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The Identification of Efficiency by Using Fuzzy Numbers

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Abstract. Traditional Data Envelopment Analysis (DEA) models for measuring the relative efficiencies of a set of Decision Making Units (DMUs) the using various inputs to produce various outputs are limited to crisp data. In real world situations, however, this assumption may not always be true. When some inputs and outputs are unknown decision variables, such as fuzzy data, rough data, interval data, the DEA model is called imprecise DEA. this paper develops a procedure to measure the efficiencies of DMUs with fuzzy observations. The basic idea is to transform a fuzzy DEA model to family of conventional crisp DEA models by applying optimistic, intermediate and pessimistic concepts. A numerical example is given to show the efficiency.

Keywords: Data Envelopment Analysis, Triangular fuzzy number, Efficiency DMUs.

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1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric method for evaluating the relative efficiency of DMUs on the basis of multiple inputs and outputs. The original DEA models [2,3] assume the inputs and outputs are measured by exact values on a ratio scale. Cooper et al [4] addressed the problem of imprecise data in DEA in its general form. The term imprecise data reflects the situation where some of the input and output data are only know to lie with bounded interval (interval number)while other data are known only up to an order. In this paper we assume the inputs and outputs have fuzzy number form.The concept of fuzzy number and arithmetic operations with these numbers were first introduce and investigated by

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Zadeh [12]. Mizumoto and Tanaka [8,9], Dubois and Prade [5]. One of the major application using imprecise data arithmetic is treating DEA models

parameters are all or partially represented by fuzzy numbers or rough data[10,11]. In this paper we proposed, optimistic, intermediate and pessimistic concepts for evaluating the relative efficiencies of DMUs which stand for DMUs with triangular fuzzy number input and output terms. By this idea, the efficiency of DMU has three parameters optimistic efficiency,intermediate efficiency and pessimistic efficiency and then by using optimistic, intermediate and pessimistic efficiency all of the DMUs are compared.

This paper is organized as follows: In section 2, definitions and notation of fuzzy set theory is reviewed. In section 3, the Optimistic, Intermediate and Pessimistic efficiency is presented. In section 4, we will classified DMUs based upon Optimistic, Intermediate and Pessimistic efficiency. A numerical example is presented in section 5. Finally, section 6 concludes the paper with the summary and conclusions.

2. Preliminaries

We first review the fundamental notation and basic definitions of fuzzy set theory initiated by Bellman and Zadeh [12].

Definition 2.1 *If X is a collection of objects denoted generically by x, then a fuzzy set in X is a set of ordered pairs:*

$$\tilde{A} = \{(x, \tilde{A}(x)) | x \in X\}$$

where $\tilde{A}(x)$ is called the membership function which associates with each $x \in X$ a number in [0,1] indicating to what degree x is a number.

Definition 2.2 *The α-level set of $\tilde{A}$ is the set$\tilde{A}_\alpha = \{x | \tilde{A}(x) \geq \alpha\}$ where $\alpha \in [0, 1]$. The lower and upper bounds of any α-level set $\tilde{A}_\alpha$ are represented by finite number* $\inf_{x \in \tilde{A}_\alpha}$ *and* $\sup_{x \in \tilde{A}_\alpha}$.

Definition 2.3 *A fuzzy set $\tilde{A}$ is convex if*

$$\tilde{A}(\lambda x + (1 - \lambda)y) \geq \min\{\tilde{A}(x)\ ,\ \tilde{A}(y)\} \qquad \forall x, y \in X\ ,\ \lambda \in [0, 1].$$

Definition 2.4 *A convex fuzzy set $\tilde{A}$ on R is a fuzzy number if the following conditions hold:*

(a) Its membership function is piecewise continuous function.

(b) There exist three intervals [a,b], [b,c] and [c,d] such that $\tilde{A}$ is increasing on [a,b], equal to 1 on [b,c], decreasing on [c,d] and equal to 0 elsewhere.

Definition 2.5 *The support of a fuzzy set $\tilde{A}$ is a set $\tilde{A}$ is a set of elements in X for which $\tilde{A}(x)$ is positive, that is,*

$$supp\tilde{A} = \{x \in X | \tilde{A}(x) > 0\}.$$

Definition 2.6 *A fuzzy number $\tilde{A} = (a^l, a^m, a^u)$ is called triangular fuzzy number, if membership function is defined as:*

$$\mu_{\tilde{A}} = \begin{Bmatrix} 0 & x < a^l \\ \frac{x-a^l}{a^m-a^l} & a^l \leqslant x \leqslant a^m \\ \frac{-x+a^u}{a^u-a^m} & a^m \leqslant x \leqslant a^u \\ 0 & x > a^l \end{Bmatrix}.$$

We next define arithmetic on triangular fuzzy numbers.
Let $\tilde{A} = (a^l, a^m, a^u)$ and $\tilde{B} = (b^l, b^m, b^u)$ be two triangular fuzzy numbers. Define,

$$\begin{aligned} &x > 0, \quad x \in R : x\tilde{A} = (xa^l, xa^m, xa^u), \\ &x < 0, \quad x \in R : x\tilde{A} = (xa^u, xa^m, xa^l), \\ &\tilde{A} + \tilde{B} = (a^l + b^l, a^m + b^m, a^u + b^u), \\ &\tilde{A} - \tilde{B} = (a^l - b^u, a^m - b^m, a^u - b^l). \end{aligned}$$

3. Optimistic, Intermediate and Pessimistic Efficiency

In original DEA measure of the efficiency of any DMU is obtained as the maximum of a ratio of weighted outputs to weighted inputs subject to the condition that the similar ratios for every DMU be less than or equal to unity. The mathematical model is as follows:

$$\begin{aligned} \theta_o = &\max \frac{\sum_{r=1}^{s} u_r Y_{ro}}{\sum_{i=1}^{k} v_i X_{io}} \\ &\text{s.t.} \\ &\frac{\sum_{r=1}^{s} u_r Y_{rj}}{\sum_{i=1}^{k} v_i X_{ij}} \leqslant 1 \;\; j = 1, 2, \ldots, n \\ &u_r \;, v_i \geqslant 0 \; i = 1, 2, \ldots, k \;,\; r = 0, 1, 2, \ldots, s \;,\; o \in \{1, 2, \ldots, n\} \end{aligned} \tag{1}$$

Here the X_{ij} , Y_{rj} (all positive) are crisp outputs and inputs. When X_{ij} , Y_{rj} are triangular fuzzy number we have following model:

$$\begin{aligned} \theta_o = &\max \frac{\sum_{r=1}^{s} u_r (y_{ro}^l, y_{ro}^k, y_{ro}^u)}{\sum_{i=1}^{k} v_i (x_{io}^l, x_{io}^m, x_{io}^u)} \\ &\text{s.t.} \\ &\frac{\sum_{r=1}^{s} u_r (y_{rj}^l, y_{rj}^m, y_{rj}^u)}{\sum_{i=1}^{k} v_i (x_{ij}^l, x_{ij}^m, x_{ij}^u)} \leqslant 1 \;\; j = 1, 2, \ldots, n \\ &u_r \;, v_i \geqslant 0 \; i = 1, 2, \ldots, k \;,\; r = 0, 1, 2, \ldots, s \;,\; o \in \{1, 2, \ldots, n\} \end{aligned} \tag{2}$$

3.1 *Optimistic Efficiency*

For any DMU of the best situation is given by the following model:

$$\begin{aligned}
\theta_o^u = \max & \frac{\sum_{r=1}^{s} u_r y_{ro}^u}{\sum_{i=1}^{k} v_i x_{io}^l} \\
\text{s.t.} & \\
& \frac{\sum_{r=1}^{s} u_r y_{rj}^l}{\sum_{i=1}^{k} v_i x_{ij}^u} \leqslant 1 \quad j \neq o \\
& \frac{\sum_{r=1}^{s} u_r y_{ro}^u}{\sum_{i=1}^{k} v_i x_{io}^l} \leqslant 1 \\
& u_r \ , v_i \geqslant 0 \ i = 1, 2, \ldots, k \ , \ r = 0, 1, 2, \ldots, s \ , \ o \in \{1, 2, \ldots, n\}
\end{aligned} \tag{3}$$

In this model DMU$_o$ has a best situation (least input most output)and for other DMUs conversely. θ_o^u is the value of optimistic efficiency for DMU$_o$.

3.2 *Intermediate Efficiency*

For any DMU of the intermediate situation is given by the following model:

$$\begin{aligned}
\theta_o^m = \max & \frac{\sum_{r=1}^{s} u_r y_{ro}^m}{\sum_{i=1}^{k} v_i x_{io}^m} \\
\text{s.t.} & \\
& \frac{\sum_{r=1}^{s} u_r y_{rj}^m}{\sum_{i=1}^{k} v_i x_{ij}^m} \leqslant 1 \quad j = 1, 2, \ldots, n \\
& u_r \ , v_i \geqslant 0 \ i = 1, 2, \ldots, k \ , \ r = 0, 1, 2, \ldots, s \ , \ o \in \{1, 2, \ldots, n\}
\end{aligned} \tag{4}$$

In model (4), all of the DMUs are in similar condition. θ_o^m is the value of intermediate efficiency for DMU$_o$.

3.3 *Pessimistic Efficiency*

For any DMU of the worst situation is given by the following model:

$$\begin{aligned}
\theta_o^l = \max & \frac{\sum_{r=1}^{s} u_r y_{ro}^l}{\sum_{i=1}^{k} v_i x_{io}^u} \\
\text{s.t.} & \\
& \frac{\sum_{r=1}^{s} u_r y_{rj}^u}{\sum_{i=1}^{k} v_i x_{ij}^l} \leqslant 1 \quad j \neq o \\
& \frac{\sum_{r=1}^{s} u_r y_{ro}^l}{\sum_{i=1}^{k} v_i x_{io}^u} \leqslant 1 \\
& u_r \ , v_i \geqslant 0 \ i = 1, 2, \ldots, k \ , \ r = 0, 1, 2, \ldots, s \ , \ o \in \{1, 2, \ldots, n\}
\end{aligned} \tag{5}$$

In model (5), DMU$_o$ has a worst situation (least output and most input) and for other DMUs conversely. θ_o^l is the value of pessimistic efficiency for DMU$_o$.
We assume that x_{ij}^l ,x_{ij}^m ,x_{ij}^u, y_{rj}^l ,y_{rj}^m ,y_{rj}^u for $i(i = 1,2,3,\dots,k)$,j $(j = 1,2,3,\dots,n)$ and r $(r = 1,2,3,\dots,s)$ are constants and strictly positive.

THEOREM 3.1 *For DMU$_o$* $o \in \{1,2,\dots,n\}$ $\theta_o^l \leqslant \theta_o^m \leqslant \theta_o^u$.

Proof prove of theorem with due attention to inputs and outputs is obvious.
■

For calculation θ_o^l, θ_o^m θ_o^u the models (1),(2), and (3) are replaced with the following LP problems respectively:

1. Optimistic efficiency

$$\begin{aligned}
\theta_o^u = \max \sum_{r=1}^{s} u_r y_{ro}^u & \\
\text{s.t.} & \\
\sum_{i=1}^{k} v_i x_{io}^l = 1 & \\
\sum_{r=1}^{s} u_r y_{rj}^l - \sum_{i=1}^{k} v_i x_{ij}^u \leqslant 0 \quad j \neq o & \\
\sum_{r=1}^{s} u_r y_{ro}^u - \sum_{i=1}^{m} v_i x_{io}^l \leqslant 0 \quad o \in \{1,2,\dots,n\} & \\
u_r\ , v_i \geqslant 0\ i = 1,2,\dots,k\ ,\ r = 0,1,2,\dots,s\ , &
\end{aligned} \tag{6}$$

2. Intermediate efficiency

$$\begin{aligned}
\theta_o^m = \max \sum_{r=1}^{s} u_r y_{ro}^m & \\
\text{s.t.} & \\
\sum_{i=1}^{k} v_i x_{io}^m = 1 & \\
\sum_{r=1}^{s} u_r y_{rj}^m - \sum_{i=1}^{k} v_i x_{ij}^m \leqslant 0 & \\
u_r\ , v_i \geqslant 0\ i = 1,2,\dots,k\ ,\ r = 0,1,2,\dots,s &
\end{aligned} \tag{7}$$

3. Pessimistic efficiency

$$\begin{aligned}
\theta_o^l = \max & \sum_{r=1}^{s} u_r y_{ro}^l \\
& \text{s.t.} \\
& \sum_{i=1}^{k} v_i x_{io}^u = 1 \\
& \sum_{r=1}^{s} u_r y_{rj}^u - \sum_{i=1}^{k} v_i x_{ij}^l \leqslant 0 \quad j \neq o \\
& \sum_{r=1}^{s} u_r y_{ro}^l - \sum_{i=1}^{k} v_i x_{io}^u \leqslant 0 \quad o \in \{1,2,\ldots,n\} \\
& u_r \ , v_i \geqslant 0 \ i = 1,2,\ldots,k \ , \ r = 0,1,2,\ldots,s \ ,
\end{aligned} \tag{8}$$

With using (4),(5), and (6) the efficiency for DMU$_o$ is a triplet instance ,(θ_o^l, θ_o^m ,θ_o^u) You should notice that this triplet (θ_o^l, θ_o^m ,θ_o^u) is not always a T.F.N. For simplicity we named it triplet efficiency.

4. Categorization and Discrimination of the Units

Considering that the optimistic, intermediate and pessimistic efficiency of any DMU lies an triangular fuzzy number, all DMUs can be divided into one of the three following categories:

Category (1) Includes all DMUs which are strong efficient in any situation:

$$E^{++} = \{DMU_j,\ j = 1,2,\ldots,n\ : \theta_j^l = 1\},$$

Category (2) Includes all DMUs which are weak efficient in any situation:

$$E^{+} = \{DMU_j,\ j = 1,2,\ldots,n\ : \theta_j^u = 1, \theta_j^m \leqslant 1, \theta_j^l < 1\},$$

Category (3) Includes all DMUs which are inefficient in any situation:

$$E^{-} = \{DMU_j,\ j = 1,2,\ldots,n\ : \theta_j^u < 1\}.$$

5. Numerical Example

We provide a numerical example to illustrate our method. Table 1 presented a data set, consisting of five DMUs each consuming one input and producing one output. The efficiency scores obtained by applying models (6), (7), and (8),represented in Table 2.

As can be observed from Table 2, DMU2 is classified in E^{++}, is strong efficient, DMU1, DMU3, and DMU4 are classified in E^{+} as they are weak efficient in any case. DMU5 is classified in E^{-}, is inefficient.

Table 1. Input and output data for numerical example.

$DMUs$	I	O
DMU_1	(4,5,7)	(1,2,3)
DMU_2	(6,7,8)	(5,7,8)
DMU_3	(2,3,4)	(1,3,8)
DMU_4	(2,5,6)	(1,2,4)
DMU_5	(2,7,10)	(2,5,7)

Table 2. The efficiency scores of DMUs.

$DMUs$	Value of efficiency
DMU_1	(0.0357,0.4,1)
DMU_2	(1,1,1)
DMU_3	(0.0714,,1.0,1)
DMU_4	(0.0417,0.4,1)
DMU_5	(0.0500,0.7,1)

6. Conclusions

In This paper, we discusses and developed an imprecise DEA procedure based upon the CCR model. We consider the inputs and outputs of DMUs as the triangular fuzzy numbers. We then find the efficiency scores that might attain in an imprecise data setting. At the next, we use these efficiency scores to optimistic, intermediate and pessimistic concepts and categorize units in three classes.

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