

## On Common Neighborhood Graphs II

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### ABSTRACT

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Let  $G$  be a simple graph with vertex set  $V(G)$ . The common neighborhood graph or congraph of  $G$ , denoted by  $con(G)$ , is a graph with vertex set  $V(G)$ , in which two vertices are adjacent if and only if they have at least one common neighbor in  $G$ . We compute the congraphs of some composite graphs. Using these results, the congraphs of several special graphs are determined.

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## 1 INTRODUCTION AND PRELIMINARIES

Let  $G$  be a simple graph with vertex set  $V(G)$  and edge set  $E(G)$ . For any vertex  $v \in V(G)$ , the set of neighbors of  $v$  is the set  $N_v(G) = \{u \in V(G) \mid uv \in E(G)\}$ . We say that  $v \in V(G)$  is an isolated vertex if  $N_G(v)$  is an empty set. The distance between the vertices  $u$  and  $v$  of  $G$  denoted by  $d_G(u, v)$ . ( $d(u, v)$  for short), is defined as the length of the shortest path connecting  $u$  and  $v$ .

The complement of a graph  $G$  is a graph  $H$  on the same vertices such that two vertices of  $H$  are adjacent if and only if they are not adjacent in  $G$ . The graph  $H$  is usually denoted by  $\bar{G}$ . The minimum length of a cycle in a graph  $G$  is called the girth of  $G$ . We

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now define several kinds of products of pairs of graphs; see [14] for details. The union of the simple graphs  $G$  and  $H$  is the graph  $G \cup H$  with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . If  $G$  and  $H$  are disjoint, then we refer to their union as a disjoint union. Suppose that  $G$  and  $H$  are two graphs with disjoint vertex sets. Their Cartesian product  $G \times H$  is a graph such that  $V(G \times H) = V(G) \times V(H)$ , and two vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  are adjacent in  $G \times H$  if and only if either  $u_1 = u_2$  and  $v_1$  is adjacent with  $v_2$ , or  $v_1 = v_2$  and  $u_1$  is adjacent with  $u_2$ . The join  $G + H$  of the graphs  $G$  and  $H$  is the graph union  $G \cup H$  together with all the edges joining  $V(G)$  and  $V(H)$ . The tensor product  $G \otimes H$  of the graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  in which  $(u_1, v_1)$  is adjacent with  $(u_2, v_2)$  whenever  $u_1 u_2 \in E(G)$  and  $v_1 v_2 \in E(H)$ . The strong product  $G \Omega H$  of  $G$  and  $H$  has the vertex set  $V(G \Omega H) = V(G) \times V(H)$  and two distinct vertices  $(u_1, v_1)$  and  $(u_2, v_2)$  of  $G \Omega H$  are adjacent if  $u_1 = u_2$  and  $v_1 v_2 \in E(G)$ , or  $u_1 u_2 \in E(G)$  and  $v_1 = v_2$ , or  $u_1 u_2 \in E(G)$  and  $v_1 v_2 \in E(H)$ . For given vertices  $y \in V(G)$  and  $z \in V(H)$ , a splice of  $G$  and  $H$  by vertices  $y$  and  $z$ ,  $(G.H)(y, z)$ , is defined by identifying the vertices  $y$  and  $z$  in the union of  $G$  and  $H$  [10]. Hou and Shiu [13] introduced an edge version of corona product as follows.

Let  $G$  and  $H$  be two graphs on disjoint sets of  $n_1, n_2$  vertices and  $m_1, m_2$  edges, respectively. The edge corona  $G \diamond H$  is defined as the graph obtained by taking one copy of  $G$  and  $m_1$  copies of  $H$ , then joining two end-vertices of the  $i$ -th edge of  $G$  to every vertex in the  $i$ -th copy of  $H$ .

Now, we define the Hajós join which is introduced in [11]. Let  $G$  and  $H$  be two graphs,  $vw \in E(G)$ , and  $xy \in E(H)$ . Then the Hajós join of these two graphs, which is denoted by  $G \Delta H$ , is a new graph that combines the two graphs by identifying vertices  $v$  and  $x$  into a single vertex, removing the two edges  $vw$  and  $xy$ , and adding a new edge  $wy$ . For example, if  $G$  and  $H$  are cycles of length  $p$  and  $q$  respectively, then the Hajós join of these two cycles is itself a cycle, of length  $p + q - 1$ .

Let  $G$  be a simple graph with vertex set  $\{v_1, v_2, \dots, v_n\}$ . The common neighborhood graph (congraph) of  $G$ , denoted by  $con(G)$ , is a graph with the vertex set  $\{v_1, v_2, \dots, v_n\}$  in which two vertices are adjacent if and only if they have at least one common neighbor in the graph  $G$  [1, 2].

Congraphs have been investigated in several earlier works [1, 2, 6, 12, 15]. In [12], we obtained some results on congraphs of graph products. In this paper we continue this study and report additional results along these lines.

It should be noted that in two earlier works [3, 4] the so-called derived graph  $G^\dagger$  of the graph  $G$  was considered. The derived graph  $G^\dagger$  has the same vertex set as the parent graph  $G$ , and two vertices of  $G^\dagger$  are adjacent if and only if their distance in  $G$  is equal to

two. It is immediately seen that  $G^\dagger = \text{con}(G)$  if and only if the parent graph  $G$  does not contain triangles. Thus, in particular,  $G^\dagger = \text{con}(G)$  holds whenever  $G$  is bipartite.

The notations used in this paper is standard and taken mainly from [5, 14]. In what follows, the graphs considered are assumed to be simple. If a graph has parallel edges, we consider these as a single edge.

## 2 COMMON NEIGHBORHOOD GRAPHS OF SOME GRAPH OPERATIONS

In this section we obtain  $\text{con}(G)$  for some operations on two graphs. We begin with the tensor product. To do this, we state the following lemma which immediately follows from the definition of the operation  $\otimes$ .

**Lemma 2.1.** Let  $(v_i, u_j)$  and  $(v_r, u_s)$  be two vertices of  $G \otimes H$ . Then  $(v_k, u_t) \in N_{G \otimes H}(v_i, u_j) \cap N_{G \otimes H}(v_r, u_s)$  if and only if  $v_k \in N_G(v_i) \cap N_G(v_r)$  and  $u_t \in N_H(u_j) \cap N_H(u_s)$ .

**Theorem 2.2.** Let  $G$  and  $H$  be two graphs without isolated vertices. Then

$$\text{con}(G \otimes H) = \text{con}(G) \Omega \text{con}(H).$$

**Proof.** Let  $(v_i, u_j)$  and  $(v_r, u_s)$  be two vertices of  $G \otimes H$  such that  $v_i \neq v_r$  and  $u_j \neq u_s$ . If  $(v_i, u_j)(v_r, u_s)$  is an edge of  $\text{con}(G \otimes H)$ , then there is a vertex  $(v_k, u_t) \in V(G \otimes H)$  such that  $(v_k, u_t) \in N_{G \otimes H}(v_i, u_j) \cap N_{G \otimes H}(v_r, u_s)$ . So by Lemma 2.1,  $v_k \in N_G(v_i) \cap N_G(v_r)$  and  $u_t \in N_H(u_j) \cap N_H(u_s)$ . Therefore  $v_i v_r \in E(\text{con}(G))$  and  $u_j u_s \in E(\text{con}(H))$ . This means that for  $v_i \neq v_r$  and  $u_j \neq u_s$  it holds that  $(v_i, u_j)(v_r, u_s)$  is an edge of  $\text{con}(G \otimes H)$  if and only if  $v_i v_r \in E(\text{con}(G))$  and  $u_j u_s \in E(\text{con}(H))$ .

Assume that  $v_i = v_r = v$ . If  $(v, u_j)(v, u_s)$  is an edge of  $\text{con}(G \otimes H)$ , then there is a vertex  $(v_k, u_t)$  such that  $(v_k, u_t) \in N_{G \otimes H}(v, u_j) \cap N_{G \otimes H}(v, u_s)$ . By Lemma 2.1, we have  $v_k \in N_G(v)$  and  $u_t \in N_H(u_j) \cap N_H(u_s)$ . So if  $v_i = v_r$ , then  $u_j u_s \in E(\text{con}(H))$ . Therefore, for  $v_i = v_r$  it holds that  $(v_i, u_j)(v_r, u_s)$  is an edge of  $\text{con}(G \otimes H)$  if and only if  $v_i = v_r$  and  $u_j u_s \in E(\text{con}(H))$ . Similarly if  $u_j = u_s$ , then  $(v_i, u_j)(v_r, u_s)$  is an edge of  $\text{con}(G \otimes H)$  if and only if  $u_j = u_s$  and  $v_i v_r \in E(\text{con}(G))$ .

Hence  $\text{con}(G \otimes H) = (\text{con}(G) \otimes \text{con}(H)) \cup (\text{con}(G) \times \text{con}(H)) = \text{con}(G) \Omega \text{con}(H)$  and this completes the proof. ■

In the following theorem, we determine the congraph of Hajós join.

**Theorem 2.3.** Let  $G$  and  $H$  be two graphs with the girth at least 5. Then

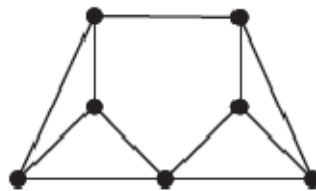
$$\begin{aligned} \text{con}(G\Delta H) = & (\text{con}(G) \cdot \text{con}(H))(v, x) - E(y + N_H(x)) - E(w + N_G(v)) - E(x + N_H(y)) \\ & - E(v + N_G(w)) \cup E((N_G(v) - \{w\}) + (N_H(x) - \{y\})) \cup E(N_G(w) + y) \\ & \cup E(N_H(y) + w), \end{aligned}$$

where for two vertices  $r, s$ , the notation  $E(r + N(s))$  denotes the edges of the join of the vertex  $r$  and the neighbors of  $s$ .

**Proof.** In the structure of Hajós join, if we don't remove two edges  $vw$  and  $xy$ , and don't add a new edge  $wy$ , we can arrive to splice of two graphs  $G$  and  $H$ . So we consider the graph  $(\text{con}(G) \cdot \text{con}(H))(v, x)$  as the base of the common neighborhood graph of the Hajós join of  $G$  and  $H$ . Then we investigate the effect of removing the two edges  $vw$  and  $xy$ , and adding a new edge  $wy$ .

Since the girth of the graph  $G$  is at least 5, when we remove the edge  $vw$ , all the edges  $wr$  and  $vs$  in  $\text{con}(G)$ ,  $r \in N_G(v)$  and  $s \in N_G(w)$ , that have  $v$  and  $w$  as the common neighbor, respectively, will be deleted. Similarly when we eliminate the edge  $xy$ , all the edges  $ya$  and  $xb$  in  $\text{con}(H)$ ,  $a \in N_H(x)$  and  $b \in N_H(y)$ , that have  $x$  and  $y$  as the common neighbor, respectively, will be deleted. Continuing this argument, when we identify the vertices  $v$  and  $x$  into a single vertex, then  $N_G(v) - w$  and  $N_H(x) - y$  will have a common neighbor. So each vertex in  $N_G(v) - w$  will be adjacent to each vertex in  $N_H(x) - y$ . By adding the new edge  $wy$ ,  $w$  will become the common neighbor between  $y$  and  $N_G(w)$  and  $y$  the common neighbor between  $w$  and  $N_H(y)$ . ■

Applying the Hajós join to two copies of  $K_4$  by identifying a vertex from each copy into a single vertex, deleting an edge incident to the combined vertex within each subgraph, and adding a new edge connecting the endpoints of the deleted edges, produces the Moser spindle, see Fig. 1. As an application we characterize the common neighborhood graph of the Moser spindle.



**Figure 1.** The Moser Spindle Graph.

**Corollary 2.4.** The common neighborhood graph of the Moser spindle is  $K_7 - e$ .

In the next theorem, we compute the common neighborhood graph of the edge corona product of graphs. One can see that the edge corona product of  $G$  with a complete graph  $K_t$ , and the common neighborhood graph of  $G$  are subgraphs of  $G \diamond H$ .

**Theorem 2.5.** Let  $G$  be a graph with  $n$  vertices and  $m$  edges and  $H$  be a graph with  $t$  vertices. Then

$$\begin{aligned} \text{con}(G \diamond H) &= (G \diamond K_t) \cup \text{con}(G) \cup \left( \bigcup_{e_k=v_i v_j} (N_G(v_i) \cup N_G(v_j)) + H_k \right) \\ &\cup \left( \bigcup_{e_i=v_l v_q, e_j=v_l v_{q'}} (H_i + H_j) \right). \end{aligned}$$

**Proof.** Let  $V = \{v_1, \dots, v_n\}$ ,  $E(G) = \{e_1, \dots, e_m\}$ , and  $V(H) = \{u_1, \dots, u_t\}$ . Denote the  $i$ -th copy of  $H$  in  $G \diamond H$ , by  $H_i$ . Each two vertices of  $H_i$  have the end vertices of  $e_i$  as common neighbors, So the induced subgraph of  $\text{con}(G \diamond H)$  on each  $H_i$  is a complete graph. On the other hand, a vertex in  $H_i$  has a common neighbor with a vertex in  $H_j$  if and only if the edges  $e_i$  and  $e_j$  are adjacent. So the induced subgraph of  $G \diamond H$  on the vertices  $H_i \cup H_j$  is  $H_i + H_j$  if and only if  $e_i$  and  $e_j$  are adjacent in  $G$  and there is no edge between  $H_i$  and  $H_j$  if  $e_i$  and  $e_j$  are not adjacent in  $G$ .

We now consider the vertices  $\{v_1, \dots, v_n\}$ . Clearly,  $v_i$  and  $v_j$  have a common neighbor  $v_k$  in  $G \diamond H$  if and only if  $v_k$  is their common neighbor in  $G$ . Also  $v_i$  and  $v_j$  have a common neighbor  $u_r$  in  $G \diamond H$ , if and only if  $v_i v_j$  is an edge of  $G$ .

Finally, a vertex  $v_s$  in  $G$  has a common neighbor with a vertex  $u_r$  in  $H_k$  if and only if  $v_s$  is in  $N_G(v_i) \cup N_G(v_j)$ , where  $e_k = v_i v_j$ . This completes the proof. ■

By definition, the edge corona  $T \diamond S_n$  of a tree  $T$  of order  $n$  and  $S_n$  is the graph obtained by taking one copy of  $T$  and  $n - 1$  copies of  $S_n$  and then joining two end-vertices of the  $i$ -th edge of  $T$  to every vertex in the  $i$ -th copy of  $S_n$ .

**Corollary 2.6.** The common neighborhood graph of the edge corona product of graphs  $K_2$  and  $S_n$  satisfies  $\text{con}(K_2 \diamond S_n) = K_{n+2}$ .

### 3. RELATION BETWEEN SOME SPECIAL GRAPHS AND THEIR CONGRAPHS

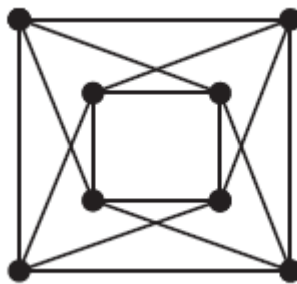
In this section we compute the common neighborhood graphs of the central graph, line graph, shadow graph, and Mycielski graph. So we should first define these graphs.

For a given graph  $G$ , the line graph of  $G$  is denoted by  $L(G)$  and the vertices of  $L(G)$  are the edges of  $G$ . Two edges of  $G$  that share a vertex are considered to be adjacent in  $L(G)$ . The subdivision graph of the graph  $G$  is denoted by  $S(G)$  and is the graph

obtained by inserting an additional vertex in each edge of  $G$ . Equivalently, each edge of  $G$  is replaced by a path of length 2.

For a given graph  $G$ , we do an operation on  $G$  by subdividing each edge exactly once and joining each pair of vertices of the original graph which were previously non-adjacent. The graph obtained by this process is said to be the central graph of  $G$ , denoted by  $C(G)$ , [17, 18, 19].

The shadow graph  $D_2(G)$  of a connected graph  $G$  is constructed by taking two copies of  $G$  say  $G'$  and  $G''$  and joining each vertex  $u'$  in  $G'$  to the neighbors of the corresponding vertex  $u''$  in  $G''$ . For example,  $D_2(C_4)$  is depicted in Fig. 2.



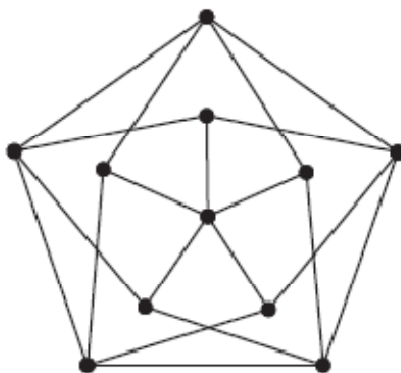
**Figure 2.** The Shadow Graph  $D_2(C_4)$ .

The Mycielski graph of  $G$  was introduced by J. Mycielski [16] for the purpose of constructing triangle-free graphs with arbitrarily large chromatic number. This graph has been much studied [7, 8, 9].

Let  $G$  be a graph with vertex set  $\{v_1, v_2, \dots, v_n\}$ . The Mycielski graph  $\mu(G)$  of  $G$  contains  $G$  itself as an isomorphic subgraph, together with  $n + 1$  additional vertices: a vertex  $u_i$  which corresponds to each vertex  $v_i$  of  $G$ , and another vertex  $w$ . Each vertex  $u_i$  is connected by an edge to  $w$ , so that these vertices form a subgraph in the form of a star  $K_{1,n}$ . In addition, for each edge  $v_i v_j$  of  $G$ , the Mycielski graph includes two edges,  $u_i v_j$  and  $v_i u_j$ . In Fig. 3 we shows Mycielski's construction applied to a 5-vertex cycle. The resulting Mycielskian is the Grötzsch graph, an 11-vertex graph with 20 edges. The Grötzsch graph is the smallest triangle-free 4-chromatic graph.

**Theorem 3.1.** Let  $G$  be a graph. Then

$$con(C(G)) = G \cup L(G) \cup con(\overline{G}) \cup \left( \bigcup_{e=uv \in E(G)} [v_e + (V(G) - (N_G(u) \cup N_G(v)))] \right)$$



**Figure 3.** The Grötzsch Graph.

**Proof.** Let  $V(G) = \{v_1, \dots, v_n\}$  and  $E(G) = \{e_1, \dots, e_m\}$ . So the set of vertices of  $C(G)$  is  $V(C(G)) = \{v_1, \dots, v_n, v_{e_1}, \dots, v_{e_m}\}$ , where  $v_{e_i}$  is the vertex inserted in the edge  $e_i$ , ( $1 \leq i \leq m$ ). We determine the graph  $con(C(G))$  in three steps:

(i) We find the edges between the vertices of  $\{v_1, v_2, \dots, v_n\}$ . The vertices  $v_i$  and  $v_j$  have a common neighbor in the set  $\{v_{e_1}, \dots, v_{e_m}\}$ , of graph  $C(G)$  if and only if  $v_i v_j$  is an edge in the graph  $G$ . Also the vertices  $v_i$  and  $v_j$  have a common neighbor in the set  $\{v_1, \dots, v_n\}$  of the graph  $C(G)$  if and only if  $v_i$  and  $v_j$  have a common neighbor in the graph  $\bar{G}$ . So the subgraph induced by the vertices  $v_1, \dots, v_n$  in the graph  $C(G)$  is  $G \cup con(\bar{G})$ .

(ii) We consider the subgraph of  $C(G)$  induced by the set  $\{v_{e_1}, \dots, v_{e_m}\}$ . It is easy to see that  $v_{e_i}$  and  $v_{e_j}$  do not have common neighbors in  $\{v_{e_1}, \dots, v_{e_m}\}$ . On the other hand,  $v_{e_i}$  and  $v_{e_j}$  have the vertex  $v_t$  as common neighbor in  $\{v_1, v_2, \dots, v_n\}$  if and only if the edges  $e_i$  and  $e_j$  have the vertex  $v_t$  as the common vertex in  $G$ . Therefore the respective induced subgraph is  $L(G)$ .

(iii) We find the edges between  $\{v_1, v_2, \dots, v_n\}$  and  $\{v_{e_1}, \dots, v_{e_m}\}$ . Let  $e_i = v_r v_s$  be an edge of  $G$ . So  $N_{C(G)}(v_{e_i}) = \{v_r, v_s\}$  and this means that  $v_{e_i}$  is adjacent in  $con(C(G))$  to the vertices that are neighbors of  $v_r$  and  $v_s$ . By the definition of  $C(G)$ , the edges between  $\{v_1, v_2, \dots, v_n\}$  and  $\{v_{e_1}, \dots, v_{e_m}\}$  are  $\cup_{e=uv \in E(G)} [v_e + (V(G) - (N_G(u) \cup N_G(v)))]$ .

Combining (i), (ii), and (iii), the theorem follows. ■

In the graph  $G$ , let  $\{e_1, \dots, e_k\}$  be all of the edges incident to vertex  $u$ . We denote the set of  $\{v_{e_1}, \dots, v_{e_k}\}$  in the graph  $L(G)$  by  $N_G(u)$ . That  $v_{e_i}$  is a vertex of  $L(G)$  corresponding to an edge  $e_i$  of  $G$ .

**Theorem 3.2.** Let  $G$  be a graph. Then  $con(L(G)) = \bigcup_{e=uv \in E(G)} (N'_G(u) + N'_G(v))$ .

**Proof.** Consider the vertices  $\{v_{e_1}, \dots, v_{e_m}\}$  in  $L(G)$ . If  $e_i = v_r v_s$  is an edge in  $G$ , then  $v_{e_i}$  can be as the common neighbor of the sets  $N'_G(v_r)$  and  $N'_G(v_s)$ . So for each  $e_i = v_r v_s$  in  $G$ ,  $N'_G(v_r) + N'_G(v_s)$  is the subgraph of  $con(L(G))$ . ■

**Theorem 3.3.** Let  $G$  be a graph without isolated vertices and  $G'$  and  $G''$  be two copies of  $G$ . Then  $con(D_2(G)) = D_2(con(G)) \cup \{v'_i v''_i \mid v'_i \in V(G'), v''_i \in V(G''), 1 \leq i \leq |V(G)|\}$ .

**Proof.** Suppose that  $V(G) = \{v_1, v_2, \dots, v_n\}$ ,  $V(G') = \{v'_1, v'_2, \dots, v'_n\}$ , and  $V(G'') = \{v''_1, v''_2, \dots, v''_n\}$ . By definition of the shadow graph, it is easy to see that  $v'_t \in N_{G'}(v'_r) \cap N_{G'}(v'_s)$  if and only if  $v'_t \in N_{G'}(v'_r) \cap N_{G'}(v'_s)$ . Similarly,  $v''_t \in N_{G''}(v''_r) \cap N_{G''}(v''_s)$  if and only if  $v''_t \in N_{G''}(v''_r) \cap N_{G''}(v''_s)$ . Therefore, the subgraph of  $con(D_2(G))$  induced on  $V(G')$  is  $con(G')$  and induced on  $V(G'')$  is  $con(G'')$ . ■

We now determine the edges between  $V(G')$  and  $V(G'')$ . To do this, for two vertices  $v_i$  and  $v_j$ ,  $i \neq j$ , we use the following facts resulting from the definition of shadow graph:

- 1)  $v'_k \in N_{G'}(v'_i) \cap N_{G''}(v''_j)$  if and only if  $v''_k \in N_{G'}(v'_i) \cap N_{G''}(v''_j)$ .
- 2)  $v'_k \in N_{G'}(v'_i) \cap N_{G'}(v'_j)$  if and only if  $v'_k \in N_{G'}(v'_i) \cap N_{G'}(v'_j)$ .

Therefore,  $uv$  is an edge of  $con(G)$  if and only if  $u'v''$  and  $v'u''$  are edges of  $con(D_2(G))$ . On the other hand, since  $G$  has no isolated vertices, for each  $i$ ,  $1 \leq i \leq |V(G)|$ ,  $v'_i v''_i$  are edges of  $con(D_2(G))$ . and the proof is completed. ■

**Corollary 3.4.** For path  $P_n$  and complete graph  $K_2$  the following equality holds:

$$con(D_2(P_n)) = con(P_n) \times K_2.$$

**Theorem 3.5.** Let  $G$  be a graph with  $n$  vertices. Then the congraph of its Mycielski graph contains  $con(G)$  as an isomorphic subgraph, together with  $n + 1$  additional vertices: a vertex  $u_i$  corresponding to each vertex  $v_i$  of  $con(G)$  such that the induced graph of  $\mu(G)$  on them is  $K_n$  and another vertex  $w$ . Each vertex  $v_i$  is connected by an edge to  $w$ , so that these vertices form a subgraph in the form of a star  $K_{1,n}$ . In addition, for each edge  $v_i v_j$  of  $con(G)$ , the common neighborhood graph of the Mycielski graph includes two edges,  $u_i v_j$  and  $v_i u_j$ .



**Proof.** Since in the Mycielski graph, each vertex  $u_i$  is connected by an edge to  $w$ , so vertex  $w$  is the common neighborhood of vertices  $u_1, \dots, u_n$  in the graph  $\mu(G)$  and this implies that the subgraph of  $\mu(G)$  induced on these vertices is  $K_n$ . It is clear that  $v_t \in N_{\mu(G)}(v_r) \cap N_{\mu(G)}(v_s)$  if and only if  $u_t \in N_{\mu(G)}(v_r) \cap N_{\mu(G)}(v_s)$ , so the subgraph of  $\mu(G)$  induced on  $\{v_1, \dots, v_n\}$  is  $con(G)$ . Also by the definition of Mycielski graph, since  $G$  has not isolated vertices, the vertices  $u_i$  are common neighborhoods of the vertices  $v_j$  and  $w$ . This implies that,  $v_i w \mid 1 \leq i \leq m$  are edges of  $con(\mu(G))$ .

Now we obtain the edges between  $\{v_1, \dots, v_n\}$  and  $\{u_1, \dots, u_n\}$ . Let the vertex  $v_k$  be the common neighbor of the vertices  $v_i$  and  $v_j$ . By the definition of Mycielski graph, we have the following cases:

**Case 1.** The vertex  $v_k$  is in the common neighborhood of vertices  $v_j$  and  $u_j$  in graph  $G$ . This implies that  $v_j u_j, 1 \leq j \leq m$  are edges of the congraph.

**Case 2.** The vertex  $v_k$  is in the common neighborhood of the vertices  $v_j$  and  $u_i, v_i$  and  $u_j$  in graph  $G$ . This implies that for each edge  $v_i v_j$  of  $con(G)$ , the common neighborhood graph of  $\mu(G)$  includes two edges  $u_i v_j$  and  $v_i u_j$  and this completes the proof. ■

As an application we compute the common neighborhood graph of Grötzsch graph.

**Corollary 3.6.** Let  $C_5 : v_1 v_2 v_3 v_4 v_5 v_1$ . Then the common neighborhood graph of the Grötzsch graph is determined via  $(w + C_5) \cup K_5 \cup \{v_i u_{i+1} \mid 1 \leq i \leq 4\} \cup \{v_{i+1} u_i \mid 1 \leq i \leq 4\} \cup v_1 u_5 \cup v_5 u_1$ .

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## On Common Neighborhood Graphs II

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### گراف‌های همسایه مشترک ۲

ادیتور رابط : تومسیلاو داسلیک

#### چکیده

فرض کنید  $G$  یک گراف ساده با مجموعه رئوس  $V(G)$  است. گراف همسایه مشترک از  $G$  که با  $con(G)$  نشان داده می‌شود، یک گراف با مجموعه رئوس  $V(G)$  است که در آن، دو رأس مجاورند اگر و فقط اگر حداقل یک همسایه مشترک در  $G$  داشته باشند. در این مقاله، ما گراف همسایه مشترک تعدادی گراف‌های مرکب را محاسبه می‌کنیم. با استفاده از این نتایج، همسایه مشترک چندین گراف خاص مشخص می‌شوند.

لغات کلیدی: گراف همسایه مشترک، گراف میسلیسکی، گراف شادو، گراف هجوس.