplained terminology and notation.

In theoretical chemistry, the physico-chemical properties of chemical compounds are often modeled by means of molecular-graph-based structure-descriptors, which are also referred to as topological indices [10, 15]. The Zagreb indices are widely studied degree-based topological indices, and were introduced by Gutman and Trinajstić' [9] in 1972. The first and the second Zagreb indices of a graph  $G$  are respectively defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \text{ and } M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v).$$

The first Zagreb index can also be expressed as a sum over edges of  $G$ ,

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

Recently, G.H. Shirdel, H. Rezapour and A.M. Sayadi [14] introduced a new version of Zagreb index named hyper-Zagreb index which is defined for a graph  $G$  as

$$HM(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^2.$$

Some new results on the hyper-Zagreb index can be found in [7, 8].

The Cartesian product  $G \times H$  of graphs  $G$  and  $H$  has the vertex set  $V(G \times H) = V(G) \times V(H)$  and  $(a, x)(b, y)$  is an edge of  $G \times H$  if  $a = b$  and  $xy \in E(H)$ , or  $ab \in E(G)$  and  $x = y$ . If  $(a, x)$  is a vertex of  $G \times H$ , then  $d_{G \times H}((a, x)) = d_G(a) + d_H(x)$ .

The composition  $G[H]$  of graphs  $G$  and  $H$  with disjoint vertex sets  $V(G)$  and  $V(H)$  and edge sets  $E(G)$  and  $E(H)$  is the graph with vertex set  $V(G) \times V(H)$  and  $(a, x)$  is adjacent to  $(b, y)$  whenever  $a$  is adjacent to  $b$  or  $a = b$  and  $x$  is adjacent to  $y$ . If  $(a, x)$  is a vertex of  $G[H]$ , then  $d_{G[H]}((a, x)) = |V(H)|d_G(a) + d_H(x)$ .

The corona product  $G \circ H$  is defined as the graph obtained from  $G$  and  $H$  by taking one copy of  $G$  and  $|V(G)|$  copies of  $H$  and then by joining with an edge each vertex of the  $i^{th}$  copy of  $H$  which is named  $(H, i)$  with the  $i^{th}$  vertex of  $G$  for  $i = 1, 2, \dots, |V(G)|$ . If  $u$  is a vertex of  $G \circ H$ , then

$$d_{G \circ H}(u) = \begin{cases} d_G(u) + |V(H)| & \text{if } u \in V(G), \\ d_H(u) + 1 & \text{if } u \in V(H, i). \end{cases}$$

G. H. Shirdel et al. [14] computed the hyper-Zagreb index of some graph operations. However, the formulae of Theorem 2, Theorem 3, and Theorem 4 of their paper for computing the hyper-Zagreb index of Cartesian product, composition, and corona product are incorrect. In this paper, we give correct expressions for the hyper-Zagreb index of the Cartesian product, composition and corona product of graphs. Readers interested in more information on computing topological indices of graph operations can be referred to [1–6, 12, 13].

## 2. RESULTS

**Theorem 2.1** *Let  $G$  and  $H$  be graphs. Then*

$$HM(G \times H) = |V(G)|HM(H) + |V(H)|HM(G) + 12M_1(G)|E(H)| + 12M_1(H)|E(G)|.$$

**Proof.** By definition of the hyper-Zagreb index, we have

$$\begin{aligned} HM(G \times H) &= \sum_{(a,x)(b,y) \in E(G \times H)} [d_{G \times H}((a, x)) + d_{G \times H}((b, y))]^2 \\ &= \sum_{a \in V(G)} \sum_{xy \in E(H)} [d_G(a) + d_H(x) + d_G(a) + d_H(y)]^2 \\ &\quad + \sum_{x \in V(H)} \sum_{ab \in E(G)} [d_H(x) + d_G(a) + d_H(x) + d_G(b)]^2 \\ &= \sum_{a \in V(G)} \sum_{xy \in E(H)} [2d_G(a) + d_H(x) + d_H(y)]^2 \\ &\quad + \sum_{x \in V(H)} \sum_{ab \in E(G)} [2d_H(x) + d_G(a) + d_G(b)]^2 \\ &= \sum_{a \in V(G)} \sum_{xy \in E(H)} [4d_G(a)^2 + (d_H(x) + d_H(y))^2 + 4d_G(a)(d_H(x) + d_H(y))] \\ &\quad + \sum_{x \in V(H)} \sum_{ab \in E(G)} [4d_H(x)^2 + (d_G(a) + d_G(b))^2 + 4d_H(x)(d_G(a) + d_G(b))] \\ &= 4|E(H)|M_1(G) + |V(G)|HM(H) + 8|E(G)|M_1(H) \\ &\quad + 4|E(G)|M_1(H) + |V(H)|HM(G) + 8|E(H)|M_1(G). \end{aligned}$$

□

As an application of Theorem 2.1, we list explicit formulae for the hyper-Zagreb index of the rectangular grid  $P_r \times P_s$ ,  $C_4$  –nanotube  $P_r \times C_q$ , and  $C_4$  –nanotorus  $C_p \times C_q$ . The formulae follow from Theorem 2.1 by using the expressions  $M_1(P_n) = 4n - 6$ ,  $n > 1$ ;  $M_1(C_n) = 4n$ ;  $HM(P_n) = 16n - 30$ ,  $n > 2$  and  $HM(C_n) = 16n$ .

**Corollary 2.2**  $HM(P_r \times P_s) = 128rs - 150r - 150s + 144$ ,  $r, s > 2$ ;

$$HM(P_r \times C_q) = 128rq - 150q, r > 2; HM(C_p \times C_q) = 128pq.$$

**Theorem 2.3** *Let  $G$  and  $H$  be graphs. Then*

$$\begin{aligned} HM(G[H]) &= |V(H)|^4HM(G) + |V(G)|HM(H) \\ &\quad + 12|V(H)|^2|E(H)|M_1(G) + 10|V(H)||E(G)|M_1(H) + 8|E(H)|^2|E(G)|. \end{aligned}$$

**Proof.** Using the definition of the hyper-Zagreb index, we have

$$\begin{aligned}
 HM(G[H]) &= \sum_{(a,x)(b,y) \in E(G[H])} [d_{G[H]}((a,x)) + d_{G[H]}((b,y))]^2 \\
 &= \sum_{x \in V(H)} \sum_{y \in V(H)} \sum_{ab \in E(G)} [|V(H)|d_G(a) + d_H(x) + |V(H)|d_G(b) + d_H(y)]^2 \\
 &\quad + \sum_{a \in V(G)} \sum_{xy \in E(H)} [|V(H)|d_G(a) + d_H(x) + |V(H)|d_G(a) + d_H(y)]^2 \\
 &= \sum_{x \in V(H)} \sum_{y \in V(H)} \sum_{ab \in E(G)} [|V(H)|^2(d_G(a) + d_G(b))^2 + d_H(x)^2 + d_H(y)^2 \\
 &\quad + 2d_H(x)d_H(y) + 2|V(H)|(d_G(a) + d_G(b))(d_H(x) + d_H(y))] \\
 &\quad + \sum_{a \in V(G)} \sum_{xy \in E(H)} [4|V(H)|^2d_G(a)^2 + (d_H(x) + d_H(y))^2 \\
 &\quad + 4|V(H)|d_G(a)(d_H(x) + d_H(y))] \\
 &= |V(H)|^4HM(G) + |V(H)||E(G)|M_1(H) + |V(H)||E(G)|M_1(H) + 8|E(H)|^2|E(G)| \\
 &\quad + 2|V(H)|^2M_1(G)(2|E(H)| + 2|E(H)|) + 4|V(H)|^2|E(H)|M_1(G) + |V(G)|HM(H) \\
 &\quad + 8|V(H)||E(G)|M_1(H).
 \end{aligned}$$

□

As an application of Theorem 2.3, we present formulae for the hyper-Zagreb index of the fence graph  $P_n[K_2]$  and the closed fence graph  $C_n[K_2]$ .

**Corollary 2.4**  $HM(P_n[K_2]) = 500n - 816, n > 2; HM(C_n[K_2]) = 500n.$

**Theorem 2.5** *Let G and H be graphs. Then*

$$\begin{aligned}
 HM(G \circ H) &= HM(G) + |V(G)|HM(H) + 5|V(H)|M_1(G) + 5|V(G)|M_1(H) + \\
 &\quad 4|V(H)|^2|E(G)| + 4|V(G)||E(H)| + 8|E(G)||E(H)| + |V(G)||V(H)|(|V(H)| + 1)^2 \\
 &\quad + 4(|V(H)| + 1)(|E(G)||V(H)| + |E(H)||V(G)|).
 \end{aligned}$$

**Proof.** By definition of the hyper-Zagreb index, we have

$$\begin{aligned}
 HM(G \circ H) &= \sum_{uv \in E(G \circ H)} [d_{G \circ H}(u) + d_{G \circ H}(v)]^2 \\
 &= \sum_{uv \in E(G)} [d_G(u) + |V(H)| + d_G(v) + |V(H)|]^2 \\
 &\quad + \sum_{uv \in E(H)} \sum_{i=1}^{|V(G)|} [d_H(u) + 1 + d_H(v) + 1]^2 \\
 &\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} [d_G(u) + |V(H)| + d_H(v) + 1]^2.
 \end{aligned}$$

It is easy to see that

$$\begin{aligned}
 \sum_{uv \in E(G)} [d_G(u) + d_G(v) + 2|V(H)|]^2 &= \sum_{uv \in E(G)} [(d_G(u) + d_G(v))^2 + 4|V(H)|^2 \\
 &\quad + 4|V(H)|(d_G(u) + d_G(v))] = HM(G) + 4|V(H)|^2|E(G)| + 4|V(H)|M_1(G).
 \end{aligned} \tag{2.1}$$

$$\begin{aligned}
 \sum_{uv \in E(H)} \sum_{i=1}^{|V(G)|} [d_H(u) + d_H(v) + 2]^2 &= \sum_{uv \in E(H)} \sum_{i=1}^{|V(G)|} [(d_H(u) + d_H(v))^2 + 4 \\
 &\quad + 4(d_H(u) + d_H(v))] = |V(G)|(HM(H) + 4|E(H)| + 4M_1(H)).
 \end{aligned} \tag{2.2}$$

$$\begin{aligned}
 \sum_{u \in V(G)} \sum_{v \in V(H)} [d_G(u) + d_H(v) + |V(H)| + 1]^2 &= \sum_{u \in V(G)} \sum_{v \in V(H)} [d_G(u)^2 + d_H(v)^2 \\
 &\quad + 2d_G(u)d_H(v) + (|V(H)| + 1)^2 + 2(|V(H)| + 1)(d_G(u) + d_H(v))] \\
 &= |V(H)|M_1(G) + |V(G)|M_1(H) + 8|E(G)||E(H)| + |V(G)||V(H)|(|V(H)| + 1)^2
 \end{aligned}$$

$$+4(|V(H)| + 1)(|E(G)||V(H)| + |E(H)||V(G)|). \quad (2.3)$$

By adding Eqs. (2.1), (2.2), and (2.3) the proof is completed.  $\square$

Using Theorem 2.5, we can compute the hyper-Zagreb index of the  $k$ -thorny cycle  $C_n \circ \overline{K}_k$ .

**Corollary 2.6**  $HM(C_n \circ \overline{K}_k) = 16n + 25nk + 10nk^2 + nk^3$ .

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