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Minimal Solution of Inconsistent Fuzzy Matrix Equations

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Abstract. Fuzzy liner systems of equations, play a major role in several applications in various area such as engineering, physics and economics. In this paper, we investigate the existence of a minimal solution of inconsistent fuzzy matrix equation. Also some numerical examples are considered.

Keywords: Fuzzy numbers, Inconsistent fuzzy matrix equations, Minimal solution.

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1. Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations with these fuzzy numbers were first introduced and investigated by Zadeh [26, 10], Dubois and Prade [13] and Nahmias [19]. Some different approaches to fuzzy numbers and the structure of fuzzy number spaces were given by Purl and Ralescu [21], Goetschell and Voxman [16, 17] and Wu and Ma [24, 25]. Fuzzy systems are used to a variety of problems ranging from control chaotic systems [14] to fuzzy metric spaces [20], fuzzy linear systems, fuzzy differential equations [3], particle physics [22] and so on. Treating fuzzy linear systems is one of the major applications of fuzzy arithmetic. Many problems in various areas such as economics, engineering and physics boil down to the solution of a linear system of equations. Since Friedman et al [15] proposed a general model for solving a $n \times n$ fuzzy linear systems whose coefficient matrix is crisp and the right-hand side is an arbitrary fuzzy number vector by the embedding approach in 1998, a large number of researches have been produced about how to solve numerically fuzzy linear systems see [1, 2, 5, 6, 7] and [11, 12]

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and so on. Asady et al. [8], who merely considered the full row rank system, used the same method to solve the $m \times n$ fuzzy linear system for mn. Later, Zheng and Wang [23, 28] discussed the mn general fuzzy linear system and the inconsistent fuzzy linear system by using generalized inverses of the coefficient matrix. Then, Abbasbandy et al [4] investigated the minimal solution of the general dual fuzzy linear system by means of matrix generalized inverses theory. However, for a fuzzy matrix equation like as $A\tilde{x} = \tilde{B}$, which always has a wide use in the control theory and control engineering, few works have been done over the past decades.

In this paper a numerical method for finding minimal solution of inconsistent fuzzy matrix equations $A\tilde{x} = \tilde{B}$ is given. where A is a $m \times n$ crisp matrix and the right-hand side matrix is an arbitrary fuzzy number matrix.

In Section 2, we recall some fundamental results on fuzzy numbers. Taking advantage of the approach in [15], we use the parametric form of fuzzy numbers to replace the general fuzzy matrix system $A\tilde{x} = \tilde{B}$ with a crisp function matrix equation $SX = Y(r)$ where S is a $2m \times 2n$ crisp matrix in Section 3. Then in section 4 the expression of inconsistent fuzzy matrix equation is given based on generalized inverses of matrix S. Moreover, the existence condition of strong minimal solution of inconsistent fuzzy matrix equations is studied. Numerical example are given in section 5.

2. Preliminaries

The minimal solution of an arbitrary linear system is formally defined such that:

1. If the system is consistent and has a unique solution, then this solution is also the minimal solution.
2. If the system is consistent and has a set solution, then the minimal solution is a member of this set that has the least Euclidean norm.
3. If the system is inconsistent and has a unique least squares solution, then this solution is also the minimal solution.
4. If the system is inconsistent and has a least squares set solution, then the minimal solution is a member of this set that has the least Frobenius norm. There are various definitions for the concept of fuzzy numbers (see [2]).

Definition 2.1 26. *A fuzzy number is a fuzzy set like which satisfies:*

a. u is upper semicontinuous,
b. $u(x) = 0$ *outside some interval* $[c, d]$,
c. There are real numbers a, b such that $c \leqslant a \leqslant b \leqslant d$ *and*
  1. $u(x)$ *is monotonic increasing on* $[c, a]$,
  2. $u(x)$ *is monotonic decreasing on* $[b, d]$,
  3. $u(x) = 1$, $a \leqslant x \leqslant b$.

Definition 2.2 15. *A fuzzy number u in parametric form is a pair* $(\underline{u}, \bar{u})$ *of functions* $\underline{u}(r), \bar{u}(r), 0 \leqslant r \leqslant 1$
which satisfies the requirements:

a. $\underline{u}(r)$ *is a bounded monotonic increasing left continuous function.*
b. $\bar{u}(r)$ *is a bounded monotonic decreasing left continuous function.*
c. $\underline{u}(r), \bar{u}(r), 0 \leqslant r \leqslant 1$.

A crisp number x is simply represented by $(\underline{u}(r), \bar{u}(r)) = (x, x),\ 0 \leqslant r \leqslant 1$. *By appropriate definitions the fuzzy number space* $\{(u(r), u(r))\}$ *becomes a convex cone* E^1 *which could be embedded isomorphically and sometrically into a Banach space.*

Definition 2.3 15. *Let* $x = (\underline{x}(r), \bar{x}(r)),\ y = (\underline{y}(r), \bar{y}(r)) \in E^1$ $0 \leqslant r \leqslant 1$ *and real number* k.

(a) $\bar{x}(r) = \bar{y}(r),\ \underline{x}(r) = \underline{y}(r) \Leftrightarrow x = y$

(b) $x + y = \big(\underline{x}(r) + \underline{y}(r),\ \bar{x}(r) + \bar{y}(r)\big)$
$x - y = \big(\underline{x}(r) + \bar{y}(r),\ \bar{x}(r) + \underline{y}(r)\big)$

(c) $kx(r) = \begin{cases} (k\underline{x}(r),\ k\bar{x}(r)) & k \geqslant 0 \\ (k\bar{x}(r),\ k\underline{x}(r)) & k < 0 \end{cases}$

Definition 2.4 *The matrix system*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1l} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2l} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{x}_{n1} & \tilde{x}_{n2} & \cdots & \tilde{x}_{nl} \end{bmatrix} = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \cdots & \tilde{b}_{1l} \\ \tilde{b}_{21} & \tilde{b}_{22} & \cdots & \tilde{b}_{2l} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{b}_{m1} & \tilde{b}_{m2} & \cdots & \tilde{b}_{ml} \end{bmatrix} \tag{1}$$

where $1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n$ *are crisp numbers and the elements* $\tilde{b}_{ij}$ *in the right-hand matrix are fuzzy numbers, i.e.,*
$\tilde{b}_{ij} \in E^1,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant 1$ *is called a general fuzzy matrix equation (GFME). Using matrix notation, we have*

$$A\tilde{x} = \tilde{B} \tag{2}$$

A fuzzy number matrix

$$\tilde{x} = (x_1, x_2, \ldots, x_l)$$

given by

$$x_j = \big((\underline{x}_{1j}(r), \bar{x}_{1j}(r)), (\underline{x}_{2j}(r), \bar{x}_{2j}(r)), \ldots, (\underline{x}_{nj}(r), \bar{x}_{nj}(r))\big)^T,\ 1 \leqslant j \leqslant l,\ 0 \leqslant r \leqslant 1$$

is called a solution of the fuzzy matrix system (2) if

$$Ax_j = b_j, \quad j = 1, 2, \ldots, l$$

Where $b_j = \big(\big(\underline{b}_{1j}(r), \bar{b}_{1j}(r)\big), \big(\underline{b}_{2j}(r), \bar{b}_{2j}(r)\big), \ldots, \big(\underline{b}_{mj}(r), \bar{b}_{mj}(r)\big)\big)^T$, is the jth column of fuzzy number matrix $\tilde{B}$.

3. The Model

Using the embedding approach in [17, 24] and the technique applied in [15] by Friedman et al we extend the matrix systems (1) into a $2m \times 2n$ crisp function matrix equation.

Theorem 3.1 *The fuzzy matrix equation (1) can be extended into a crisp matrix equation as follows:*

$$\begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1,2n} \\ s_{21} & s_{22} & \cdots & s_{2,2n} \\ \vdots & \vdots & \vdots & \vdots \\ s_{2m,1} & s_{2m,2} & \cdots & s_{2m,2n} \end{bmatrix} \begin{bmatrix} \underline{x}_{11} & \underline{x}_{12} & \cdots & \underline{x}_{1l} \\ \underline{x}_{21} & \underline{x}_{22} & \cdots & \underline{x}_{2l} \\ \vdots & \vdots & \vdots & \vdots \\ \underline{x}_{n1} & \underline{x}_{n2} & \cdots & \underline{x}_{nl} \\ -\bar{x}_{11} & -\bar{x}_{12} & \cdots & -\bar{x}_{1l} \\ -\bar{x}_{21} & -\bar{x}_{22} & \cdots & -\bar{x}_{2l} \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{x}_{n1} & -\bar{x}_{n2} & \cdots & -\bar{x}_{nl} \end{bmatrix} = \begin{bmatrix} \underline{y}_{11} & \underline{y}_{12} & \cdots & \underline{y}_{1l} \\ \underline{y}_{21} & \underline{y}_{22} & \cdots & \underline{y}_{2l} \\ \vdots & \vdots & \vdots & \vdots \\ \underline{y}_{m1} & \underline{y}_{m2} & \cdots & \underline{y}_{ml} \\ -\bar{y}_{11} & -\bar{y}_{12} & \cdots & -\bar{y}_{1l} \\ -\bar{y}_{21} & -\bar{y}_{22} & \cdots & -\bar{y}_{2l} \\ \vdots & \vdots & \vdots & \vdots \\ -\bar{y}_{m1} & -\bar{y}_{m2} & \cdots & -\bar{y}_{ml} \end{bmatrix} \quad (3)$$

where

$$\tilde{x}_{kj} = \left(\underline{x}_{kj}(r), \bar{x}_{kj}(r)\right), \qquad k = 1, 2, \ldots, n \ j = 1, 2, \ldots, l$$
$$\tilde{y}_{kj} = \tilde{b}_{kj} = \left(\underline{x}_{kj}(r), \bar{x}_{kj}(r)\right), \qquad k = 1, 2, \ldots, m \ j = 1, 2, \ldots, l$$

Where b_j, x_j denote the jth column of unknown matrix $\tilde{x}$ and fuzzy number matrix $\tilde{B}$, respectively and $s = (s_{ij})$, $1 \leqslant j \leqslant 2m$, $1 \leqslant j \leqslant 2n$, and s_{ij} are determined as follows:

$$a_{ij} \geqslant 0 \qquad s_{ij} = a_{ij}, \quad s_{m+i,n+j} = a_{ij}$$
$$a_{ij} < 0 \qquad s_{i,j+n} = -a_{ij}, \quad s_{m+i,j} = -a_{ij} \ 1 \leqslant j \leqslant m, \ 1 \leqslant j \leqslant n$$

and any $S_k l$ which is not determined by the above items is zero, $1 \leqslant k \leqslant 2m, 1 \leqslant l \leqslant 2n$.
Moreover, S is nonnegative and

$$S = \begin{bmatrix} B & C \\ C & B \end{bmatrix}$$

where $A = B - C$.

Writing (3) in matrix form, we have

$$SX(r) = Y(r) \quad (4)$$

In order to solve the original fuzzy matrix equation (1), we must consider the model matrix equation (3). There are some main results about solvability of the Eq. (3) and the original fuzzy system (1).

Lemma 3.2 27. *The $2m \times 2n$ crisp system of linear equation $Sx = y$ exists solution if and only if the rank of matrix S equals to that of matrix (S, y), i.e.,*
Rank(S)=Rank(S, Y)
when Rank(S)<Rank(S, Y), the system does not have any solution,
when Rank(S)=Rank(S, Y)= $2n$, the system has a unique solution,
when Rank(S)=Rank(S, Y)< $2n$, the system has an infinite of solutions.

Theorem 3.3 *The model matrix equation $SX = Y(r)$ has solution if and only if Rank(s)=Rank$(S, Y(r))$, $r \in [0, 1]$.*

Corollary 3.4 *Let T be $p \times q$ real column full rank or row full rank. There exists a $p \times p$ orthogonal matrix U, a $q \times q$ orthogonal matrix V and a $p \times q$ diagonal* *matrix* $\sum$ *with* $\langle\sum\rangle_{ij} = 0$ *for* $i \neq j$ *and* $\langle\sum\rangle_{ii} = \sigma_i > 0$ *with*

$$\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_s \geqslant 0$$

where $s = \min p, q$, *such that the singular value decomposition,*

$$T = U \sum V^+$$

is valid. And if $\sum^+$ *is that* $q \times p$ *matrix whose only non-zero entries are* $\langle\sum^+\rangle_{ii} = 1/\sigma_i$ *for* $1 \leqslant i \leqslant s$, *then* $T^+ = V \sum U^+$ *is the unique, pseudo-inverse of* T.

We refer the reader to [9] for more information on finding pseudo-inverse of an arbitrary matrix, and when we work with full rank matrices, there are not any problem and all calculations are stable and well-posed.

COROLLARY 3.5 *The matrix* S *is row full rank (for* $m \leqslant n$*) or column full rank (for* $n < m$*) if and only if the matrices* $A = B - C$ *and* $B + C$ *are both row full rank (for* $m \leqslant n$*) or column full rank (for* $n < m$*).*

In order to solve the linear fuzzy system (1). we must calculate S^+. The next result is taken from the theory of block matrices and provides the structure of S^+. For finding S^+, we must find the pseudo-inverse of two real full rank $m \times n$ matrices by following theorem.

THEOREM 3.6 *The pseudo-inverse of nonnegative full rank matrix*

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$$

is

$$S^+ = \begin{pmatrix} D & E \\ E & D \end{pmatrix} \tag{5}$$

where

$$D = \frac{1}{2}[(B + C)^+ + (B - C)^+], \; E = \frac{1}{2}[(B + C)^+ - (B - C)^+].$$

Proof Let S^+ be the pseudo-inverse, it is unique. Without loss of generality, suppose that

$$S^+ = \begin{pmatrix} D & E \\ E & D \end{pmatrix}$$

We know $SS^+S = S$ Hence

$$\begin{pmatrix} B & C \\ C & B \end{pmatrix} \begin{pmatrix} D & E \\ E & D \end{pmatrix} \begin{pmatrix} B & C \\ C & B \end{pmatrix} = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$$

and get

$$BDB + BEC + CDC + CEB = B, \;\; BDC + BEB + CDB + CEC = C \tag{6}$$

By adding and then by subtracting the-two parts of (6), we obtain

$$(B+C)(D+E)(B+C) = (B+C), \quad (B-C)(D-E)(B-C) = (B-C)$$

also, we can show

$$(D+E)(B+C)(D+E) = (D+E), \qquad (D-E)(B-C)(D-E) = (D-E)$$
$$[(B+C)(D+E)]^t = (B+C)(D+E), \qquad [(B-C)(D-E)]^t = (B-C)(D-E)$$
$$[(D+E)(B+C)]^t = (D+E)(B+C), \qquad [(D-E)(B-C)]^t = (D-E)(B-C)$$

Thus S^+ must have the structure. given by (5). In order to calculate E and D in (5), we have

$$(B+C)^+ = D+E, \quad (B-C)^+ = D-E$$

and consequently,

$$D = \frac{1}{2}[(B+C)^+ + (B-C)^+], \quad E = \frac{1}{2}[(B+C)^+ - (B-C)^+]$$

■

COROLLARY 3.7 18. *The minimal solution of $SX = Y$ is obtained by*

$$X = S^+Y \tag{7}$$

The following result provides necessary and coefficient conditions for existing a fuzzy matrix solution.

THEOREM 3.8 *The solution X of (7) is a fuzzy matrix for an arbitrary fuzzy matrix Y if and only if S^+ is non-negative, i.e.*

$$(S^+) \leqslant 0, \ 1 \leqslant i \leqslant 2m, 1 \leqslant j \leqslant 2n$$

Proof The same as the proof of Theorem 3 in [4]. ■

4. Inconsistent Fuzzy Matrix Equation

DEFINITION 4.1 29. *If the crisp matrix equation (3) does not have solution, the associated fuzzy matrix equation $A\tilde{x} = \tilde{B}$, i.e.,*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1l} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2l} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{x}_{n1} & \tilde{x}_{n2} & \cdots & \tilde{x}_{nl} \end{bmatrix} = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \cdots & \tilde{b}_{1l} \\ \tilde{b}_{21} & \tilde{b}_{22} & \cdots & \tilde{b}_{2l} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{b}_{m1} & \tilde{b}_{m2} & \cdots & \tilde{b}_{ml} \end{bmatrix}$$

where the coefficient matrix $A = (a_{ij})$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ is crisp and the right-hand matrix $\tilde{B} = (\tilde{b}_{ij})$ is fuzzy, i.e.,

$$\tilde{b}_{ij} \in E^1, \ 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$$

is called an inconsistent fuzzy matrix equation (IFME). Let's consider the following examples.

Example 4.2 The matrix A of the fuzzy matrix system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} = \begin{bmatrix} (1+r, 3-r) & (2+r, 3) \\ (r, 2-r) & (-1, -r) \\ (4+r, 8-2r) & (r, 2-r) \end{bmatrix}$$

is nonsingular, while

$$S = \begin{bmatrix} 1\,1\,1\,0\,0\,0 \\ 1\,1\,0\,0\,0\,1 \\ 1\,0\,0\,0\,1\,1 \\ 0\,0\,0\,1\,1\,1 \\ 0\,0\,1\,1\,1\,0 \\ 0\,1\,0\,1\,0\,1 \end{bmatrix}$$

in its extended matrix equation $SX = Y(r)$ is singular. This example illustrates that a fuzzy matrix system, which is represented even if by a nonsingular matrix A, may has no solution or has an infinite number of solutions.

Example 4.3 Consider the fuzzy matrix system

$$\begin{bmatrix} 1\,1 & 1 \\ 1\,1 & -1) \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} = \begin{pmatrix} (r, 2-r) & (-1+r, -r) \\ (0, 1-r) & (1+r, 3-r) \end{pmatrix}$$

The extended 4×6 matrix is

$$S = \begin{bmatrix} 1\,1\,1\,0\,0\,0 \\ 1\,1\,0\,0\,0\,1 \\ 0\,0\,0\,1\,1\,1 \\ 0\,0\,1\,1\,1\,0 \end{bmatrix}$$

and the augmented matrix is

$$\begin{bmatrix} 1\,1\,1\,0\,0\,0 & r & -1+r \\ 1\,1\,0\,0\,0\,0 & 0 & 1+r \\ 0\,0\,0\,1\,1\,1 & r-2 & r \\ 0\,0\,1\,1\,1\,0 & r-1 & -3+r \end{bmatrix}$$

Actually, the original system is inconsistent since

$$\text{Rank}(S, Y(r)) = 4, \quad \text{Rank}(S) = 3$$

The above two examples show that the fuzzy matrix equation without solution exists in some cases. So it is very necessary to seek their minimal solution for this type of fuzzy matrix system.

Definition 4.4 *Let* $X = \{(\underline{x}_{ij}(r), -\bar{x}_{ij}(r))\}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant l$ *denote the* *minimal solution of Eq. 4. The fuzzy number matrix*

$$U = \{(\underline{u}_{ij}(r), -\bar{u}_{ij}(r))\},\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l$$

defined by

$$\underline{u}_{ij}(r) = \min\{\underline{x}_{ij}(r), \bar{x}_{ij}(r), \underline{x}_{ij}(1), \bar{x}_{ij}(1)\}$$
$$\bar{u}_{ij}(r) = \max\{\underline{x}_{ij}(r), \bar{x}_{ij}(r), \underline{x}_{ij}(1), \bar{x}_{ij}(1)\}$$

is called the minimal fuzzy solution of Eq. 4. If $\{(\underline{x}_{ij}(r), \bar{x}_{ij}(r))\}$ *are all fuzzy numbers then U is called a strong minimal fuzzy solution. Otherwise, U is a weak minimal fuzzy solution.*

5. Numerical Examples

Example 5.1 Consider the following fuzzy systems

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} (1+r, 3-r) & (-1+r, 2-r) \\ (r, 2-r) & (0, 1-r) \end{bmatrix}$$

The extended 4×4 matrix S is

$$S = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1+r & -1+r \\ 0 & 1 & 1 & 0 & r & 0 \\ 0 & 1 & 1 & 0 & r-3 & r-2 \\ 1 & 0 & 0 & 1 & r-2 & r-1 \end{bmatrix}$$

The original system is inconsistent, therefore we have

$$S^+ = \begin{pmatrix} -0.7071 & 0 & 0.7071 & 0 \\ 0 & 0.7071 & 0 & 0.7071 \\ 0 & 0.7071 & 0 & -0.7071 \\ -0.7071)0 & -0.7071 & 0 & \end{pmatrix} \times$$

$$\begin{pmatrix} 0.5000 & 0 & 0 & 0 \\ 0 & 0.5000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} -0.7071 & 0 & 0 & -0.7071 \\ 0 & 0.7071 & 0.7071 & 0 \\ 0 & -0.7071 & 0.7071 & 0 \\ -0.7071)0 & 0 & 0.7071 & \end{pmatrix}$$

Thus

$$X = \begin{pmatrix} \underline{x}_{11}(r) & \underline{x}_{12}(r) \\ \underline{x}_{21}(r) & \underline{x}_{22}(r) \\ -\bar{x}_{11}(r) & -\bar{x}_{12}(r) \\ -\bar{x}_{21}(r) & -\bar{x}_{22}(r) \end{pmatrix} = S^+Y(r) = S^+ \begin{pmatrix} 1+r & -1+r \\ r & 0 \\ -3+r & 2+r \\ -2+r & -1+r \end{pmatrix} = \begin{pmatrix} \frac{-1}{4}+\frac{1}{2}r & \frac{-1}{2}+\frac{1}{2}r \\ \frac{-3}{4}+\frac{1}{2}r & \frac{-1}{2}+\frac{1}{4}r \\ \frac{-3}{4}+\frac{1}{2}r & \frac{-1}{2}+\frac{1}{4}r \\ \frac{-1}{4}+\frac{1}{2}r & \frac{-1}{2}+\frac{1}{2}r \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} (\frac{-1}{4}+\frac{1}{2}r, \frac{3}{4}-\frac{1}{2}r) & (\frac{-1}{2}+\frac{1}{2}r, \frac{1}{2}-\frac{1}{4}r) \\ (\frac{-3}{4}+\frac{1}{2}r, \frac{1}{4}-\frac{1}{2}r) & (\frac{-1}{2}+\frac{1}{4}r, \frac{1}{2}-\frac{1}{2}r) \end{pmatrix}$$

which is a strong minimal fuzzy solution.

Example 5.2 Consider the fuzzy matrix system

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = \begin{pmatrix} (r, 2-r) & (1+r, 3-r) \\ (0, 1-r) & (r, 2-r) \end{pmatrix}$$

The extended 4×6 matrix is

$$S = \begin{pmatrix} 1\,1\,1\,0\,0\,0 \\ 1\,1\,0\,0\,0\,1 \\ 0\,0\,0\,1\,1\,1 \\ 0\,0\,1\,1\,1\,0 \end{pmatrix}$$

and the augmented matrix is

$$\begin{pmatrix} 1\,1\,1\,0\,0\,0 & r & 1+r \\ 1\,1\,0\,0\,0\,1 & 0 & r \\ 0\,0\,0\,1\,1\,1 & -2+r & -3+r \\ 0\,0\,1\,1\,1\,0 & -1+r & -2+r \end{pmatrix}$$

Which implies that the original system is in consistent since

$$\text{Rank}(S) = 3, \quad \text{Rank}(S, Y(r)) = 4$$

$$S^{+} = \begin{pmatrix} -0.4082 & 0.5000 & -0.0000 & 0.4964 & 0.4105 & 0.4105 \\ -0.4082 & 0.5000 & -0.0000 & -0.7608 & -0.0476 & -0.0476 \\ -0.4082 & -0.0000 & -0.7071 & 0.2644 & -0.3629 & -0.3629 \\ -0.4082 & -0.5000 & 0.0000 & -0.1322 & 0.6815 & -0.3185 \\ -0.4082 & -0.5000 & 0.0000 & -0.1322 & -0.3185 & 0.6815 \\ -0.4082 & 0.0000 & 0.7071 & 0.2644 & -0.3629 & -0.3629 \end{pmatrix} \times$$

$$\begin{pmatrix} 0.4082 & 0 & 0 & 0 \\ 0 & 0.5000 & 0 & 0 \\ 0 & 0 & 0.7071 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} -0.5000 & 0.5000 & -0.5000 & -0.5000 \\ -0.5000 & 0.5000 & 0.5000 & 0.5000 \\ -0.5000 & -0.5000 & 0.5000 & -0.5000 \\ -0.5000 & -0.5000 & -0.5000 & 0.5000 \end{pmatrix}$$

Then the minimal solution of system is

$$X = S^{+}Y(r) = S^{+} \begin{pmatrix} r & r-1 \\ 1+r & r \\ r-3 & r-2 \\ r-3 & r-1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{8} + \frac{1}{8}r & \frac{5}{12} + \frac{1}{3}r \\ \frac{1}{8} + \frac{1}{8}r & \frac{5}{12} + \frac{1}{3}r \\ \frac{1}{2}r & \frac{1}{6} + \frac{1}{3}r \\ -\frac{5}{8} + \frac{3}{8}r & -\frac{13}{12} + \frac{1}{3}r \\ -\frac{5}{8} + \frac{3}{8}r & -\frac{13}{12} + \frac{1}{3}r \\ 1 - \frac{1}{2}r & -\frac{5}{6} + \frac{1}{3}r \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = \begin{pmatrix} (\frac{1}{8}+\frac{1}{8}r, \frac{5}{8}-\frac{3}{8}r) & (\frac{5}{12}+\frac{1}{3}r, \frac{13}{12}-\frac{1}{3}r) \\ (\frac{1}{8}+\frac{1}{8}r, \frac{5}{8}-\frac{3}{8}r) & (\frac{5}{12}+\frac{1}{3}r, \frac{13}{12}-\frac{1}{3}r) \\ (1-\frac{1}{2}r, \frac{1}{2}r) & (\frac{1}{6}+\frac{1}{3}r, \frac{5}{6}-\frac{1}{3}r) \end{pmatrix}$$

Obviously, x_{31} is not fuzzy number and hence we can obtain the weak minimal fuzzy solution as follows:

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix} = \begin{pmatrix} (\frac{1}{8}+\frac{1}{8}r, \frac{5}{8}-\frac{3}{8}r) & (\frac{5}{12}+\frac{1}{3}r, \frac{13}{12}-\frac{1}{3}r) \\ (\frac{1}{8}+\frac{1}{8}r, \frac{5}{8}-\frac{3}{8}r) & (\frac{5}{12}+\frac{1}{3}r, \frac{13}{12}-\frac{1}{3}r) \\ (\frac{1}{2}r, 1-\frac{1}{2}r) & (\frac{1}{6}+\frac{1}{3}r, \frac{5}{6}-\frac{1}{3}r) \end{pmatrix}$$

6. Conclusions

In this paper, we propose a general model for solving a fuzzy matrix equation with $n \times l$ variables. The original system with matrix coefficient A is replaced by a $(2m) \times (2n)$ crisp linear matrix equation S which may singular even if A is nonsingular. For finding the pseudo-inverse of S, we find the pseudo-inverse of two $m \times n$ matrices. Also, a condition for the existence of a fuzzy solution to the fuzzy general linear system, is presented.

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