

ON an Extension of a Quadratic Transformation Formula due to Gauss

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Abstract. The aim of this research note is to prove the following new transformation formula

$$(1-x)^{-2a} {}_3F_2 \left[\begin{matrix} a, a + \frac{1}{2}, d+1 \\ c+1, d \end{matrix} ; -\frac{4x}{(1-x)^2} \right] = {}_4F_3 \left[\begin{matrix} 2a, 2a-c, a-A+1, a+A+1 \\ c+1, a-A, a+A \end{matrix} ; -x \right]$$

for $d = c$, we get a known quadratic transformations due to Gauss. The result is derived with the help of the generalized Gauss's summation theorem available in the literature.

Keywords: Gauss hypergeometric function, ${}_2F_1$ Hypergeometric function, Contiguous function relation, Linear recurrence relation.

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1. Introduction and Results Required

We start with the following very interesting and useful quadratic transformation formula due to Gauss [4] viz.

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$$(1-x)^{-2a} {}_2F_1 \left[\begin{matrix} a, a + \frac{1}{2}, \\ c, \end{matrix} ; -\frac{4x}{(1-x)^2} \right] = {}_2F_1 \left[\begin{matrix} 2a, 2a + 1 - c, \\ c, \end{matrix} ; -x \right] \quad (1)$$

Bailey [2] derived this result by employing Gauss's summation theorem [3] viz.

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (2)$$

provided that $Re(c-a-b) > 0$.

The aim of this research note is to continue what we did in [1], and provide the extension of (1) by the same technique developed by Bailey. For this the following results will be required in our present investigations:

- Binomial theorem

$$(1-x)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n \quad (3)$$

- Elementary identities

$$(2a+2n)_m = \frac{(2a)_{m+n}}{2^{2n} (a)_n (a+\frac{1}{2})_n} \quad (4)$$

$$(2a)_{m+n} = (2a+m)_n (2a)_m \quad (5)$$

$$(m-n)! = \frac{(-1)^n m!}{(-m)_n} \quad (6)$$

- Rainville [6, Lemma 10, p. 56 eq. (1)]

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n-k) \quad (7)$$

- Extension of Gauss's summation theorem [5]

$${}_3F_2 \left[\begin{matrix} a, b, d+1 \\ c, d \end{matrix} ; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[(c-a-b-1) + \frac{ab}{d} \right] \quad (8)$$

provided $Re(c-a-b) > 1$.

2. New Transformation Formula

The extension of the Gauss's transformation formula (1) to be established is

$$\begin{aligned}
 & (1-x)^{-2a} {}_3F_2 \left[\begin{matrix} a, a + \frac{1}{2}, d + 1 \\ c + 1, d \end{matrix} ; -\frac{4x}{(1-x)^2} \right] \\
 & = {}_4F_3 \left[\begin{matrix} 2a, 2a - c, a - A + 1, a + A + 1 \\ c + 1, a - A, a + A \end{matrix} ; -x \right]
 \end{aligned} \tag{9}$$

2.1 Derivation

In order to derive (9), we proceed as follows. Start with the left-hand side of (9), we have

$$\text{L.H.S.} = (1-x)^{-2a} {}_3F_2 \left[\begin{matrix} a, a + \frac{1}{2}, d + 1 \\ c + 1, d \end{matrix} ; -\frac{4x}{(1-x)^2} \right]$$

express ${}_3F_2$ as a series, we have

$$= \sum_{n=0}^{\infty} \frac{(a)_n (a + \frac{1}{2})_n (d + 1)_n (-1)^n 2^{2n} x^n}{(c + 1)_n (d)_n n!} (1-x)^{-(2a+2n)}$$

using (3), we have

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2a + 2n)_m (a)_n (a + \frac{1}{2})_n (d + 1)_n (-1)^n 2^{2n}}{m! (c + 1)_n (d)_n n!} x^{n+m}$$

using (4), we have

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2a)_{2n+m} (d + 1)_n (-1)^n}{(c + 1)_n (d)_n n! m!} x^{n+m}$$

changing m by $m - n$ and using (7)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^m \frac{(2a)_{m+n} (d + 1)_n (-1)^n}{(c + 1)_n (d)_n n! (m - n)!} x^m$$

using (5) and (6), we have

$$= \sum_{m=0}^{\infty} \frac{(2a)_m}{m!} x^m \sum_{n=0}^m \frac{(-m)_n (2a + m)_m (d + 1)_n}{(c + 1)_n (d)_n n!}$$

summing up the inner series

$$= \sum_{m=0}^{\infty} \frac{(2a)_m}{m!} x^m {}_3F_2 \left[\begin{matrix} -m, 2a + m, d + 1 \\ c + 1, d \end{matrix} ; 1 \right],$$

using (8) by taking $a = -m$, $b = 2a + m$, $d = d$ and $c = c + 1$, we get after much simplification

$$= \sum_{m=0}^{\infty} \frac{(a-A)(a+A)(-1)^m (2a)_m (2a-c)_m (a-A+1)_m (a+A+1)_m x^m}{d(2a-c)(c+1)_m (a-A)_m (a+A)_m m!}$$

finally summing up the series, we get

$$= {}_4F_3 \left[\begin{matrix} 2a, 2a-c, a-A+1, a+A+1 \\ c+1, a-A, a+A \end{matrix} ; -x \right].$$

= R.H.S.

This completes the proof of (9). ■

2.2 Special Case

In (9), if we take $d = c$, we get the Gauss's quadratic transformation formula (1). Hence (9) can be regarded as an extension of (1).

References

- [1] Rakha M. A., Rathie N., Chopra P., On an extension of a quadratic transformation formula due to Kummer, *Math. Commun.*, **14** (2009) 207-209.
- [2] Bailey W. N., Products of generalized hypergeometric series, *Proc. London Math. Soc.*, **2** (1928) 242 - 254.
- [3] Bailey W. N., *Generalized hypergeometric series*, Cambridge University Press, Cambridge (1935).
- [4] Gauss C. F., Disquisitiones generales circa seriem infinitam ..., *Comm. soc. reg. sci. Gött. rec.*, **II** (1876) 123-162.
- [5] Prudnikov A. P., Brychkov Iu. A., Marichev O. I., *More Special Functions, Integrals and Series*, Gordon and Breach, New York, **3** (1990).
- [6] Rainville E. D., *Special functions*, The Macmillan Company, New York, (1960).