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The Use of the He's Iteration Method for Solving Nonlinear Equations Using CADNA Library

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Abstract. In this paper, we apply the Newton's and He's iteration formulas in order to solve the nonlinear algebraic equations. In this case, we use the stochastic arithmetic and the CESTAC method to validate the results. We show that the He's iteration formula is more reliable than the Newton's iteration formula by using the CADNA library.

Keywords: Newton's iteration method, He's iteration method, Nonlinear equations, Stochastic arithmetic, CESTAC method,CADNA Library.

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1. Introduction

The stochastic arithmetic [6-10,17] covers a large part of the properties of exact arithmetic, properties which are lost in the usual floating-point arithmetic. The CADNA (control of accuracy and debugging for numerical application) library [18] is a tool for automatic implementation of the stochastic arithmetic in any Fortran or C programming language. By the use of CADNA library, it is possible during the run of a program, to detect the numerical instabilities and to stop correctly any iterative process. In short, the stochastic arithmetic serves to validate the results provided by a computer, and to assure the user the reliability of scientific computations.

The basic idea of the CESTAC (control et estimation stochastique des arrondis de calculs) method [17] is to replace the usual floating-point arithmetic with a random arithmetic. Consequently, each result appears as a random variable. Some applications of the stochastic arithmetic have been presented in [1-4,9,11].

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In this paper, we apply the CADNA library order to find the real root of a nonlinear equation based on the stochastic arithmetic. In this case, we consider the well known Newton's iteration formula [5] and the He's iteration formula [12-15]. In some examples. we illustrate that the He's iteration formula is faster and more accurate than the Newton's iteration formula.

In section 2, a brief description of the stochastic arithmetic and the CESTAC method are given. In section 3, the He's iteration formula is introduced for solving a nonlinear equation. In section 4, some numerical examples are tested by the CADNA library and compare the results of both methods with each other and show the effectiveness of the He's iteration method.

2. CESTAC Method-Stochastic Arithmetic

Let F be the set of all the values representable in the computer. Thus, any value $r \in$
$mathbbR$ is represented in the form of $R \in F$ in the computer. It has been mentioned in [17] that in a binary floating-point arithmetic with p mantissa bits, the rounding error stems from assignment operator is

$$R = r - \epsilon 2^{E-p}\alpha. \tag{1}$$

In relation (1), ϵ is the sign of r and $2^{-p}\alpha$ is the lost part of the mantissa due to round-off error and E is the binary exponent of the result. In single precision case, $p = 24$ and in double precision case, $p = 53$. Also if the floating-point arithmetic is as rounding to $+\infty$ or $-\infty$ then $-1 \leqslant \alpha \leqslant 1$.

According to (1) , if we want to perturb the last mantissa bit of the value r, it is sufficient that we change α in the interval $[-1, 1]$. In the CESTAC method if the arithmetic is considered as rounding to $+\infty$ or $-\infty$, α can be considered as a random variable uniformly distributed on $[-1, 1]$. Thus R, the calculated result, is a random variable and its precision depend s on its mean (μ) and its standard deviation (σ).

The idea of the CESTAC method is to consider that every result $R \in F$ of a floating-point operation corresponds to two informatical results, one rounded off from below (R^-) , the second rounded off from above (R^+), each of them representing the exact arithmetical result $r \in \mathbb{R}$, with equal validity. If a computer program is performed N times, the distribution of the results $R_i, i = 1, ..., N$ is quasi-Gaussian which their mean is equal to the exact value r, that is $E(R) = r$ [6-8,17]. These N samples are used for estimating the values μ and σ.

In practice, the samples R_i are obtained by perturbation of the last mantissa bit (or previous bits if necessary) of every result R, then the mean of random samples R_i, that is $\overline{R} = \frac{\sum_{i=1}^{N} R_i}{N}$, is considered as the result of an arithmetical operation. If $N = 3$, it has been proved in [16] that the number of exact significant digits common to $\overline{R}$ and to the exact value r can be estimated by,

$$C_{\overline{R},r} = \log_{10} \frac{|\,\overline{R}\,|}{\sigma} - 0.39. \tag{2}$$

In relation (2), σ is the standard deviation of the samples R_i which is given by, $\sigma = \sqrt{\frac{\sum_{i=1}^{N}\left(R_i - \overline{R}\right)^2}{N-1}}$.

In the CESTAC method if $C_{\overline{R},r} \leqslant 0$ or $\overline{R} = 0$ then R is called an informatical or stochastic zero. In this case, we write $R = @.0$ and it means the informatical result R is insignificant.

In order to simultaneous implementation of the CESTAC method we should substitute a stochastic arithmetic in place of the floating-point arithmetic. In this way every arithmetical operation is performed N times synchronously before running the next operation. If $N = 3$, the relation (2) can be used to estimate the number of exact significant digits of any result of any arithmetical operation. By using of the stochastic arithmetic, sudden losses of accuracy, numerical instabilities, and the appearance of an insignificant result (stochastic zero) are detected [17]. CADNA library is a tool for automatic implementation of the stochastic arithmetic, i.e. the automatic synchronous implementation of the CESTAC method. It enables a user to run the scientific code with this new arithmetic on the computer, without having to rewrite or even substantially modify the initial source code. This library which was proposed by Chesneaux was created thanks to the any FORTRAN and C programming languages. The detail information about the CADNA library and its properties have been explained in [17,18].

3. He's Iteration Method

Consider the nonlinear equation

$$f(x) = 0. \tag{3}$$

The well-known Newton's iteration formula for solving (3) is [5],

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \ n = 0, 1, 2, ... \tag{4}$$

The iteration formula (4) is widely used in numerical calculations. But, this method has some disadvantages [5]:

1) It is sensitive to initial guess x_0,

2) In the case $f'(x_n) \approx 0$, it may become an invalid method. To overcome these shortcomings, many modifications of the Newton's iteration method were proposed. Among these methods the He's iteration method is more effective [12-15]. This method is based on the general Lagrange's multiplier which is used to re-derive the above formulation. If x_n is an approximate root of Eq. (3) then, $f(x_n) \neq 0$. Now, we write a correction equation in the form,

$$x_{n+1} = x_n + \lambda f(x_n), \tag{5}$$

where, λ is a general Lagrange multiplier [15], which can be identified optimally by setting $\frac{dx_{n+1}}{dx_n} = 0$. Therefore, we can identify the multiplier as follows:

$$\lambda = \frac{-1}{f'(x_n)}. \tag{6}$$

By substituting the identified multiplier in (5), we obtain the well-known Newton's iteration formula. The above idea was first proposed by Inokuti [16], and was further developed to the well-known variational iteration method by He [12-15]. Now, we rewrite the correction equation (5) in an alternative way [15] as follows,

$$x_{n+1} = x_n + \lambda f(x_n)e^{\alpha x_n}, \tag{7}$$

where, α is a free parameter. By setting $\frac{dx_{n+1}}{dx_n} = 0$, we have

$$1 + \alpha\lambda e^{\alpha x_n} f(x_n) + \lambda e^{\alpha x_n} f'(x_n) = 0. \tag{8}$$

From (8), we can identify the multiplier in the form of,

$$\lambda = -\frac{1}{e^{\alpha x_n}(f'(x_n) + \alpha f(x_n))}. \tag{9}$$

Therefore, we obtain the following iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \alpha f(x_n)}. \tag{10}$$

The free parameter α can be used as a control parameter to adjust convergence in each step if necessary. The relation (10) is called the He's iteration formula which is very effective when $f'(x_n)$ is small.

4. Numerical Examples

In this section, four examples are evaluated to illustrate the effectiveness of the He's iteration method mentioned in (10) by using the CADNA library. The programs have been provided with C++. The computed values have obtained by using the Newton's iteration method and He's iteration method in the stochastic arithmetic. The termination criterion in both methods is $|x_n - x_{n+1}| = @.0$. It means that when the difference between two sequential results of the sequence is equal to the informatical zero then the computations must be stop. In this case, the continuation of execution is useless and n is the optimal step and x_n is the optimal approximation of the exact root.

Example 4.1 In this example, the root of the equation $\sin x = 0$, with $x_0 = 1.6$ and $\alpha = -1$ is computed. The exact root is $x = \pi$. In this case $f'(x_0)$ is a small value. The results are shown in tables 1 and 2 in single precision. The tables show that the He's iteration method is more accurate than the Newton's iteration method. In this case, the approximate root of the equation is $0.3141592E + 01$ with optimal step $n = 5$ using the He's iteration method. But, the obtained result in the Newton's method is not reliable in this step.

Table 1. Newton's iteration method in the stochastic arithmetic for example 4.1

n	x_n	$\|x_n - x_{n+1}\|$	$\|x_n - x\|$
1	0.35832E+02	0.34232E+02	0.32690407E+02
2	0.3255E+02	0.328E+01	0.29408407E+02
3	0.304E+02	0.24E+01	0.27258407E+02
4	0.320E+02	0.15E+01	0.28858407+E02
5	0.313E+02	0.6E+00	0.28158407E+02
6	0.314160E+02=@.0	@.0	0.28274407E+02

Table 2. He's iteration method in the stochastic arithmetic for example 4.1

n	x_n	$\|x_n - x_{n+1}\|$	$\|x_n - x\|$
1	0.2571617E+01	0.9716171E+00	0.5699756E+00
2	0.2962208E+01	0.390590E+00	0.1793846E+00
3	0.3115707E+01	0.1534991E+00	0.0258856E+00
4	0.3140945E+01	0.25237E-01	0.6E-6
5	0.3141592E+01	0.646E-03	@.0
6	0.3141592E+01	@.0	@.0

Example 4.2 In this example, the numerical solution of the equation $x^{10} - 1 = 0$ with $x_0 = 0.5$ and $\alpha = -1$ is considered. The exact solution is $x = 1$. The results are shown in tables 3 and 4 in single precision. Tables 3 and 4 illustrate that the He's iteration method is more faster than the Newton's iteration method. In this case, the optimal step is $n = 43$ for the Newton's method, but the optimal step is $n = 14$ for the He's iteration method.

Table 3. Newton's iteration method in the stochastic arithmetic for example 4.2

n	x_n	$\|x_n - x_{n+1}\|$	$\|x_n - x\|$
1	0.516499 E+02	0.511499 E+02	0.506499E+02
2	0.464849 E+02	0.516499E+01	0.454849E+02
3	0.418364 E+02	0.464849 E+01	0.408364E+02
⋮	⋮	⋮	⋮
31	0.218955 E+01	0.243246 E+00	0.118955E+01
32	0.197068 E+01	0.218868 E+00	0.987068E+00
⋮	⋮	⋮	⋮
41	0.1000024 E+01	0.2292 E-02	0.24E-05
42	0.100000 E+01	0.23 E-04	@.0
43	0.1000000 E+01	@.0	@.0

Table 4. He's iteration method in the stochastic arithmetic for example 4.2

n	x_n	$\|x_n - x_{n+1}\|$	$\|x_n - x\|$
1	0.517910 E +00	0.101791 E +01	0.48209E+00
2	0.1489135 E +01	0.9712247 E +00	0.1489135E+00
⋮	⋮	⋮	⋮
8	0.100671 E + 01	0.20439 E - 01	0.671E-02
⋮	⋮	⋮	⋮
12	0.1000005 E +01	0.30 E -04	0.5E-06
13	0.1000001 E +01	0.4 E -05	@.0
14	0.100000 E +01	@.0	@.0

Example 4.3 In this example, we approximate the root of the equation $xsinx + cosx = 0$ with $x_0 = 1.0$ and $\alpha = -1$. The results are shown in tables 5 and 6 in single precision. Tables 5 and 6 show that the He's iteration method determines an accurate solution but the Newton's iteration method is not an appropriate method to find the root of the equation. As we observe, the optimal step in the He's iteration method is $n = 4$ with approximate root $0.2798386E + 01$. But, the result of the Newton's method is not correct.

Example 4.4 In this example, the numerical solution of the nonlinear equation $e^{sinx} - x = 0$ with $x_0 = 1.0$ and $\alpha = -1$ is considered. The results are shown in tables 7 and 8 in single precision. One can see the faster convergence of the He's iteration method.

Table 5. Newton's iteration method in the stochastic arithmetic for example 4.3

n	x_n	$\|x_n - x_{n+1}\|$	$\|x_n - x\|$
1	-0.155740E+01	0.255740+E01	0.43558E+01
2	0.7377E+02	0.7532E+02	0.709716E+02
3	0.56E+01	0.1E+02	0.28016E+01
4	0.56E+02	@ .0	—-

Table 6. He's iteration method in the stochastic arithmetic for example 4.3

n	x_n	$\|x_n - x_{n+1}\|$	$\|x_n - x\|$
1	0.2642092E+01	0.164209E+01	0.156308E+00
2	0.2785316E+01	0.14322E+00	0.013084+00
3	0.2798278E+01	0.1296E-01	0.122E-03
4	0.2798386E+01	0.10E-03	0.14E-04
5	0.2798385E+01	@.0	@.0

Table 7. Newton's iteration method in the stochastic arithmetic for example 4.4

n	x_n	$\|x_n - x_{n+1}\|$	$\|x_n - x\|$
1	-0.420867E+01	0.520867E+01	0.642777E+01
2	-0.11470E+01	0.30616E+01	0.33661E+01
3	0.70852E+00	0.1855584E+01	0.48942E+00
4	-0.19440E+01	0.26525E+01	0.41631E+00
⋮	⋮	⋮	⋮
18	0.2199422E+01	0.577091E-01	0.016678E+00
19	0.2219192E+01	0.2977E-03	0.92E-04
20	0.2219107E+01	0.850 E-06	0.7E-05
21	0.2219107E+01	@.0	@.0

Table 8. He's iteration method in the stochastic arithmetic for example 4.4

n	x_n	$\|x_n - x_{n+1}\|$	$\|x_n - x\|$
1	0.213760E+01	0.123760E+01	0.534E-02
2	0.2218827E+01	0.1877E-01	0.273E-03
3	0.22191E+01	0.280E-03	@.0
4	0.2219107E+01	@.0	@.0

5. Conclusion

In this work, we observed that the He's iteration formula can be faster and more accurate method in comparison with the Newton's iteration method to solve a nonlinear equation. In this case, we applied the stochastic arithmetic and the CESTAC method to show the accuracy of the results. For this purpose, we use the CADNA library to validate the results. We concluded that the He's iteration formula is a reliable method for finding the roots of a nonlinear equation, but the Newton's method maybe an invalid method. So, if we choose an appropriate value for the parameter α in the He's iteration formula, we can ensure that the obtained sequence is convergent and we achieve to the solution of the equation rapidly. Also, in the He's iteration method the optimal number of iteration is less than the Newton's iteration method with an accurate value for the root of a nonlinear equation. Throughout this paper, we write $E = \varepsilon(h)$ as shorthand for the inequality $\mid E \mid \leqslant ch^{\delta}$ that c and δ are positive constants.

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