



## Non-Polynomial Spline Solutions for Special Nonlinear Fourth-Order Boundary Value Problems

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**Abstract.** We present a sixth-order non-polynomial spline method for the solutions of two-point nonlinear boundary value problem  $u^{(4)} + f(x, u) = 0, u(a) = \alpha_1, u''(a) = \alpha_2, u(b) = \beta_1, u''(b) = \beta_2$ , in off step points. Numerical method of sixth-order with end conditions of the order 6 is derived. The convergence analysis of the method has been discussed. Numerical examples are presented to illustrate the applications of method, and to compare the computed results with other known methods.

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**Keywords:** Non-polynomial spline, Boundary formula, Convergence analysis.

### Index to information contained in this paper

1. Introduction
2. Non-Polynomial Quintic Spline Functions
3. End Condition
4. Convergence Analysis
5. Numerical Results
6. Conclusion

### 1. Introduction

Consider the special nonlinear fourth-order boundary value problem given by

$$u^{(4)} + f(x, u) = 0, a < x < b, a, b, x \in \mathfrak{R}, \quad (1)$$

subjected to boundary conditions

$$u(a) = \alpha_1, u''(a) = \alpha_2, u(b) = \beta_1, u''(b) = \beta_2. \quad (2)$$

It is assumed that  $f(x, u)$  is real and continuous on  $[a, b]$ , and  $\alpha_i, \beta_i, i = 1, 2$  are finite real constants. The existence and uniqueness of the real valued function  $u(x)$  which satisfies (1)-(2) has been given in [1]. E. H. Twizell in [20] derived a fourth-order finite difference method for the numerical solution of (1)-(2). C. P. Katti [11] has given a sixth order finite difference method for the two-point boundary value problem (1) with boundary conditions of first-order derivatives.

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In the case of linear differential equations (1), the class of fourth-order two-point boundary value problems have been solved by some authors see Usmani [22], [23], [24], and [25], Usmani et al. [26], Rashidinia et al. [16]-[18] and references therein. Numerical methods based on the finite difference of the various orders by which the solution of (1) are approximated over a finite set of grid points have been developed by Chawla et al. [5]-[6], Jain et al. [10]-[9] and references therein. Daele et al. [7] introduced a new second order method for solving the boundary value problems (1) based on non-polynomial spline function. Al-Said et al. [2]-[3] developed numerical methods for solutions of fourth-order obstacle problems with collocation, finite difference and spline techniques. S. S. Siddiqi and G. Akram [19] analyzed a system of fourth-order boundary value problems using non-polynomial spline functions. M.A. Ramadan et al [15] developed quintic non-polynomial spline solutions for fourth-order two-point boundary value problem. Siraj-ul-Islam et al. [21] developed numerical methods based on quartic non-polynomial splines for solution of of a system of third-order boundary-value problems. M.A.Khan et al. [13] developed and analyzed a class of methods based on non-polynomial sextic spline functions for the solution of a special fifth-order boundary-value problems. Khan et al.[12] have used parametric quintic spline function for the solution of a system of fourth-order boundary-value problems. Numerical methods for nonlinear fourth-order boundary value problems have been study by Mohamed Alihajji and Kamel Al-khaled [4]. Wazwaz [27] applied ADM for solving a special  $2m$  order boundary value problem of the form  $u^{(2m)}(x) = f(x, u)$ ,  $0 < x < b$ .

In this paper non-polynomial quintic spline relations have been derived using off-step points. We apply such non-polynomial quintic spline functions that have polynomial and trigonometric parts to develop new numerical method for obtaining smooth approximations to the solutions (1)-(2). Non-polynomial quintic spline formulation is derived in section 2. We develop the  $O(h^{10})$  methods at end conditions in section 3. In section 4, convergence analysis is proved. Finally, in section 5, Numerical examples are given to illustrate the applications of the method.

## 2. Non-Polynomial Quintic Spline Functions

We introduce the set of grid points in the interval  $[a, b]$

$$x_0 = a, \quad x_{l-\frac{1}{2}} = a + (l - \frac{1}{2})h, \quad h = \frac{b-a}{N}, \quad l = 1, 2, \dots, N, \quad x_N = b.$$

Non-polynomial quintic spline function  $S_l(x)$  which interpolates  $u(x)$  at the mesh points  $x_{l-\frac{1}{2}}$ ,  $l = 1, 2, \dots, N$ , depends on a parameter  $\tau$  and reduces to ordinary quintic spline  $S_l(x)$  in  $[a, b]$  as  $\tau \rightarrow 0$ .

For each segment  $[x_{l-\frac{1}{2}}, x_{l+\frac{1}{2}}]$ ,  $l = 1, 2, \dots, N-1$ , the quintic spline  $S_l(x)$ , is defined as

$$S_l(x) = \sum_{i=0}^3 a_{li}(x - x_l)^i + e_l \sin \tau(x - x_l) + f_l \cos \tau(x - x_l), \quad l = 0, 1, 2, \dots, N, \quad (3)$$

where  $a_{li}$ , ( $i = 0, 1, 2, 3$ ),  $e_l$  and  $f_l$  are constants and  $\tau$  is free parameter.

Let  $u_l$  be an approximation to  $u(x_l)$ , obtained by the segment  $S_l(x)$  of the mixed spline function passing through the points  $(x_{l-\frac{1}{2}}, u_{l-\frac{1}{2}})$  and  $(x_{l+\frac{1}{2}}, u_{l+\frac{1}{2}})$ , to obtain the necessary conditions for the coefficients introduced in (3), we do not only require that  $S_l(x)$  satisfies interpolatory conditions at  $x_{l-\frac{1}{2}}, x_{l+\frac{1}{2}}$

but also the continuity of second and four derivatives at the common nodes  $(x_l, u_l)$ .

To derive expression for the coefficients, we first denote:

$$S_l(x_{l\pm\frac{1}{2}}) = u_{l\pm\frac{1}{2}}, \quad S_l''(x_{l\pm\frac{1}{2}}) = M_{l\pm\frac{1}{2}}, \quad S_l^{(4)}(x_{l\pm\frac{1}{2}}) = F_{l\pm\frac{1}{2}}. \quad (4)$$

From algebraic manipulation we get the following expression:

$$\begin{aligned} a_{l0} &= -\frac{(8 + \theta^2)[F_{l-\frac{1}{2}} + F_{l+\frac{1}{2}}] + \tau^2\theta^2[M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}}] - 8\tau^4[u_{l-\frac{1}{2}} + u_{l+\frac{1}{2}}]}{16\tau^4}, \\ a_{l1} &= \frac{(24 + \theta^2)[F_{l-\frac{1}{2}} - F_{l+\frac{1}{2}}] + \tau^2\theta^2[M_{l-\frac{1}{2}} - M_{l+\frac{1}{2}}] - 24\tau^4[u_{l-\frac{1}{2}} - u_{l+\frac{1}{2}}]}{24\theta\tau^3}, \\ a_{l2} &= \frac{[F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}}] + \tau^2[M_{l+\frac{1}{2}} - M_{l-\frac{1}{2}}]}{4\tau^2}, \\ a_{l3} &= \frac{[F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}}] + \tau^2[M_{l+\frac{1}{2}} - M_{l-\frac{1}{2}}]}{6h\tau^2}, \\ e_l &= \frac{F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}}}{2\tau^4 \sin(\frac{\theta}{2})}, \\ f_l &= \frac{F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}}}{2\tau^4 \cos(\frac{\theta}{2})}, \end{aligned}$$

where  $\theta = \tau h$  and  $l = 1, 2, \dots, N - 1$ .

Using the continuity of first and third derivatives at  $(x_l, u_l)$ , that are  $S'_{l-1}(x_l) = S'_l(x_l)$  and  $S'''_{l-1}(x_l) = S'''_l(x_l)$ , we obtain the following relations:

$$\begin{aligned} M_{l-\frac{3}{2}} + 22M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}} &= \frac{24}{h^2}(u_{l-\frac{3}{2}} - 2u_{l-\frac{1}{2}} + u_{l+\frac{1}{2}}) - h^2\left[\left(\frac{24}{\theta^4} - \frac{12}{\theta^3 \sin(\frac{\theta}{2})}\right.\right. \\ &\left. + \frac{1}{\theta^2}\right)F_{l-\frac{3}{2}} + 2\left(-\frac{24}{\theta^4} + \frac{12 \cos(\theta)}{\theta^3 \sin(\frac{\theta}{2})} + \frac{11}{\theta^2}\right)F_{l-\frac{1}{2}} + \left(\frac{24}{\theta^4} - \frac{12}{\theta^3 \sin(\frac{\theta}{2})} + \frac{1}{\theta^2}\right)F_{l+\frac{1}{2}}, \quad (5) \end{aligned}$$

$$\begin{aligned} M_{l-\frac{3}{2}} - 2M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}} &= h^2\left[\frac{1 - \cos(\frac{\theta}{2})}{2\theta^2 \cos(\frac{\theta}{2})}F_{l-\frac{3}{2}} + 2\left(\frac{\cos(\frac{\theta}{2}) - \cos(\theta)}{2\theta^2 \cos(\frac{\theta}{2})}\right)F_{l-\frac{1}{2}}\right] \\ &\quad + \frac{1 - \cos(\frac{\theta}{2})}{2\theta^2 \cos(\frac{\theta}{2})}F_{l+\frac{1}{2}}. \quad (6) \end{aligned}$$

From (5) and (6) we have

$$h^2 M_{l-\frac{1}{2}} = (u_{l-\frac{3}{2}} - 2u_{l-\frac{1}{2}} + u_{l+\frac{1}{2}}) + \left(-\frac{1}{\tau^4} - \frac{\theta(-24 + \theta^2)}{48\tau^4 \sin(\frac{\theta}{2})}\right) F_{l-\frac{3}{2}} + \left(\frac{2}{\tau^4} - \frac{\theta^2}{\tau^4}\right) \quad (7)$$

$$+ \frac{\theta(-24 + \theta^2) \cos(\theta)}{24\tau^4 \sin(\frac{\theta}{2})} F_{l-\frac{1}{2}} + \left(-\frac{1}{\tau^4} - \frac{\theta(-24 + \theta^2)}{48\tau^4 \sin(\frac{\theta}{2})}\right) F_{l+\frac{1}{2}}.$$

The elimination of  $M_l$ 's using (6) and (7) gives

$$u_{l-\frac{5}{2}} - 4u_{l-\frac{3}{2}} + 6u_{l-\frac{1}{2}} - 4u_{l+\frac{1}{2}} + u_{l+\frac{3}{2}} = h^4 [\alpha(F_{l-\frac{5}{2}} + F_{l+\frac{3}{2}}) + \beta(F_{l-\frac{3}{2}} + F_{l+\frac{1}{2}})] \quad (8)$$

$$+ \gamma F_{l-\frac{1}{2}}, \quad l = 3(1)N - 2,$$

where

$$\alpha = \frac{1}{\theta^4} - \frac{1}{2\theta^3 \sin(\frac{\theta}{2})} + \frac{1}{48\theta \sin(\frac{\theta}{2})},$$

$$\beta = -\frac{4}{\theta^4} + \frac{1 + \cos(\theta)}{\theta^3 \sin(\frac{\theta}{2})} + \frac{11 - \cos(\theta)}{24\theta \sin(\frac{\theta}{2})},$$

$$\gamma = \frac{6}{\theta^4} - \frac{1 + 2\cos(\theta)}{\theta^3 \sin(\frac{\theta}{2})} + \frac{1 - 22\cos(\theta)}{24\theta \sin(\frac{\theta}{2})}.$$

When  $\tau \rightarrow 0$ , that  $\theta \rightarrow 0$ , then  $(\alpha, \beta, \gamma) \rightarrow \frac{1}{1920}(1, 236, 1446)$ , and the relations defined by (8) reduce into quintic polynomial spline function. Now by using the spline relation (8) and discretize the given system (1) at the grid points  $x_l$ . We obtain  $(N - 4)$  nonlinear equation in the  $(N)$  unknowns  $u_{l-\frac{1}{2}}$ ,  $l = 1, 2, \dots, N$  as

$$(u_{l-\frac{5}{2}} + \alpha h^4 f(x_{l-\frac{5}{2}}, u_{l-\frac{5}{2}})) + (-4u_{l-\frac{3}{2}} + \beta h^4 f(x_{l-\frac{3}{2}}, u_{l-\frac{3}{2}})) + \quad (9)$$

$$(6u_{l-\frac{1}{2}} + \gamma h^4 f(x_{l-\frac{1}{2}}, u_{l-\frac{1}{2}})) + (-4u_{l+\frac{1}{2}} + \beta h^4 f(x_{l+\frac{1}{2}}, u_{l+\frac{1}{2}})) +$$

$$(u_{l+\frac{3}{2}} + \alpha h^4 f(x_{l+\frac{3}{2}}, u_{l+\frac{3}{2}})) + t_l = 0, \quad l = 3(1)N - 2.$$

Taylor expansion of the local truncation errors  $t_l$ ,  $l = 3, \dots, N - 2$ , associated with our method are given by

$$t_l = (1 - 2\alpha - 2\beta - \gamma) h^4 u_l^{(4)} + \left(\frac{-1}{2} + \alpha + \beta + \frac{1}{2}\gamma\right) h^5 u_l^{(5)}$$

$$+ \left(\frac{7}{24} - \frac{17}{4}\alpha - \frac{5}{4}\beta - \frac{1}{8}\gamma\right) h^6 u_l^{(6)} + \left(-\frac{5}{48} + \frac{49}{24}\alpha + \frac{13}{24}\beta + \frac{\gamma}{48}\right) h^7 u_l^{(7)} +$$

$$\left(\frac{23}{640} - \frac{353\alpha}{192} - \frac{41\beta}{192} - \frac{\gamma}{384}\right) h^8 u_l^{(8)} + \left(-\frac{23}{2304} + \frac{1441\alpha}{1920} + \frac{121\beta}{1920} + \frac{\gamma}{3840}\right) h^9 u_l^{(9)}$$

$$+ \left(\frac{2497}{967680} - \frac{8177\alpha}{23040} - \frac{73\beta}{4608} - \frac{\gamma}{46080}\right) h^{10} u_l^{(10)} + O(h^{11}). \quad (10)$$

For different choices of parameters  $\alpha, \beta$  and  $\gamma$  we can obtain classes of the methods such as:

### Second-Order Method

For  $\alpha = \frac{9}{240}, \beta = \frac{1}{24000}$  and  $\gamma = 1 - 2\alpha - 2\beta$  gives  $t_l = \frac{133}{8000}h^6u_l^{(6)} + O(h^7), l = 3, \dots, N - 2.$

### Fourth-Order Method

For  $\alpha = \frac{1}{24}, \beta = 0$  and  $\gamma = 1 - 2\alpha - 2\beta$  gives  $t_l = \frac{-31}{720}h^8u_l^{(8)} + O(h^9), l = 3, \dots, N - 2.$

### Sixth-Order Method

For  $\alpha = \frac{-1}{720}, \beta = \frac{31}{180}$  and  $\gamma = 1 - 2\alpha - 2\beta$  gives  $t_l = \frac{1}{3024}h^{10}u_l^{(10)} + O(h^{11}), l = 3, \dots, N - 2.$

## 3. End Condition

To obtain the unique solution of the nonlinear system (9) we need four more equations to be associated with the system. By using Taylor series and method of undetermined coefficients the boundary formulas associated with boundary conditions for the sixth-order method can be determine as follows. In order to obtain the six-order boundary formula we define the following identities:

$$c_0u_0 + \sum_{j=1}^3 c_j u_{j-\frac{1}{2}} + \lambda h^2 u_0'' + \rho_0 h^4 u_0^{(4)} + h^4 \sum_{j=1}^5 \rho_j u_{j-\frac{1}{2}}^{(4)} + t_1^* = 0, \quad (11)$$

$$c'_0u_0 + \sum_{j=1}^4 c'_j u_{j-\frac{1}{2}} + \lambda' h^2 u_0'' + h^4 \sum_{j=1}^6 \rho'_j u_{j-\frac{1}{2}}^{(4)} + t_2^* = 0, \quad (12)$$

$$c'^*_0u_N + \sum_{j=1}^4 c'^*_j u_{N-j+\frac{1}{2}} + \lambda'^* h^2 u_N'' + h^4 \sum_{j=1}^6 \rho'^*_j u_{N-j+\frac{1}{2}}^{(4)} + t_{N-1}^* = 0, \quad (13)$$

$$c^*_0u_N + \sum_{j=1}^3 c^*_j u_{N-j+\frac{1}{2}} + \lambda^* h^2 u_N'' + h^4 \rho^*_0 u_N^{(4)} + h^4 \sum_{j=1}^5 \rho^*_j u_{N-j+\frac{1}{2}}^{(4)} + t_N^* = 0. \quad (14)$$

In order that  $t_1^*, t_2^*, t_{N-1}^*$  and  $t_N^*$  are  $O(h^{10})$  we find that

$$(c_0, c_1, c_2, c_3, \lambda) = (c_0^*, c_1^*, c_2^*, c_3^*, \lambda^*) = (-6, 10, -5, 1, \frac{5}{4}),$$

$$(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5) = \left( \frac{8041}{7620480}, \frac{-16255}{43008}, \frac{-293953}{1161216}, \frac{2999}{71680}, \frac{-3763}{301056}, \frac{3035}{1741824} \right),$$

$$(\rho_0^*, \rho_1^*, \rho_2^*, \rho_3^*, \rho_4^*, \rho_5^*) = \left( \frac{8041}{7620480}, \frac{-16255}{43008}, \frac{-293953}{1161216}, \frac{2999}{71680}, \frac{-3763}{301056}, \frac{3035}{1741824} \right),$$

$$(c'_0, c'_1, c'_2, c'_3, c'_4, \lambda') = (c'^*_0, c'^*_1, c'^*_2, c'^*_3, c'^*_4, \lambda'^*) = (2, -5, 6, -4, 1, \frac{1}{4}),$$

$$(\rho'_1, \rho'_2, \rho'_3, \rho'_4, \rho'_5, \rho'_6) = \left( \frac{-132283}{884736}, \frac{-7263443}{10321920}, \frac{-1775513}{15482880}, \frac{-632117}{15482880}, \frac{172129}{10321920}, \frac{-85373}{30965760} \right),$$

$$(\rho_1^*, \rho_2^*, \rho_3^*, \rho_4^*, \rho_5^*, \rho_6^*) = \left( \frac{-132283}{884736}, \frac{-7263443}{10321920}, \frac{-1775513}{15482880}, \frac{-632117}{15482880}, \frac{172129}{10321920}, \frac{-85373}{30965760} \right),$$



The sufficient condition for  $\Lambda^{-1} \geq 0$  is that  $\xi > r$  where

$$r = \frac{x - \sqrt{x^2 - 4}}{2}, (x \geq 2).$$

*Proof* Suppose that  $\Lambda$  is nonsingular and we denote  $Q = (q_{ij})_{N \times N}$  be the inverse of  $\Lambda$ . The identity  $\Lambda Q = I$  is expressed by the equations

$$\begin{cases} \xi q_{1,j} - q_{2,j} = \delta_{1,j}, & (i) \\ -q_{i-1,j} + xq_{i,j} - q_{i+1,j} = \delta_{i,j}, & i = 2, 3, \dots, N-1, \quad (ii) \\ \xi q_{N,j} - q_{N-1,j} = \delta_{N,j}, & (iii) \end{cases} \quad (21)$$

for each  $j = 1, 2, \dots, N$ , where  $\delta_{i,j}$  is the Kronecker delta. By using [14] the elements  $q_{i,j}$  can be written as:

$$-e_{i-1} + xe_i - e_{i+1} = 0, \quad i = 2, 3, \dots, N-1, \quad e_1 = 0, \quad \text{and} \quad e_N = 1, \quad (22)$$

$$-t_{i-1,j} + xt_{i,j} - t_{i+1,j} = \delta_{i,j}, \quad i = 2, 3, \dots, N-1, \quad t_{1,j} = 0, \quad \text{and} \quad t_{N,j} = 0, \quad (23)$$

for each  $j = 2, 3, \dots, N-1$ , and  $t_{i,1} = t_{i,N} = 0$ . The solutions to (22) and (23) are given in [8] and [14]. We have

$$q_{i,j} = t_{i,j} + \psi_j e_{N-i+1} + \chi_j e_i, \quad (24)$$

where  $\psi_j$  and  $\chi_j$  must be determined. We known that the expression  $q_{i,j}$  satisfy the equations (21)(ii) in general. It satisfies equations (21)(i) and (21)(iii). By substituting equation (24) into (21)(i) and (21)(iii) we get

$$(\xi - e_{N-1})\psi_j - e_2\chi_j = t_{2,j} + \delta_{1,j}, \quad (25)$$

$$(-e_2)\psi_j + (\xi - e_{N-1})\chi_j = t_{N-1,j} + \delta_{N,j}. \quad (26)$$

By using (25) and (26) we get:

$$\begin{pmatrix} \xi - e_{N-1} & -e_2 \\ -e_2 & \xi - e_{N-1} \end{pmatrix} \begin{pmatrix} \psi_j \\ \chi_j \end{pmatrix} = \begin{pmatrix} t_{2,j} + \delta_{1,j} \\ t_{N-1,j} + \delta_{N,j} \end{pmatrix}, \quad (27)$$

if equation (27) has a unique solution, then the inverse of matrix  $\Lambda$  is  $Q = (q_{i,j})_{N \times N}$ . The sufficient condition on  $\xi$  and  $x$  so that  $\Lambda^{-1} \geq 0$  can be obtained as follows.

We assume that  $x \geq 2$  so that  $t_{i,j} \geq 0$  for  $i = 1, 2, \dots, N, j = 1, 2, \dots, N$ . From expression (24), the  $\psi_j$  and  $\chi_j$  should be nonnegative, thus

$$\begin{pmatrix} \xi - e_{N-1} & -e_2 \\ -e_2 & \xi - e_{N-1} \end{pmatrix}^{-1} \geq 0, \quad (28)$$

by simplified (28) we get

$$\begin{pmatrix} \xi - r & \\ & \xi - r \end{pmatrix}^{-1} > 0, \quad (29)$$

where  $r = \frac{x - \sqrt{x^2 - 4}}{2}$ . If  $N$  is sufficiently large, then condition (29) implies condition (28). Thus  $x \geq 2$  and condition (29) implies that  $\Lambda^{-1} \geq 0$  for  $N$  sufficiently large if  $\xi > r$ .

Consequently the matrix  $P$  which is special case of the matrix  $\Lambda$  for  $\xi = 3$  and  $x = 2$  is monotone matrix. Thus the matrix  $A_0 = P^2$ , is a monotone matrix. By using Lemma 1, in [24] the symmetric matrix  $A_0$ , is irreducible and monotone and

$$\|A_0^{-1}\| \leq \frac{[5(b-a)^4 + 10(b-a)^2h^2 + 9h^4]}{384h^4}. \quad (30)$$

The  $N$  components vectors  $\mathbf{f}^{(1)}$  and  $R^{(1)}$ , are given by

$$\mathbf{f}^{(1)} = (f_{\frac{1}{2}}^{(1)}, \dots, f_{N-\frac{1}{2}}^{(1)})^t, \quad (31)$$

where  $\mathbf{f}_l^{(1)}(U^{(1)}) = f(x_l, u_l^{(1)})$ ,  $l = \frac{1}{2}, \frac{3}{2}, \dots, N - \frac{1}{2}$ , and

$$R^{(1)} = \begin{pmatrix} 6\alpha_1 - \frac{5}{4}h^2\alpha_2 - h^4(\frac{8041}{7620480})f(x_0, \alpha_1) \\ -2\alpha_1 - \frac{h^2}{4}\alpha_2 \\ 0 \\ \vdots \\ 0 \\ -2\beta - \frac{1}{4}h^2\beta_2 \\ 6\beta_1 - \frac{5}{4}h^2\beta_2 - h^4(\frac{8041}{7620480})f(x_N, \beta_1) \end{pmatrix}. \quad (32)$$

$$A_0\bar{U}^{(1)} + h^4B\mathbf{f}^{(1)}(\bar{U}^{(1)}) = R^{(1)} + t^{(1)}, \quad (33)$$

where the vector  $\bar{U}^{(1)} = u(x_{l-\frac{1}{2}})$ ,  $l = 1, 2, \dots, N$ , is the exact solution and  $t^{(1)} = [t_{\frac{1}{2}}^{(1)}, t_{\frac{3}{2}}^{(1)}, \dots, t_{N-\frac{1}{2}}^{(1)}]^T$ , is the local truncation errors.

From (16) and (33) we have:

$$[A]E^{(1)} = [A_0 + h^4BF_k(U^{(1)})]E^{(1)} = t^{(1)}, \quad (34)$$

where

$$E^{(1)} = \bar{U}^{(1)} - U^{(1)} = [e_{\frac{1}{2}}^{(1)}, e_{\frac{3}{2}}^{(1)}, \dots, e_{N-\frac{1}{2}}^{(1)}]^T, \quad (35)$$

$$\mathbf{f}^{(1)}(\bar{U}^{(1)}) - \mathbf{f}^{(1)}(U^{(1)}) = F_k(U^{(1)})E^{(1)},$$

and  $F_k(U^{(1)}) = \text{diag}\{\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}\}$ ,  $l = \frac{1}{2}, \frac{3}{2}, \dots, N - \frac{1}{2}$ , is a diagonal matrix of order  $N$ .

■

LEMMA 4.2 If  $M$  is a square matrix of order  $N$  and  $\|M\| < 1$ , then  $(I + M)^{-1}$  exists and  $\|(I + M)^{-1}\| \leq \frac{1}{(1-\|M\|)}$ .

LEMMA 4.3 The matrix  $[A_0 + h^4BF_k(U^{(1)})]$  in (34) is nonsingular, provided  $Y < \frac{1981808640}{5306209\lambda}$ , where  $\lambda = [5(b-a)^4 + 10(b-a)^2h^2 + 9h^4]$  and  $Y = \max|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}|$ ,  $l = \frac{1}{2}, \frac{3}{2}, \dots, N - \frac{1}{2}$ . (The norm referred to is the  $L_\infty$  norm).

*Proof* We know that  $[A_0 + h^4BF_k(U^{(1)})] = A_0[I + h^4A_0^{-1}BF_k(U^{(1)})]$ . But the inverse of  $A_0$  exists we have to show that the inverse of  $[I + h^4A_0^{-1}BF_k(U^{(1)})]$  exists. By



using lemma 4.2, we have

$$h^4 \|A_0^{-1} B F_k(U^{(1)})\| \leq h^4 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\| < 1, \quad (36)$$

by using (19) we obtain  $\|B\| \leq \frac{5306209}{5160960}$  and also we have  $\|F_k(U^{(1)})\| \leq Y = \max\left|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}\right|, l = \frac{1}{2}, \frac{3}{2}, \dots, N - \frac{1}{2}$ , and then by using (30) and (36) we obtain

$$Y < \frac{1981808640}{5306209\lambda}.$$

As a consequence of Lemmas 4.3 and 4.2 the nonlinear system (16) has a unique solution if  $Y < \frac{1981808640}{5306209\lambda}$ . ■

To show that the matrix  $[A = A_0 + h^4 B F_k(U^{(1)})]$  is monotone. we will prove the following theorem:

**THEOREM 4.4** *If  $Y < \frac{1981808640}{5306209\lambda}$ , then the matrix  $A$  given by (34) is monotone where  $\lambda = 5(b-a)^4 + 10(b-a)^2 h^2 + 9h^4$ .*

*Proof* : From (34) we have

$$A = A_0 + h^4 B F_k(U^{(1)}) = P^2 + h^4 B F_k(U^{(1)}),$$

hence  $AP^{-2} = I + h^4 B F_k(U^{(1)})P^{-2}$ , so that

$$\begin{aligned} P^2 A^{-1} &= (I + h^4 B F_k(U^{(1)})P^{-2})^{-1} = \\ &= I - (h^4 B F_k(U^{(1)})P^{-2}) + (h^4 B F_k(U^{(1)})P^{-2})^2 - (h^4 B F_k(U^{(1)})P^{-2})^3 + \dots \\ &= [I - (h^4 B F_k(U^{(1)})P^{-2})][I + (h^4 B F_k(U^{(1)})P^{-2})^2 + (h^4 B F_k(U^{(1)})P^{-2})^4 + \dots]. \end{aligned}$$

Also if

$$\rho(h^4 B F_k(U^{(1)})P^{-2}) < 1$$

then, the two infinite series convergence. Let

$$\|F_k(U^{(1)})\| \leq Y = \max\left|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}\right|, l = \frac{1}{2}, \frac{3}{2}, \dots, N - \frac{1}{2},$$

then

$$A^{-1} = [P^{-2} - P^{-2} h^4 B F_k(U^{(1)})P^{-2}][I + (h^4 B F_k(U^{(1)})P^{-2})^2 + (h^4 B F_k(U^{(1)})P^{-2})^4 + \dots],$$

where the infinite series is nonnegative. Hence to show that  $A$  is monotone, it sufficient to show that  $[P^{-2} - P^{-2} h^4 B F_k(U^{(1)})P^{-2}] > 0$ . Here we have

$$P^{-2} > P^{-2} h^4 B F_k(U^{(1)})P^{-2} \Rightarrow I > h^4 B F_k(U^{(1)})P^{-2} \Rightarrow$$

$$\text{Archive of SID} \quad \|h^4 A_0^{-1} B F_k(U^{(1)})\| \leq h^4 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\| < 1. \quad (37)$$

By substituting  $\|B\|$ ,  $\|P^{-2}\|$  and  $\|F_k(U^{(1)})\|$  into (37) we get

$$Y < \frac{1981808640}{5306209\lambda}$$

■

**THEOREM 4.5** Let  $u(x_{l-\frac{1}{2}})$  be the exact solution of the boundary value problem (1) with boundary conditions (2) and we assume  $u_{l-\frac{1}{2}}$ ,  $l = 1, 2, \dots, N$  be the numerical solution obtained by solving the nonlinear system (16). Then we have

$$\|E^{(1)}\| \equiv O(h^6), \text{ (provided } Y < \frac{1981808640}{5306209\lambda}, \alpha = \frac{-1}{720}, \beta = \frac{31}{180}, \gamma = \frac{79}{120} \text{)}.$$

*Proof* We can write the error equation (34) in the following form

$$\begin{aligned} E^{(1)} &= (A_0 + h^4 B F_k(U^{(1)}))^{-1} t^{(1)} = (I + h^4 A_0^{-1} B F_k(U^{(1)}))^{-1} A_0^{-1} t^{(1)}, \\ \|E^{(1)}\| &\leq \|(I + h^4 A_0^{-1} B F_k(U^{(1)}))^{-1}\| \|A_0^{-1}\| \|t^{(1)}\|, \end{aligned}$$

It follows that

$$\|E^{(1)}\| \leq \frac{\|A_0^{-1}\| \|t^{(1)}\|}{1 - h^4 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\|}, \quad (38)$$

provided that  $h^4 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\| < 1$ . Also we have

$$\|t^{(1)}\| \leq \frac{360041}{154828800} h^{10} M_{10}, \quad (\alpha = \frac{-1}{720}, \beta = \frac{31}{180}, \gamma = \frac{79}{120}), \quad (39)$$

where  $M_{10} = \max |u^{(10)}(\xi)|$ ,  $a \leq \xi \leq b$ .

Substituting  $\|A_0^{-1}\|$ ,  $\|F_k(U^{(1)})\|$ ,  $\|B\|$  and  $\|t^{(1)}\|$  from above relations in (38) and simplifying we obtain

$$\|E^{(1)}\| \leq \frac{1858157199360\lambda h^6 M_{10}}{306841053560832000 - 821553972019200\lambda Y} \equiv O(h^6), \quad (40)$$

where  $\lambda = 5(b-a)^4 + 10(b-a)^2 h^2 + 9h^4$ . It is a sixth-order convergent method provided

$$Y < \frac{1981808640}{5306209\lambda} \quad (41)$$

■

Consequently it follows that the numerical method under consideration is sixth-order convergent process.

## Archive of SID 5. Numerical Results

In this section we present the results obtained by applying the numerical method discussed in pervious sections to the following two-point boundary-value problems.

*Example 5.1* We consider the differential equation

$$u^{(4)} + u^2 = -8x \cos(x) - 13 \sin(x) + x^2 \sin(x) + (x^2 - 1)^2 (\sin(x))^2, \\ 0 < x < 1, \quad (42)$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = 2 \sin(1) + 4 \cos(1). \quad (43)$$

The analytical solution is  $u(x) = (x^2 - 1) \sin(x)$ .

*Example 5.2* Consider the differential equation

$$u^{(4)} - 6e^{4u} = -\frac{12}{(1+x)^4}, 0 < x < 1, \quad (44)$$

with the boundary conditions:

$$u(0) = 0, u(1) = \ln(2), u''(0) = -1, \quad u''(1) = \frac{-1}{4}. \quad (45)$$

The analytical solution is  $u(x) = \ln(1+x)$ .

*Example 5.3* We consider the differential equation

$$u^{(4)} - 5u^3 = 96x \cos(x) - 16x(-1+x^2) \cos(x) + 24 \sin(x) - 48x^2 \sin(x) \\ - 24(-1+x^2) \sin(x) + (-1+x^2)^2 \sin(x) - 5(-1+x^2)^6 \sin(x)^3, \\ 0 < x < 1, \quad (46)$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = 8 \sin(1). \quad (47)$$

The analytical solution is  $u(x) = (x^2 - 1)^2 \sin(x)$ .

We solved examples 5.1, 5.2 and 5.3 by using non-polynomial quintic spline method with step lengths  $h = 2^{-m}$ ,  $m = 2, 3, 4, 5, 6, 7$  for  $\alpha = \frac{-1}{720}$ ,  $\beta = \frac{31}{180}$ ,  $\gamma = \frac{79}{120}$ . The maximum absolute errors in solutions for our method are listed in table 1.

Table 1. Maximum absolute errors in solution.

$m$	Example 5.1	Example 5.2	Example 5.3
3	$5.22793 \times 10^{-10}$	$3.38556 \times 10^{-7}$	$3.56386 \times 10^{-8}$
4	$8.07062 \times 10^{-12}$	$1.18519 \times 10^{-9}$	$5.78254 \times 10^{-10}$
5	$1.85186 \times 10^{-13}$	$3.83505 \times 10^{-11}$	$1.30425 \times 10^{-11}$
6	$3.61431 \times 10^{-15}$	$1.01539 \times 10^{-12}$	$2.18424 \times 10^{-13}$
7	$8.01699 \times 10^{-15}$	$1.81576 \times 10^{-14}$	$2.61800 \times 10^{-16}$

Example 5.4 Consider the differential equation

$$u^{(4)} + xu = -(8 + 7x + x^3)e^x, \quad 0 < x < 1, \quad (48)$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = -4e. \quad (49)$$

The analytical solution for this boundary value problem is  $u(x) = x(1 - x)e^x$ .

Example 5.5 Consider the following problem ,

$$u^{(4)} - u = -8x \cos(x) - 12 \sin(x), \quad 0 < x < 1, \quad (50)$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = 4 \cos(1) + 2 \sin(1). \quad (51)$$

The exact solution is given by  $u(x) = (x^2 - 1) \sin(x)$ . Examples 5.4 and 5.5 solved by using non-polynomial quintic spline method with step lengths  $h = 2^{-m}$ ,  $m = 3, 4, 5$  for  $\alpha = \frac{-1}{720}, \beta = \frac{31}{180}, \gamma = \frac{79}{120}$ . The maximum absolute errors in solutions for our method are listed in table 2.

Table 2. Maximum absolute errors in solution.

$m$	Example 5.4	Example 5.5
3	$1.691902 \times 10^{-9}$	$5.248385 \times 10^{-10}$
4	$2.23407 \times 10^{-11}$	$8.071303 \times 10^{-12}$
5	$2.020517 \times 10^{-13}$	$1.990957 \times 10^{-14}$

Table 3. Maximum absolute errors in solution Example 5.2

$m$	Second-order in [20]	Fourth-order in [20]	Method A in [1]	Method B in [1]
3	$1.9 \times 10^{-4}$	$3.7 \times 10^{-6}$	$1.4 \times 10^{-5}$	$1.4 \times 10^{-5}$
4	$4.6 \times 10^{-5}$	$2.9 \times 10^{-7}$	$8.3 \times 10^{-7}$	$8.3 \times 10^{-7}$
5	$1.1 \times 10^{-5}$	$1.9 \times 10^{-8}$	$5.4 \times 10^{-8}$	$5.4 \times 10^{-8}$

Table 4. Maximum absolute errors in solution Example 5.4

$m$	Sixth-order in [17]	Sixth-order in [10]	Sixth-order in [23]	Sixth-order in [23]
3	$2.47 \times 10^{-9}$	$1.91 \times 10^{-7}$	$2.66 \times 10^{-6}$	$3.86 \times 10^{-7}$
4	$3.93 \times 10^{-11}$	$3.12 \times 10^{-9}$	$4.68 \times 10^{-8}$	$6.59 \times 10^{-9}$
5	$3.25 \times 10^{-13}$	$4.98 \times 10^{-11}$	$7.72 \times 10^{-10}$	$1.05 \times 10^{-10}$
6	-	-	$8.01 \times 10^{-12}$	$9.81 \times 10^{-12}$

Table 5. Maximum absolute errors in solution Example 5.4 and 5.5 in [15].

$m$	Example 5.4 Fourth-order (1)	Example 5.4 Fourth-order (2)	Example 5.5 Fourth-order (1)	Example 5.5 Fourth-order (2)
3	$1.91 \times 10^{-7}$	$2.09 \times 10^{-7}$	$5.96 \times 10^{-8}$	$6.48 \times 10^{-8}$
4	$3.12 \times 10^{-9}$	$7.92 \times 10^{-9}$	$9.10 \times 10^{-10}$	$2.29 \times 10^{-9}$
5	$5.02 \times 10^{-11}$	$1.27 \times 10^{-9}$	$1.42 \times 10^{-12}$	$3.66 \times 10^{-10}$

We have developed new non-polynomial quintic spline method at mid point for finding the numerical solution of nonlinear fourth-order boundary value problems. The boundary condition corresponding to the sixth-order method is developed. The convergence analysis of our presented method is discussed based on monotonicity of the coefficient matrix.

## References

- [1] R. P. Agarwal and G. Akrivis. Boundary value problems occurring in plate de ction theory,. Journal of Computational and Applied Mathematics, (1982), 8:145-154.
- [2] E. Al-Said and M. Noor. Quartic spline method for solving fourth order obstacle boundary value problems. Journal of Computational and Applied Mathematics, 143:107-116, 2002.
- [3] E. Al-Said, M. Noor, and T. M. Rassias. Cubic splines method for solving fourth-order obstacle boundary value problems. Applied Mathematics and Computation, 174:180-187, 2006.
- [4] M. ali hajji and K. al khaled. Numerical methods for nonlinear fourth-order boundary value problems with applications. International Journal of Computer Mathematics, 85(1):83104, 2008.
- [5] M. Chawla and C. Katti. Finite difference method for two-point boundary-value problems involving high order differential equations. BIT, 19:2733, 1979.
- [6] M. Chawla and Subramanian. High accuracy quintic spline solution of fourth- order two-point boundary-value problems. International Journal of Computer Mathematics,, 31:8794, 1989.
- [7] M. V. Daele, G. V. Berghe, and H. Meyer. A smooth approximation for the solution of a fourth order boundary value problem based on nonpolynomial splines. Journal of Computational and Applied Mathematics, 51:383-394, 1994.
- [8] C. F. Fischer and R. A. Usmani. Properties of some tridiagonal matrices and their application to boundary value problems. SIAM Journal on Numerical Analysis,, 6:127142, 1969.
- [9] M. Jain. Numerical Solution of Differential Equations. (2nd edn) NewYork:Wiley, 1984.
- [10] M. Jain, S. Iyannger, and J. Soldhanha. Numerical solution of a fourth-order ordinary differential equation. Journal of Engineering Mathematics, 11:373-380, 1977.
- [11] C. P. Katti. Five-diagonal sixth-order methods for two-point boundary value problems involving fourth order differential equations. Mathematics of Computation, 35(152):1177-1179, 1980.
- [12] A. Khan, T. Aziz, and M. A. Noor. Parametric quintic spline approach to the solution of a system of fourth-order boundary-value problems. J. Opt. Theory Appl., 122:69-82, 2004.
- [13] M. Khan, S. ul Islam, S. Tirmizi, E. Twizell, and S. Ashraf. A class of methods based on non-polynomial sextic spline functions for the solution of a special fifth-order boundary-value problems. J. Math. Anal. Appl., 321:651-660, 2006.
- [14] D. Meek. The inverses of some matrices deviating slightly from a symmetric, tridiagonal, toeplitz form. SIAM Journal on Numerical Analysis, 17(1):39-43, 1980.
- [15] M. Ramadan, I. Lashien, andW. Zahra. Quintic nonpolynomial spline solutions for fourth order boundary value problem. Communication in Nonlinear Science and Numerical Simulation, 14:1105-1114, 2009.
- [16] J. Rashidinia and A. Golbabaee. Convergence of numerical solution of a fourth-order boundary value problem. Applied Mathematics and Computations, 171:1296-1305, 2005.
- [17] J. Rashidinia and R. Jalilian. Non-polynomial spline for solution of boundary- value problems in plate de ction theory. International Journal of Computer Mathematics, 84:1483-1494, 2007.
- [18] J. Rashidinia, R. Mohammadi, R. Jalilian, and M. Ghasemi. Convergence of cubic-spline approach to the solution of a system of boundary-value problems. Applied Mathematics and Computation, 192:319-331, 2007.
- [19] S. S. Siddiqi and G. Akram. Solution of the system of fourth order boundary value problems using nonpolynomial spline technique. Applied Mathematics and Computation, 185:128-135, 2007.
- [20] E. H. Twizell. A fourth-order extrapolation method for special nonlinear fourth-order boundary value problems. Communications in Applied Numerical Methods, 2:593-602, 1986.
- [21] S. ul Islam, I. A. Tirmizi, and M. A. Khan. Quartic non-polynomial spline approach to the solution of a system of third-order boundary-value problems. J. Math. Anal. Appl., 335:1095-1104, 2007.
- [22] R. Usmani. Smooth spline approximations for the solution of a boundary value problem with engineering applications. Journal of Computational and Applied Mathematics, 6:93-98, 1980.
- [23] R. Usmani. On a class of numerical integration methods for a problem of plate de ction theory. International Journal of Computer Mathematics, 11:305-318, 1982.
- [24] R. Usmani. The use of quartic splines in the numerical solution of a fourth-order boundary value problems. Journal of Computational and Applied Mathematics, 44:187-199, 1992.
- [25] R. Usmani. On the numerical integration of a boundary value problem involving a fourth order linear differential equation. BIT, 17:227-234, 1997.
- [26] R. Usmani and S. A. Warsi. Smooth spline solutions for boundary value problems in plate de ction theory. Computers and Mathematics with Applications, 6:205-211, 1980.
- [27] A. Wazwaz. Approximate solution to boundary value problems of higher order by the modified decomposition method. Computers and Mathematics with Applications, 40:679-691, 2000.