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Propagation Models and Fitting Them for the Boolean Random Sets

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Abstract

In order to study the relationship between random Boolean sets and some explanatory variables, this paper introduces a Propagation model. This model can be applied when corresponding Poisson process of the Boolean model is related to explanatory variables and the random grains are not affected by these variables. An approximation for the likelihood is used to find pseudo-maximum likelihood estimates of propagation model parameters when the grains are nonrandom circle with unknown radii.

Keywords: random closed set; hitting functional; Poisson point process; Boolean model; lower positive tangent points.

1. Introduction

Outcomes of many random phenomena are objects or images which can be studied as sets in R^d. For examples, consider the shape of a tumor or region affected by cancer in medicine, the area hitting by a meteorite or a bomb, the region covered by some plants or fire in a forest and the activated region of brain by some stimulant in neuroscience and so on. Thanks to the improvement of computer capabilities in storing and analyzing these images, probability and statistics experts pervasively, face these random phenomena and their realizations. To independently study the random behavior of these phenomena [6] and [4], , the theory of random closed sets is introduced. A vast literature and some models are also extended for modelling and generating the realization of these phenomena of which, the most important one is the Boolean model and some others are [1,2,3,7,10,11].

In the next Section we will discuss random closed sets and Boolean model. Most of the researchers' studies in the field of random sets are focused on parameter estimation and studying characteristics of a random set using a realization of it. As examples see [2,9,11].

However, in many studies , there are some auxiliary information as well as the recording images. These information may affect the random set distribution. For example gender, age and other biological factors may affect the distribution of tumour shape. Hence, the study of relationships between random sets and auxiliary variables can be used to predict the behavior of the random sets. For this aim as well as extending statistical methods to random sets in this paper introduces regression models for Boolean random sets. Models depending on the kind of auxiliary variables, will be called growth, propagation and growth$-$propagation models. A method for fitting propagation regression model when the grains are unknown and non-random circles is also presented.

2. Random sets and Boolean models

In [6] and [4] theories, a random closed set is defined on the base set E which in general is a locally compact, Hausdorff and separable space. Let $\mathfrak{F}$ be the set of all closed subsets of E and $\mathcal{K}$ be the set of all compact subsets of E. For any $A \subset E$, define $\mathfrak{F}_A = \{F \in \mathfrak{F} : F \cap A \neq \emptyset\}$, $\mathfrak{F}^A = \{F \in \mathfrak{F} : F \cap A = \emptyset\}$.It can be shown that the collection of sets in form of $\mathfrak{F}^K \cap \mathfrak{F}_{G_1} \cap \cdots \cap \mathfrak{F}_{G_n}$ for all $n = 1,2,\ldots$ where $K \in \mathcal{K}$ and $G_1, \ldots, G_n$ are open subset of E is a

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topologic base on E. Generated topology with this base is called the hit-or-miss topology. Let Σ be the σ-field generated by the open sets of the hit-or-miss topology and $(\Omega, \mathrm{A}, \mathrm{P})$ be a probability space. A random closed set Y is defined as a measurable mapping from (Ω, A) to (Φ, Σ). The induced probability by Y on Σ is:

$$P_Y(B) = P(Y^{-1}(B)) \qquad \forall B \in \Sigma.$$

This probability function is too complicated to use. Fortunately, between this probability function and hitting function of Y i.e. $T_Y(K) = P_Y(Y \cap K \neq \varnothing) = P(\{Y \cap K \neq \varnothing\}), \quad K \in \mathcal{K}$ there exits a one to one correspondence [4]. Hitting function which has the same rule of cumulative distribution function of a random variable is more suitable to study than the distribution of the random set Y.

In the sequel we will assume $E = \mathrm{P}^d$ and review some characteristics and parameters of a random closed set. A random set Y on P^d is called stationary if its hitting functional is invariant under translation and is called isotropic if its hitting function is invariant under rotation.

One of the parameters used to describe the random set is volume fraction. The volume fraction p is the mean fraction of volume occupied by Y in a region of unit volume. For stationary random sets, this quantity does not depend on the choice of the region, and it can be shown that $p = P_Y(o \in Y) = T_Y(\{o\})$ i.e. it is probability of hitting Y to origin.

One of the models that can generate such random closed sets is ``Boolean model". A Boolean model (or Boolean random set) is formed by placing random closed sets at the points of a homogeneous Poisson point process and taking the union of these sets. To be exact, if D is a homogeneous Poisson point process with intensity λ and for $i = 1,2,\ldots$; Z_i's are independent copies of random closed set Z_0 a Boolean model Y is defined as

$$Y = \bigcup_{d_i \in D} (Z_i \oplus d_i) \qquad (1)$$

where $Z_i \oplus d_i = \{z + d_i : z \in Z_i\}$. The points of the Poisson process are called the germs and the associated random sets the grains.

It can be shown (for example see [11]) the hitting functional of a Boolean model is:

$$T_Y(K) = 1 - \exp\{-\lambda E[\| Z_0 \oplus \breve{K} \|]\},$$

where $\breve{K} = \{-k : k \in K\}$, $Z_0 \oplus \breve{K} = \{z - k : z \in Z_0, k \in K\}$ and $\| Z_0 \oplus \breve{K} \|$ is the Lebesgue measure of $\| Z_0 \oplus \breve{K} \|$. From this equation For a Boolean random set we have $p = 1 - \exp\{-\lambda E[\| Z_0 \|]\}$.

Finally we get familiar with the point process of tangent points which are used in constructing estimator of λ and the grain distribution (for example see [2] and [8]) which in this paper are used for fitting the propagation model. If Z_i is almost surely convex and u is unique vector in upwards direction then the first touched point of the hyperplane with normal vector u with Z_i, is called a lower positive tangent point. Some of these tangent points are covered by other grains while other points are visible. These exposed, or observable, tangent points form a point process with intensity $\lambda(1-p) = \lambda \exp\{-\lambda E[\| Z_0 \|]\}$ which is also called a lower positive point process, see [8]. Thus we can simply estimate λ, by

$$\hat{\lambda} = \frac{n^+}{\| W \| (1 - \hat{p})}, \qquad (2)$$

where n^+ is the number of lower positive tangent points in window W and $\hat{p} = \frac{\| Y \cap W \|}{\| W \|}$ is an unbiased estimator of p.

3. Propagation model

Assume that $X = (x_1, x_2, \ldots, x_k)'$ is a vector of explanatory variables which we want to study its relation with the Boolean random set Y and evaluate effects of its elements on Y. The fact that the Boolean random sets are affected by two random sources; the Poisson process D and the distribution of the Z_0, leads us to classify the explanatory variables into three general categories:

- The explanatory variables that just affect D which We will call these propagation variables.
- The explanatory variables that just have effect on Z_0. Which We will call growth variables.
- Propagation - growth explanatory variables are the ones that affect both D and Z_0.

The natural extension of Boolean model (1) which shows the relationship of that with X is

$$Y_X = \bigcup_{d_i \in D_X} (Z_{iX} \oplus d_i), \qquad (3)$$

where conditional on X, Z_{iX} for $i = 1,2,\ldots$, are iid copies of Z_{0X} and D_X is a Poisson point process with the intensity λ_X. Obviously the vector of explanatory variables can contain all above three types of variables. In fact, establishing the type of explanatory variables affecting Y is one of the main goals of the analysis. In this paper all of explanatory variables are assumed to be propagation variables. In this case the model (3) reduces to

$$Y_X = \bigcup_{d_i \in D_X} (Z_i \oplus d_i). \tag{4}$$

In addition we suppose D_X is a Poisson point process with the intensity $\lambda_X = f(X, \boldsymbol{\beta})$ where f is a positive value and differentiable function and $\boldsymbol{\beta}$ is a vector of unknown parameters of the model, and Z_i's are i.i.d. copies of Z_0. We call this model a propagation model.

Example 1: Figure 1, shows 8 simulated realization of model (4) in 1×1 windows, together with value of an explanatory variable x, when $\lambda_i = \exp(\beta_0 + \beta_1 x_i)$ and Z_0 is nonrandom circle with radius R, for $\beta_0 = 7.1$, $\beta_1 = -3$ and $R = 0.0785$.

In the next section we shall introduce a method for fitting this model to the observation given in Figure 1, in general observations $(Y_i, X_i), i = 1, 2, \ldots, n$, where Y_i is a realization of the Boolean model Y_{X_i} in window W_i and $X_i = (x_{i1}, \ldots, x_{ik})$.

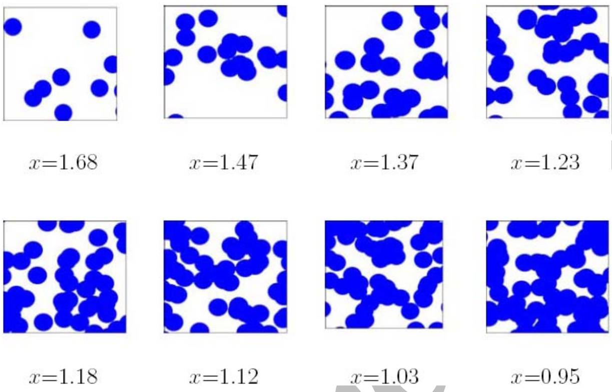

Fig. 1. Realization of propagation model.

4. Estimation of parameters

when the distribution of grains is completely known, in [5] ,three methods for fitting propagation model are presented. In two methods they used the idea that the n_i s, the number of Poisson process in windows W_i s, have Poisson distribution with mean λ_i. They substitute an appropriate estimate for nonobservable n_i s. In the third method, they used number of lower positive tangent points in W_i s (which are observed but their exact distribution is not known) and the approximate Poisson distribution. Here the third method will be adjusted in order to find the maximum likelihood estimate of propagation model parameters when grains are nonrandom circle with unknown radii. In fact if n_i^+ s are exposed to lower positive tangent points in windows W_i s we will use the following approximation:

$$n_i^+ \sim Poisson\ (\| W_i \| \lambda_i \exp\{-E[\| Z_0 \|]\lambda_i\}). \tag{5}$$

when Z_0 is a circle with radius R, $\| Z_0 \| = \pi R^2$ and if let $\lambda_i^* = \| W_i \| \lambda_i \exp\{-\pi R^2 \lambda_i\}$, then log-likelihood function is

$$l(\boldsymbol{\beta}, c) = \sum_{i=1}^{n} \{-\lambda_i^* + (n_i^+) \ln \lambda_i^*\} - \sum_{i=1}^{n} \ln n_i^+ .$$

And likelihood equations are

$$\begin{cases} \sum (n_i^+ - \lambda_i^*)(\frac{1}{\lambda_i} - \pi R^2) \frac{\partial \lambda_i}{\partial \beta_j} = 0, \qquad j = 1, \ldots, p \\ \sum (n_i^+ - \lambda_i^*) \lambda_i = 0 \end{cases} \tag{6}$$

also Fisher information matrix elements are

$$\begin{aligned} -E[\frac{\partial^2 l(\boldsymbol{\beta}, R)}{\partial \beta_i \partial \beta_j}] &= \sum_{k=1}^{n} \lambda_k^* (\frac{1}{\lambda_k} - \pi R^2)^2 \frac{\partial \lambda_k}{\partial \beta_i} \times \frac{\partial \lambda_k}{\partial \beta_j}, \\ -E[\frac{\partial^2 l(\boldsymbol{\beta}, R)}{\partial \beta_i \partial R}] &= -2\pi R \sum_{k=1}^{n} \lambda_k \lambda_k^* (\frac{1}{\lambda_k} - \pi R^2) \frac{\partial \lambda_k}{\partial \beta_i} \\ -E[\frac{\partial^2 l(\boldsymbol{\beta}, R)}{(\partial R)^2}] &= (2\pi R)^2 \sum_{k=1}^{n} \lambda_k^2 \lambda_k^* . \end{aligned} \tag{7}$$

Example 2: In the special case where $\lambda_i = \exp(\beta_0 + \beta_1 X_i)$, likelihood equations (6) have the following form

$$\begin{cases} \sum (n_i^+ - \lambda_i^*)(1 - \pi R^2 \lambda_i) = 0 \\ \sum x_i (n_i^+ - \lambda_i^*)(1 - \pi R^2 \lambda_i) = 0 \\ \sum (n_i^+ - \lambda_i^*) \lambda_i = 0 \end{cases} \tag{8}$$

in this case with using (7) Fisher information matrix is

$$\begin{bmatrix} \sum \lambda_i^* (1 - \pi R^2 \lambda_i)^2 & \sum x_i \lambda_i^* (1 - \pi R^2 \lambda_i)^2 & -2\pi R \sum \lambda_i \lambda_i^* (1 - \pi R^2 \lambda_i) \\ \sum x_i \lambda_i^* (1 - \pi R^2 \lambda_i)^2 & \sum x_i^2 \lambda_i^* (1 - \pi R^2 \lambda_i)^2 & -2\pi R \sum x_i \lambda_i \lambda_i^* (1 - \pi R^2 \lambda_i) \\ -2\pi R \sum \lambda_i \lambda_i^* (1 - \pi R^2 \lambda_i) & -2\pi R \sum x_i \lambda_i \lambda_i^* (1 - \pi R^2 \lambda_i) & (2\pi R)^2 \sum \lambda_i^2 \lambda_i^* \end{bmatrix} \tag{9}$$

Table 1
Values of auxiliary variable and number of lower positive tangent points corresponding to images in Figure 1.

i	1	2	3	4	5	6	7	8
x	1.68	1.47	1.37	1.23	1.18	1.12	1.03	0.95
n^+	7	14	19	13	22	20	24	14

Table 1 shows the number of lower positive tangent points for images given in the Figure 1. Replacing this values in equations (8) and solving these equations, the following maximum likelihood estimates are obtained,

$$\hat{\beta}_0 = 7.86, \quad \hat{\beta}_1 = -3.38, \quad \hat{R} = 0.0753,$$

Replacing these values in matrix (9) and calculating its inverse the following asymptotic variances are obtained,

$$Var\,(\hat{\beta}_0) = 1.047, \quad Var\,(\hat{\beta}_1) = 0.795, \quad Var\,(\hat{R}) = 0.005,$$

so the fitted propagation model for images given in the Figure 1 is

$$\hat{Y}_x = \bigcup_{d_i \in \hat{D}_x} (\hat{Z}_i \oplus d_i)$$

where $\hat{D}_x$ is Poisson point process with intensity $\hat{\lambda} = \exp(7.86 - 3.38x)$ and $\hat{z}_i$s are circles with radii $\hat{R} = 0.0753$.

As it can be seen the fitted model is also so close to the model generating the observations in the Figure 1. Our simulation studies show that although distribution of (5) is an approximation, the obtained maximum likelihood estimators have the usual large sample properties of maximum likelihood estimators, i.e., they are asymptotically unbiased with covariance matrix equal to the inverse of the Fisher information matrix and they have asymptotic normal distribution.

5. Conclusion

This paper introduces a propagation model to determine the relationship between Boolean random sets and some explanatory variables. Using an approximate likelihood function, this paper provides pseudo ml estimation for parameters for a propagation model. Despite using such approximate likelihood function, we conjecture, from several simulation studies, that the properties of the exact mle, for large sample, have been met by our pseudo mle. Our study shows that such model and method appropriate for practical applications. More researches have to be done regarding goodness of fit of model and model selection.

6. References

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