

# A New Family of High-Order Difference Schemes for the Solution of Second Order Boundary Value Problems

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## ARTICLE INFO

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### Article History:

Received 9 August 2017

Accepted 1 February 2018

Published online 30 August 2018

Academic Editor: Hassan Yousefi-Azari

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### Keywords:

Boundary value problems

Finite difference methods

Bratu's problem

Troesch's problem

High accuracy

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## ABSTRACT

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Many problems in chemistry, nanotechnology, biology, natural science, chemical physics and engineering are modeled by two point boundary value problems. In general, analytical solution of these problems does not exist. In this paper, we propose a new class of high-order accurate methods for solving special second order nonlinear two point boundary value problems. Local truncation errors of these methods are discussed. To illustrate the potential of the new methods, we apply them for solving some well-known problems, including Troesch's problem, Bratu's problem and certain singularly perturbed problem. Bratu's and Troesch's problems, may be used to model some chemical reaction-diffusion and heat transfer processes. We also compare the results of this work with some existing results in the literature and show that the new methods are efficient and applicable.

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## 1. INTRODUCTION

The study through boundary value problem is an interesting in recent years. This interest can be attributed due to its wide range of application in scientific research. In general, nonlinear boundary value problems do not always have solutions which we can obtain using analytical methods. Therefore, techniques for rapidly computing approximate solutions of boundary value problem are very importance.

In this paper, we introduce two fast and accurate numerical schemes for the solution of second-order nonlinear differential equations of the form

$$y'' = f(x, y), \quad a < x < b, \quad (1)$$

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DOI: 10.22052/ijmc.2018.94933.1306

subject to the boundary conditions:

$$y(a) = \alpha, \quad y(b) = \beta, \quad (2)$$

where  $a, b, \alpha$  and  $\beta$  are the given constants. The existence and uniqueness of the solutions to problem (1)-(2) are discussed in [1]. The literature on the numerical approximation of solutions of boundary value problems is large and still growing rapidly. Among the most recent works concerned with numerical methods, we can consider direct implicit block method [2], Chebyshev finite difference method [3], sinc collocation method [4, 5], compact finite difference method [6], non-standard finite difference method [7, 8] and rational finite difference method [9, 10]. Also, Ramos [11] presented a non-standard explicit algorithm for initial-value problems.

In this paper a new class of novel non-classical difference methods is proposed for the solution of problem (1)-(2). Our methods are based on the idea behind in [10, 11]. Two point boundary value problems (1)-(2) covers many interesting problems. Three of these important problems, which we consider in this paper, are as follows:

### 1.1 TROESCH'S PROBLEM

Troesch's problem is defined by

$$\begin{cases} y'' - \mu \sinh(\mu y(x)) = 0, & 0 \leq x \leq 1, \\ y(0) = 0, \quad y(1) = 1, \end{cases} \quad (3)$$

where  $\mu$  is a positive constant. This problem arises in an investigation of the confinement of a plasma column under radiation pressure [12]. Also, this problem comes from the theory of gas porous electrodes [13]. Moreover, as pointed out in [14], Troesch's problems may be used to model some chemical reaction-diffusion and heat transfer processes.

The known closed-form solution of this problem in terms of the Jacobi elliptic function is (see [15])

$$y(x) = \frac{2}{\mu} \sinh^{-1} \left\{ \frac{y'(0)}{2} \operatorname{sc} \left( \mu x \left| 1 - \frac{1}{4} y'(0)^2 \right. \right) \right\}.$$

Here  $y'(0) = 2\sqrt{1-m}$ , and the constant  $m$  satisfies the transcendental equation

$$\frac{\sinh\left(\frac{\mu}{2}\right)}{\sqrt{1-m}} = \operatorname{sc}(\mu|m),$$

where,  $\operatorname{sc}(\mu|m)$  is the Jacobi elliptic function. As is said in [16], this problem is inherently unstable and difficult, especially when the sensitivity parameter  $\mu$  is large. Therefore, Troesch's problem has become a widely used test problem, and has been studied extensively. In the last decade, variational spline method [14], discontinuous Galerkin finite element method [17], variational iteration method [18], shooting method [19], B-spline

collocation method [20], Christov collocation method [21], sinc-Galerkin method [22], nonstandard finite difference method [7], finite difference method [23] and homotopy analysis method [24] are used to solve this problem.

### 1.2 BRATU’S PROBLEM

The classical Bratu’s problem is given as:

$$\begin{cases} y'' + \lambda \exp(y) = 0, & 0 \leq x \leq 1, \\ y(0) = y(1) = 0, \end{cases} \quad (4)$$

where  $\lambda$  is a constant. For  $\lambda > 0$ , the analytical solution to this problem reads [24, 25, 26, 27],

$$y(x) = -2 \ln \left[ \frac{\cosh \left( \left( x - \frac{1}{2} \right) \theta / 2 \right)}{\cosh(\theta / 4)} \right], \quad (5)$$

where  $\theta$  satisfies  $\theta = \sqrt{2\lambda} \cosh(\theta/4)$ . It is well known that, this problem has zero, one, or two solutions when  $\lambda > \lambda_c$ ,  $\lambda = \lambda_c$  and  $\lambda < \lambda_c$ , respectively. Here  $\lambda_c$ , called the critical value, is given by  $\lambda_c = 3.513830719$  [24, 25].

The Bratu model appears in a large variety of applications such as the model of thermal reaction process, questions in geometry and relativity about the Chandrasekhar model, radiative heat transfer, nanotechnology and the fuel ignition model of the thermal combustion theory (for example, we refer the reader to see [24, 25, 26, 27, 28, 29, 30], and the references therein). Various numerical methods such as homotopy analysis method [24], Adomian decomposition method [25, 28], sinc-Galerkin method [26], B-spline method [27], pseudospectral method [29] and finite difference method [29] have been applied to this problem. Also, recently, Temimi and Ben-Romdhane [30] proposed an iterative finite difference method to solve the Bratu’s problem.

### 1.3 SINGULARLY PERTURBED PROBLEM

We consider a class of singularly perturbed boundary value problems given in [6, 31, 32] as

$$\begin{cases} -\epsilon y''(x) + p(x)y(x) = q(x), & 0 \leq x \leq 1, p(x) > 0, \\ y(0) = \alpha, \quad y(1) = \beta, \end{cases} \quad (6)$$

where  $\alpha, \beta$  are given constants and  $\epsilon \in (0, \epsilon_0)$ ,  $\epsilon_0 \ll 1$ , is a small perturbation parameter. Further,  $p(x)$  and  $q(x)$  are assumed to be sufficiently continuously differentiable functions.

This type of problem occurs in many fields of science and engineering (see [6, 31, 32]). As pointed out in [32], usual numerical treatment of singular-perturbation problems gives major computational difficulties. This problem, has been studied by several researchers. Gelu et al. [6] used sixth-order compact finite difference method and Rashidinia et al. [31] employed quantic spline method. Khan et al. [32] solved this problem by sixth-order method based on sextic splines. Also, we refer the interested readers to [33, 34, 35, 36, 37]. The organization of the rest of this paper is as follows. In Section 2, the methods are described and also local truncation errors are discussed. In section 3, the numerical results of applying the methods of this paper on three test problems are presented. Finally a conclusion is drawn in Section 4.

## 2. THE PROPOSED METHODS

To approximate the solution of problem (1)-(2), first of all, the domain  $[a, b]$  is divided into  $N$  equal subintervals of fixed mesh length  $h = (b - a)/N$ . The grid points are given by  $x_i = a + ih, i = 0, \dots, N$ , in which  $N$  is a positive integer. For convenience let  $y^{(k)}(x_i) = y_i^{(k)}$ , and  $f^{(k)}(x_i, y(x_i)) = f_i^{(k)}, k = 0, 1, 2, \dots$ . Now, following the ideas in [11, 10], we suggest the following difference equation

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\frac{h^2}{1+g(h)}} = f_i, \tag{7}$$

equivalently,

$$(y_{i+1} - 2y_i + y_{i-1})(1 + g(h)) = h^2 f_i, \tag{8}$$

where  $g(h) \neq -1$  is a sufficiently differentiable unknown function that has to be determined. Expanding  $g(h)$  in Taylor's expansion about  $h = 0$  and also expanding  $y_{i+1}$  and  $y_{i-1}$  on the left side of Eq. (8) in the neighborhood of  $x_i$  by Taylor's expansion, we obtain

$$\left( h^2 y_i'' + \frac{h^4}{12} y_i^{(4)} + \frac{h^6}{360} y_i^{(6)} + \dots \right) \left( 1 + g(0) + hg'(0) + \frac{h^2}{2} g''(0) + \dots \right) = h^2 f_i. \tag{9}$$

Now, we rewrite Eq. (9) as follows

$$\begin{aligned}
 & h^2[y_i''(1 + g(0)) - f_i] + h^3[y_i''g'(0)] + h^4 \left[ \frac{y_i''(g''(0))}{2} + \frac{y_i^{(4)}(1 + g(0))}{12} \right] \\
 & + h^5 \left[ \frac{y_i''g^{(3)}(0)}{6} + \frac{y_i^{(4)}g'(0)}{12} \right] \\
 & + h^6 \left[ \frac{y_i''g^{(4)}(0)}{24} + \frac{y_i^{(4)}g''(0)}{24} + \frac{y_i^{(6)}(1 + g(0))}{360} \right] + O(h^7) \\
 & = 0.
 \end{aligned} \tag{10}$$

In order to obtain a fourth-order scheme, the coefficients of  $h^2$ ,  $h^3$  and  $h^4$  in Eq.(10) must be zero. So, we have

$$g(0) = 0, \quad g'(0) = 0, \quad g''(0) = -\frac{1}{6} \frac{y_i^{(4)}}{y_i''}. \tag{11}$$

By substituting the above values in the Taylor series of  $g(h)$  we obtain

$$g(h) = -\frac{h^2 y_i^{(4)}}{12 y_i''} + O(h^3). \tag{12}$$

From Eqs.(8) and (12) we get

$$(y_{i+1} - 2y_i + y_{i-1}) \left( 1 - \frac{h^2 y_i^{(4)}}{12 y_i''} \right) - h^2 f_i = 0. \tag{13}$$

Therefore, using Eq. (13) and having in mind the problem (1)-(2), we obtain the numerical method given by

$$\text{Scheme 1: } \begin{cases} (y_{i+1} - 2y_i + y_{i-1}) \left( 1 - \frac{h^2 f_i^{(2)}}{12 f_i''} \right) = h^2 f_i, & i = 1, 2, \dots, N - 1, \\ y_0 = \alpha, y_N = \beta. \end{cases} \tag{14}$$

Similarly, in order to obtain a sixth-order scheme, the coefficients of  $h^2, h^3, h^4, h^5$  and  $h^6$  in Eq.(10) must be zero. So, we obtain

$$\begin{aligned}
 g(0) = g'(0) = g^{(3)}(0) = 0, \quad g''(0) &= -\frac{1}{6} \frac{y_i^{(4)}}{y_i''}, \\
 g^{(4)}(0) &= -\frac{y_i^{(4)}g''(0)}{y_i''} + y_i^{(6)} \frac{1 + g(0)}{15y_i''}.
 \end{aligned} \tag{15}$$

Therefore,

$$g(h) = -\frac{h^2}{12} \frac{y_i^{(4)}}{y_i''} + \frac{h^4}{y_i''} \left( \frac{1}{144} \frac{(y_i^{(4)})^2}{y_i''} - \frac{y_i^{(6)}}{360} \right) + O(h^5). \quad (16)$$

Employing Eqs. (1), (2), (16) and (8), we obtain the numerical method given by

$$\text{Scheme 2: } \begin{cases} (y_{i+1} - 2y_i + y_{i-1}) \left( 1 - \frac{h^2}{12} \frac{f_i^{(2)}}{f_i} + \frac{h^4}{f_i} \left( \frac{(f_i^{(2)})^2}{144f_i} - \frac{f_i^{(4)}}{360} \right) \right) = h^2 f_i, \\ i = 1, 2, \dots, N-1, \\ y_0 = \alpha, y_N = \beta. \end{cases} \quad (17)$$

### 2.1 LOCAL TRUNCATION ERROR

It follows from the construction of the methods in Eqs. (14) and (17) that the new *Scheme 1* and *Scheme 2* are at least of fourth-order and sixth-order respectively. In fact, for *Scheme 1*, let us define

$$\text{LTE}_i^1 = (y(x_i + h) - 2y(x_i) + y(x_i - h)) \left( 1 - \frac{h^2}{12} \frac{f^{(2)}(x_i, y(x_i))}{f(x_i, y(x_i))} \right) - h^2 f(x_i, y(x_i)). \quad (18)$$

After expanding each term on the right side of Eq. (18) in Taylor series about  $x_i$  and collecting terms in  $h$  we get

$$\text{LTE}_i^1 = \left( -\frac{1}{144} \frac{(y^{(4)}(x_i))^2}{y''(x_i)} + \frac{1}{360} y^{(6)}(x_i) \right) h^6 + O(h^8). \quad (19)$$

Similarly, for *Scheme 2*, we have

$$\text{LTE}_i^2 = \left( \frac{1}{1728} \frac{(y^{(4)}(x_i))^3}{(y''(x_i))^2} - \frac{1}{2160} \frac{y^{(4)}(x_i)y^{(6)}(x_i)}{y''(x_i)} + \frac{1}{20160} y^{(8)}(x_i) \right) h^8 + O(h^{10}). \quad (20)$$

### 3. NUMERICAL RESULTS

In this section, to validate the application of the presented methods to problem (1)–(2), we consider three test problems. We have computed the numerical results by Maple programming.

**Example 1.** (*Troesch's problem*) In this example we will consider Troesch's problem given in Eq. (3) for different values of the parameter  $\mu$ . We solved this problem, by

applying the techniques described in Section 2. Taking  $\mu = 0.5$  and  $\mu = 1$ , in Tables 1 and 2 we compare our results with the exact solutions given in [7]. Also, in Table 3 the numerical solution obtained by *Scheme 1* and *Scheme 2* for  $\mu = 5$  is compared with the numerical approximation of the exact solutions given by a Fortran code [20] and the numerical solution obtained by B-spline collocation method [20]. From Tables 1–3 we see that *Scheme1* and *Scheme 2* yields a reasonable numerical solution for  $\mu = 0.5, 1$  and 5. As said in [20, 23], the stiffness ratio near  $x = 1$  increases as  $\mu$  increases. For this reason, most common numerical methods fail to provide enough accurate solutions for large values of  $\mu$ . In Table 4 the numerical solution obtained by the *Scheme 2* with  $N = 300$ , for  $\mu = 10, 30$ , is compared with the results obtained in [20] by the adaptive collocation method over a non-uniform mesh using  $N = 330$  and those obtained in [23] by finite difference method (FDM) for mesh size  $N = 2000$ . It can be seen from Table 4 that the results obtained using *Scheme 2* have a good agreement with the results obtained in [20, 23].

**Table 1:** Results for Troesch’s problem ( $\mu = 0.5$ ).

x	Exact	<i>Scheme1</i>		<i>Scheme2</i>	
		N = 10	N = 20	N = 10	N = 20
0.1	0.0959443493	5.0(-10)	1.0(-10)	8.0(-10)	1.0(-10)
0.2	0.1921287477	1.0(-9)	1.0(-10)	1.4(-9)	1.0(-10)
0.3	0.2887944009	1.3(-9)	1.0(-10)	2.0(-9)	0
0.4	0.3861848464	1.7(-9)	1.0(-10)	1.0(-10)	0
0.5	0.4845471647	1.8(-9)	1.0(-10)	2.7(-9)	0
0.6	0.5841332484	1.9(-9)	1.0(-10)	2.8(-9)	0
0.7	0.6852011483	1.8(-9)	1.0(-10)	2.7(-9)	1.0(-10)
0.8	0.7880165227	1.5(-9)	1.0(-10)	2.3(-9)	1.0(-10)
0.9	0.8928542161	9.0(-9)	0	1.3(-9)	0

**Example 2.** (*Bratu’s problem*) As the second example, we consider Bratu’s problem given in Eq. (4) for different values of the parameter  $\lambda$ . Taking  $\lambda = 1, 2$ , Tables 5 and 6, show the numerical solution obtained by our methods with  $N = 200$  compared to the exact solution given by Eq. (5), as well as to the values computed by iterative finite difference (IFD) method with  $N = 1000$  given in [30] and B-spline method given in [27]. Moreover, for the critical value  $\lambda = 3.51$ , in Table 7 the numerical solution obtained by the present methods with  $N = 300$ , is compared with the B-spline method [27] and IFD method [30]. As pointed by [30], many existing numerical methods for Bratu’s problem fail to compute the solution for  $\lambda = 3.51$ . From Tables 5–7, we see that the present methods are in excellent agreement with the exact values and the IFD method. Also, the present methods are clearly reliable if compared with the B-spline method.

**Table 2:** Results for Troesch’s problem ( $\mu = 1$ ).

x	Exact	Scheme 1		Scheme 2	
		N = 10	N = 20	N = 10	N = 20
0.1	0.0846612565	2.6(-8)	1.7(-9)	4.3(-8)	2.7(-9)
0.2	0.1701713582	5.2(-8)	3.3(-9)	8.4(-8)	5.4(-9)
0.3	0.2573939080	7.6(-8)	4.7(-9)	1.2(-7)	7.8(-9)
0.4	0.3472228551	9.7(-8)	6.1(-9)	1.5(-7)	1.0(-8)
0.5	0.4405998351	1.1(-7)	7.0(-9)	1.8(-7)	1.1(-8)
0.6	0.5385343980	1.2(-7)	7.6(-9)	2.0(-7)	1.2(-8)
0.7	0.6421286091	1.2(-7)	7.5(-9)	2.0(-7)	1.2(-8)
0.8	0.7526080939	1.0(-7)	6.5(-9)	1.7(-7)	1.1(-8)
0.9	0.8713625196	6.9(-8)	4.1(-9)	1.1(-7)	7.3(-9)

**Example 3.** Consider the following singularly perturbed problem [6, 31]:

$$\begin{cases} -\epsilon y'' + y = x, & 0 \leq x \leq 1, \\ y(0) = 1, \quad y(1) = 1 + \exp\left(\frac{1}{\sqrt{\epsilon}}\right). \end{cases} \quad (21)$$

The exact solution of this problem is

$$y(x) = x + \exp\left(-\frac{x}{\sqrt{\epsilon}}\right). \quad (22)$$

This problem is solved in [6] by sixth-order compact finite difference method. Also, in [31] the authors used quintic spline method to solve this problem. For the purpose of comparison in Table 8, we compare maximum absolute errors of our methods, for different values of  $\epsilon$  and  $N$ , together with the maximum absolute errors given in [6, 31].

Furthermore, we have calculated the computational orders of our methods (denoted by C-order) with the following formula:

$$\frac{\log(E_N) - \log(E_{2N})}{\log(2)},$$

where  $E_N$  and  $E_{2N}$  are maximum absolute errors obtained using  $N$  and  $2N$  mesh intervals, respectively. The results are summarized in Tables 9 and 10. From Tables 9 and 10, we see that the computational and theoretical orders of Scheme 1 and Scheme 2 are very close to each other, i.e the order of *Scheme 1* and *Scheme 2* are  $O(h^4)$  and  $O(h^6)$ , respectively.

#### 4. CONCLUSION

In this paper, a new family of schemes for numerically solving two point boundary value problems is presented. We showed that, the order of *Scheme 1* and *Scheme 2* are  $O(h^4)$  and  $O(h^6)$ , respectively. These schemes are used for solving Troesch’s problem, Bratu’s



problem and certain singularly perturbed problem. According to the numerical results, *Scheme 1* and *Scheme 2* can handle these kind of problems effectively and the comparison show that the proposed methods are in good agreement with the existing results in the literature. Also numerical results confirm the theoretical results of the proposed techniques.

**Table 3:** Comparison of numerical solutions for Troesch’s problem ( $\mu = 5$ ).

$x$	Fortran code [20]	<i>Scheme 1</i> N = 20	<i>Scheme 2</i> N = 20	B-spline [20]
0.2	0.01075342	0.01071950	0.01070406	0.01002027
0.4	0.03320051	0.03309592	0.03304801	0.03099793
0.6	0.25821664	0.25735421	0.25695699	0.24170496
0.8	0.45506034	0.45335039	0.45258050	0.42461830

**Table 4:** Comparison of numerical solutions for Troesch’s problem ( $\mu = 10, 30$ ).

$x$	$\mu = 10$			$\mu = 30$	
	<i>Scheme 2</i>	B-spline[20]	FDM	<i>Scheme 2</i>	FDM[23]
	N = 300	N = 330	N = 2000	N = 300	N = 2000
0	0	0	0	0	0
0.1	4.204824(-5)	4.207335(-5)	4.211194(-5)	3.614375(-13)	2.500056(-13)
0.2	1.297676(-4)	1.298517(-4)	1.299642(-4)	7.277661(-12)	5.033929(-12)
0.3	3.584358(-4)	3.586905(-4)	3.589786(-4)	1.461766(-10)	1.011094(-10)
0.4	9.764246(-4)	9.771828(-4)	9.779034(-4)	2.936036(-9)	2.030831(-9)
0.5	2.655001(-3)	2.657239(-3)	2.659022(-3)	5.897186(-8)	4.079021(-8)
0.6	7.218002(-3)	7.224571(-3)	7.228934(-3)	1.184481(-6)	8.192908(-7)
0.7	1.963429(-2)	1.965351(-2)	1.966406(-2)	2.379094(-5)	1.645584(-5)
0.8	5.364813(-2)	5.370517(-2)	5.373034(-2)	4.778560(-4)	3.305241(-4)
0.9	1.518614(-1)	1.520568(-1)	1.521140(-1)	9.614584(-3)	6.644214(-3)
0.95	2.757046(-1)	2.761735(-1)		4.460814(-2)	3.026175(-2)
0.97	3.713175(-1)	3.721473(-1)		8.991531(-2)	5.753674(-2)
0.98	4.468330(-1)	4.481030(-1)		1.441330(-1)	8.223035(-2)
0.99	5.714501(-1)	5.739404(-1)		5.218877(-1)	1.269861(-1)
1	1	1	1	1	1

**Table 5:** Comparison of numerical solutions for Bratu’s problem ( $\lambda = 1$ ).

$x$	Exact	<i>Scheme 1</i>	<i>Scheme 2</i>	B-spline[27]	IDF[30]
0.1	0.049846791245	0.049846791245	0.049846791245	0.0498438103	0.049846791445
0.2	0.089189934629	0.089189934628	0.089189934629	0.0891844690	0.089189934988
0.3	0.117609095768	0.117609095767	0.117609095768	0.1176017599	0.117609096243
0.4	0.134790253884	0.134790253883	0.134790253884	0.1347817559	0.134790254431
0.5	0.140539214400	0.140539214399	0.140539214400	0.1405303221	0.140539214971
0.6	0.134790253884	0.134790253883	0.134790253884	0.1347817559	0.134790254430
0.7	0.117609095768	0.117609095767	0.117609095768	0.1176017599	0.117609096243
0.8	0.089189934629	0.089189934628	0.089189934629	0.0891844690	0.089189934988
0.9	0.049846791245	0.049846791245	0.049846791245	0.0498438103	0.049846791444

**Table 6:** Comparison of numerical solutions for Bratu’s problem ( $\lambda = 2$ ).

$x$	Exact	<i>Scheme 1</i>	<i>Scheme 2</i>	B-spline[27]	IDF[30]
0.1	0.114410743268	0.114410743264	0.114410743265	0.1143935651	0.114410743957
0.2	0.206419116488	0.206419116481	0.206419116483	0.2063865190	0.206419117764
0.3	0.273879311826	0.273879311817	0.273879311820	0.2738344125	0.273879313548
0.4	0.315089364226	0.315089364215	0.315089364220	0.3150365062	0.315089366227
0.5	0.328952421341	0.328952421330	0.328952421335	0.3288968072	0.328952423437
0.6	0.315089364226	0.315089364215	0.315089364220	0.3150365062	0.315089366228
0.7	0.273879311826	0.273879311817	0.273879311820	0.2738344125	0.273879313550
0.8	0.206419116488	0.206419116481	0.206419116483	0.2063865190	0.206419117767
0.9	0.114410743268	0.114410743264	0.114410743265	0.1143935651	0.114410743961

**Table 7:** Comparison of numerical solutions for Bratu’s problem ( $\lambda = 3.51$ ).

$x$	Exact	<i>Scheme 1</i>	<i>Scheme 2</i>	B-spline[27]	IDF[30]
0.1	0.364335803565	0.364335803086	0.364335802967	0.357388461	0.364335803565
0.2	0.677869705682	0.677869704751	0.677869704528	0.664283874	0.677869705683
0.3	0.922214197098	0.922214195783	0.922214195480	0.902930838	0.922214197097
0.4	1.078634240752	1.078634239178	1.078634238825	1.055419782	1.078634240752
0.5	1.132617978282	1.132617976616	1.132617976246	1.107989815	1.132617978283
0.6	1.078634240752	1.078634239178	1.078634238825	1.055419782	1.078634240752
0.7	0.922214197097	0.922214195783	0.922214195480	0.902930838	0.922214197097
0.8	0.677869705682	0.677869704751	0.677869704528	0.664283874	0.677869705683
0.9	0.364335803565	0.364335803086	0.364335802967	0.357388461	0.364335803565

**Table 8:** Comparison of maximum absolute errors for Example 3.

$\epsilon$	N = 16	N = 32	N = 64
<i>Scheme 1</i>			
1/16	2.96(-6)	1.85(-7)	1.15(-8)
1/32	1.19(-5)	7.45(-7)	4.67(-8)
1/64	4.74(-5)	2.98(-6)	1.87(-7)
1/128	1.78(-4)	1.19(-5)	7.46(-7)
<i>Scheme 2</i>			
1/16	7.34(-9)	1.14(-10)	1.79(-12)
1/32	5.90(-8)	9.25(-10)	1.45(-11)
1/64	4.71(-7)	7.41(-9)	1.16(-10)
1/128	3.54(-6)	5.90(-8)	9.25(-10)
Method of [6]			
1/16	8.03(-9)	1.26(-10)	1.97(-12)
1/32	6.41(-8)	1.01(-9)	1.59(-11)
1/64	5.06(-7)	8.10(-9)	1.27(-10)
1/128	3.72(-6)	6.42(-8)	1.01(-9)
Method of [31]			
1/16	2.96(-6)	1.85(-7)	1.15(-8)
1/32	1.18(-5)	7.54(-7)	4.67(-8)
1/64	4.74(-5)	2.96(-6)	1.86(-7)
1/128	1.78(-4)	1.18(-5)	7.46(-7)

**Table 9:** Errors and computational orders obtained by *Scheme 1*, for Example 3.

$N$	$\epsilon = 1/16$		$\epsilon = 1/32$		$\epsilon = 1/64$	
	$E_N$	C-order	$E_N$	C-order	$E_N$	C-order
16	2.96(-6)	--	1.19(-5)	--	4.74(-5)	--
32	1.85(-7)	3.9999	7.45(-7)	3.9975	2.98(-6)	3.9915
64	1.15(-8)	4.0078	4.67(-8)	3.9957	1.87(-7)	3.9942
128	7.26(-10)	3.9855	2.92(-9)	3.9993	1.16(-8)	4.0108

**Table 10:** Errors and computational orders obtained by *Scheme 2*, for Example 3.

$N$	$\epsilon = 1/16$		$\epsilon = 1/32$		$\epsilon = 1/64$	
	$E_N$	C-order	$E_N$	C-order	$E_N$	C-order
16	7.34(-9)	--	5.90(-8)	--	4.71(-7)	--
32	1.14(-10)	6.0086	9.25(-10)	5.9951	7.41(-9)	5.9901
64	1.79(-12)	5.9929	1.45(-11)	5.9953	1.16(-10)	5.9972
128	2.80(-14)	5.9983	2.26(-13)	6.0035	1.81(-12)	6.0019

**ACKNOWLEDGEMENT.** The authors would like to thank the three referees for their comments and suggestions that have improved the paper.

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# A new family of high-order difference schemes for the solution of second order boundary value problems

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## فانواده‌ای جدید از طرح‌های تفاضلی مرتبه بالا برای حل مسائل مقدار

### مرزی مرتبه دوم

ادیتور رابطا : مسن یوسفی آذری

#### چکیده

مدل ریاضی بسیاری از مسائل در شیمی، نانوتکنولوژی، بیولوژی، علوم طبیعی، ریاضی-شیمی و مهندسی را می‌توان به صورت مسائل مقدار مرزی دونقطه‌ای در نظر گرفت. در حالت کلی، جواب تحلیلی برای این مسائل وجود ندارد. در این مقاله، ما یک رده از روش‌های با دقت بالا برای حل حالت خاصی از مسائل مقدار مرزی دونقطه‌ای غیرخطی ارائه می‌کنیم و خطای موضعی بررسی آنها مورد بحث قرار می‌گیرد. برای نشان دادن اهمیت این روش‌های جدید، از آنها برای حل چند مسأله معروف شامل مسأله تروش، مسأله براتو و برخی از مسائل منفرد خاص استفاده می‌شود. مسائل براتو و تروش در مدل‌سازی برخی از واکنش‌های شیمیایی انتشار و فرایندهای انتقال حرارت رخ می‌دهند. همچنین ما نتایج حاصل از این مقاله را با برخی از نتایج موجود مقایسه کرده و نشان می‌دهیم که روش‌های جدید، کارآمد و قابل اجرا هستند.

**لغات کلیدی:** مسأله مقدار مرزی، روش‌های تفاضلات متناهی، مسأله براتو، مسأله تروش، دقت بالا