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Alternative Mixed Integer Programming for Finding Efficient BCC Unit

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Abstract. Data Envelopment Analysis (DEA) cannot provide adequate discrimination among efficient decision making units (DMUs). To discriminate these efficient DMUs is an interesting research subject. The purpose of this paper is to develop the mix integer linear model which was proposed by Foroughi (Foroughi A.A. A new mixed integer linear model for selecting the best decision making units in data envelopment analysis. Computers & Industrial Engineering 60 (2011) 550-554) to present new alternative mix integer programming DEA (MIP-DEA) models which can be used to improve discrimination power of DEA and select the most BCC-efficient decision making unit (DMU). We will demonstrate that proposed model is able to select DMU throughout the real data sets.

Keywords: data envelopment analysis, mixed integer programming, efficiency, common weights, ranking.

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1. Introduction

Data Envelopment Analysis (DEA), as developed by Charnes et al. [5] measures the relative efficiencies among the decision making units (DMUs) with multiple-input and multiple output as a linear programming formulation. The procedure does not require a priori weights on inputs and outputs. On the other hand, as DEA applications increase, several variants of the original DEA model called the CCR model, which assumes a constant returns-to-scale, have also been proposed. CCR model are extended to BCC model by Banker et al. [4], which admits the Variable Returns to Scale (VRS) and distinguishes between technical and scale inefficiencies.

A range of DEA models have been developed that measure efficiency and capacity in different ways. These largely fall into the categories of being either input-

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oriented or output-oriented models. With input-oriented DEA, the linear programming model is configured so as to determine how much the input use of a DMU could contract if used efficiently in order to achieve the same output level. For the measurement of capacity, the only variables used in the analysis are the fixed factors of production. As these cannot be reduced, the input-oriented DEA approach is less relevant in the estimation of capacity utilization. Modifications to the traditional input-oriented DEA model, however, could be done such that it would be possible to determine the reduction in the levels of the variable inputs conditional on fixed outputs and a desired output level.

In contrast, with output-oriented DEA, the linear programming is configured to determine a DMU's potential output given its inputs if it operated efficiently as DMUs along the best practice frontier. Output-oriented models are "...very much in the spirit of neo-classical production functions defined as the maximum achievable output given input quantities" (Fre et al. [7], p. 95).

Whilst DEA has generated a good deal of attention it does have revealed some drawbacks. One of drawbacks is that it produces plural decision making units (DMUs) having the full efficient status denoted by unity (or 100%). To discriminate between these efficient DMUs is an interesting research subject. Tone [14] calls this problem the "super-efficiency problem". This problem becomes more serious if the number of inputs or outputs is increased. This lack of discrimination is because specialized DMUs may have the efficient status due to a single input or output, even though that input or output may be seen as relatively unimportant. Previously, various efforts have been devoted to develop methods without a priori information to improve discrimination in DEA. Sexton et al. [12] first introduce the concept of cross-efficiency in DEA by using peer evaluation instead of a self-evaluation. Andersen and Petersen [3] present the procedure referred to Super Efficiency-CCR (SE-CCR) model for ranking efficient units. Their basic idea is to compare the unit under evaluation with all other units in the sample, i.e., the DMU itself is excluded. Doyle and Green [6] further extend the work by Sexton et al. [12] by introducing aggressive and benevolent cross-efficiency referred to Cross-Efficiency Model (CEM). Tofallis [15] addresses the discrimination problem by presenting the profiling method. He uses the original DEA but taking one input at a time and only with related outputs. Seiford and Zhu [11] develop a supper-efficiency DEA model referred to SE-BCC model. Li and Reeves [9] propose a multiple criteria approach to DEA referred to MCDEA. Recently, Tone [14] proposes the super-efficiency model (referred to SE-SBM model) using the slacks-based measure of efficiency. However, all of these research works in DEA literature must be run n times, once for each unit, to get the relative efficiency of all DMUs.

DEA provides weights that are DMU-specific, and therefore it allows for individual circumstances of operation of the DMUs. Aside from the factors affecting performance considered in the efficiency analysis, there are often considerable variations in goals, policies, etc., among DMUs, which may justify the different weights for the same factor. The variation in weights in DEA maybe thus justified by the different circumstances under which the DMUs operate, and which are not captured by the chosen set of inputs and outputs factors (see Roll et al. [10] for discussions).

There are, however, situations in which the different DMUs experience similar circumstances and, therefore, using input and output weights that differ substantially across DMUs may not be warranted. When that is the case, both the inputs and the outputs should be aggregated by using weights that are common to all the DMUs. Common set of weights (CSW, as first denoted in Roll et al. [10]) is the usual approach in engineering and in most economic efficiency analyses. It has the appeal of a fair and impartial evaluation in the sense that each variable is attached the same weight in the assessments of all the DMUs. Nevertheless, the choice itself of such weights often raises serious difficulties, and in many cases there is no universally agreed-upon the weights to be used as pointed out in Doyle and Green [6]. It should also be noted that, unlike DEA, CSW allows us to rank the DMUs. The fact that DEA uses different profiles of weights in the assessments of the different DMUs makes impossible to derive an ordering of the units based on the resulting efficiency scores.

Recently, Amin and Toloo [2] proposed an integrated MILP model that evaluated all units by common set of weights for performance attributes which converged to the most efficient DMU without solving the LP n times. Toloo and Nalchigar [16] extended it to variable return to scale situation for finding the best BCC-efficient DMU. The proposed models eliminated the requirement of using a parameter in the objective function which is used by Ertay et al. (2006) and all units are evaluated by the common set of optimal weights. Amin [1] explained a drawback of the MILP model of Amin and Toloo [2] and proposed an improved integrated model to overcome it. Foroughi [8] illustrated that in some cases this improved model could be infeasible and proposed a new mix integer model for selecting the most CCR-efficient DMU which is useful just for constant return to scale situations. In this paper, new alternative model is presented to determine most BCC-efficient DMU for some situations that return to scale is variable.

The paper unfolds as follows. Section 2 we describe the output-oriented BCC model and the model proposed by Foroughi [8] as background models. Sections 3 introduce a new output-oriented model for finding most BCC-efficient DMU. Illustrative example and conclusion are discussed in Section 4 and Section 5, respectively.

2. BCC Model

Consider n DMUs that are to be evaluated in terms of m inputs and s outputs. Let x_{ij} $(i = 1, \ldots, m)$ and y_{rj} $(r = 1, \ldots, s)$ be the input and output values of DMU$_j$ $(j = 1, \ldots, n)$, v_i $(i = 1, \ldots, m)$ and u_r $(r = 1, \ldots, s)$ be the input and output weights for the n DMUs, the output-oriented type of model that attempts to maximize output while using no more than the observed amount of any input. Banker et al. [4] (BCC), extended the earlier work of Charnes et al. [5] by providing for variable returns to scale (VRS). The BCC input oriented (BCC-I) model evaluates the efficiency of DMUo, DMU under consideration, by solving the following linear program:

$$\begin{aligned}
&\max \sum_{r=1}^{s} u_r y_{ro} - u_o \\
&\text{s.t.} \\
&\qquad \sum_{i=1}^{m} v_i x_{io} = 1 \\
&\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} - u_o \leqslant 0 \quad j = 1, \ldots, n \\
&\qquad u_o \quad \text{free} \\
&\qquad u_r \geqslant \varepsilon \quad r = 1, \ldots, s \\
&\qquad w_i \geqslant \varepsilon \quad i = 1, \ldots, m
\end{aligned} \tag{1}$$

According to Banker et al. [4], the best output-oriented BCC-efficiency (BCC-Oo) of each DMU can be measured by the following BCC model, which was named by the acronym of the three authors:

$$
\begin{array}{lll}
\theta^* = \min & \sum_{i=1}^{m} v_i x_{io} - v_o & \\
& \text{s.t.} & \\
& \sum_{r=1}^{s} u_r y_{ro} = 1 & \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + v_o \leqslant 0 & j = 1, \ldots, n \\
& w_o & \text{free} \\
& u_r \geqslant \varepsilon & r = 1, \ldots, s \\
& w_i \geqslant \varepsilon & i = 1, \ldots, m
\end{array}
\tag{2}
$$

where DMUo refers to the DMU under evaluation, and v_i $(i = 1, \ldots, m)$ and u_r $(r = 1, \ldots, s)$ are decision variables also *varepsilon* is the positive non-Archimedean value. In order to determine the output-oriented BCC-efficiency of all DMUs, BCC-O_o and BCC-I models must be solved for each DMU, respectively. As a result, the optimal weights will vary from one DMU to another and more than one DMU will be evaluated as DEA efficient. How to distinguish between these DEA efficient DMUs has been a hot research topic and attracted considerable research interest in the DEA literature.

The following mix integer linear model, which was firstly proposed by Foroughi [8] that can select the most CCR-efficient DMU and produced a full ranking for the DMUs on the same basis:

$$
\begin{array}{lll}
d^* = \max & d & \\
& \text{s.t.} & \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} - t_j \leqslant 0 & j = 1, \ldots, n \\
& -\sum_{r=1}^{s} u_r y_{rj} + \sum_{i=1}^{m} w_i x_{ij} + t_j \leqslant 1 & j = 1, \ldots, n \\
& \sum_{i=1}^{m} w_i x_{ij} \leqslant 1, & j = 1, \ldots, n \\
& \sum_{j=1}^{n} t_j = 1 & \\
& t_j \in \{0, 1\} & j = 1, \ldots, n \\
& u_r \in U & \\
& w_i \in W &
\end{array}
\tag{3}
$$

Where W, U are assumed as the set of all acceptable multipliers which the simple sets of these are as follows: $W(\varepsilon) = \{\{w_i\} | w_i \geqslant \varepsilon,\ i = 1, \ldots, m\}$ & $U(\varepsilon) = \{\{u_r\} | u_r \geqslant \varepsilon,\ r = 1, \ldots, s\}$. Where $\varepsilon \in [0, \varepsilon^*]$. It have been improved that ε^* is always positive and the maximum non-Archimedean epsilon, ε^*, is obtained from the following LP:

$$
\begin{aligned}
\varepsilon^* = \max \quad & \varepsilon \\
& \text{s.t.} \\
& \sum_{i=1}^{m} w_i x_{io} \leqslant 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} w_i x_{ij} - u_o \leqslant 0 \; j = 1, \ldots, n \\
& u_o \qquad\qquad\qquad\qquad\qquad\quad \text{free} \\
& u_r - \varepsilon \geqslant 0 \qquad\qquad\qquad\quad r = 1, \ldots, s \\
& w_i - \varepsilon \geqslant 0 \qquad\qquad\qquad\quad i = 1, \ldots, m
\end{aligned} \tag{4}
$$

Nevertheless Foroughi [8] proposed model is feasible even if Amin, Toloo [2] model is infeasible and can be selecting the best DMU from the set of DMU that DM preferred, it is based on constant return to scale.

3. Proposed Model

The model proposed by Foroughi [8] is based on CCR model and is not appropriate for situations in which DMUs operating in variable return to scale. In this paper we propose two new MIP-DEA models which are useful for these situations. The input-oriented model proposes as:

$$
\begin{aligned}
d^* = \max \quad & d \\
& \text{s.t.} \\
& \sum_{r=1}^{s} u_r y_{rj} - u_o - \sum_{i=1}^{m} w_i x_{ij} + d \leqslant 0 \qquad j = 1, \ldots, n \\
& -\sum_{r=1}^{s} u_r y_{rj} + u_o + \sum_{i=1}^{m} w_i x_{ij} + t_j \leqslant 1 \; j = 1, \ldots, n \\
& \sum_{j=1}^{n} t_j = 1 \\
& u_o \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{free} \\
& t_j \in \{0, 1\} \qquad\qquad\qquad\qquad\qquad\; j = 1, \ldots, n \\
& \{u_r\} \in U \\
& \{w_i\} \in W
\end{aligned} \tag{5}
$$

Similarly, we can also construct an output-oriented MIP model for finding the most BCC-efficient DMU under VRS, which can be formulated as:

$$
\begin{aligned}
d^* = \max \quad & d \\
\text{s.t.} \quad & \\
& \sum_{r=1}^{s} u_r y_{rj} + v_o - \sum_{i=1}^{m} w_i x_{ij} - t_j + d \leqslant 0 && j = 1, \ldots, n \\
& -\sum_{r=1}^{s} u_r y_{rj} - v_o + \sum_{i=1}^{m} w_i x_{ij} + t_j \leqslant 1 && j = 1, \ldots, n \\
& \sum_{r=1}^{s} u_r y_{rj} = v_o \geqslant 0 && j = 1, \ldots, n \\
& \sum_{j=1}^{n} t_j = 1 \\
& v_o && \text{free} \\
& t_j \in \{0, 1\} && j = 1, \ldots, n \\
& \{u_r\} \in U \\
& \{w_i\} \in W
\end{aligned}
\tag{6}
$$

where W and U are the set of all acceptable weights. It is also assumed that $W(\varepsilon) = \{\{w_i\} | w_i \geqslant \varepsilon, i = 1, \ldots, m\}$ and $U(\varepsilon) = \{\{u_r\} | u_r \geqslant \varepsilon, r = 1, \ldots, s\}$ where $\varepsilon \in [0, \varepsilon^*]$.

The main idea of proposed Model is trying to find only one most efficient DMU, but in situations in which return to scale is variable and maximizing input (output) whereas consuming no more than the observed amount of any output (input). Wherein added free variable are v_o and u_o, enhance the capability of model for acting in variable return to scale and the $\sum_{r=1}^{s} u_r y_{rj} + v_o \geqslant 0$, $j = 1, \ldots, n$ constraints are imposed to ensure that the total output for each DMU is always nonnegative since negative outputs make no sense at all. Therefore DMU$_j$ is most BCC-efficient if and only if $t_j = 1$.

The following theorems prove the validity and some properties of the proposed model.

THEOREM 3.1 *Model (5) is always feasible.*

Proof *Suppose that DMU$_p$ is documented as strong BCC-efficient units by Model (1) and (w_p^*, u_p^*, u_o^*) be the optimal solution corresponding it and $u^* y_p = 1$ (it can be easily proved that such index exists, ties are broken arbitrary).Let*

$$\bar{d} = 1,\ \bar{\mathbf{u}} = u_p^*,\ \bar{\mathbf{w}} = w_p^*,\ u_o = u_o^*$$

$$\bar{t}_j = \begin{cases} 1, & j = p \\ 0, & j \neq p \end{cases}$$

Clearly $(\bar{d}, \bar{\mathbf{w}}, \bar{\mathbf{u}}, u_o, \bar{\mathbf{t}})$ is a feasible solution of Model (5). ■

THEOREM 3.2 *Model (6) is always feasible.*

Proof *The proof is similar to that of Theorem 1. Let DMU$_p$ is recognized as strong Bcc-efficient units by Model (2) and (w_p^*, u_p^*, v_o^*) be the optimal solution corresponding it and $u^* y_p = 1$ (it can be easily proved that such index exists, ties are broken* *arbitrary).Let*

$$\bar{d} = 1,\ \bar{\mathbf{u}} = u_p^*,\ \bar{\mathbf{w}} = w_p^*,\ v_o = v_o^*$$

$$\bar{t}_j = \begin{cases} 1, & j = p \\ 0, & j \neq p \end{cases}$$

Clearly $(\bar{d}, \bar{\mathbf{w}}, \bar{\mathbf{u}}, v_o, \bar{\mathbf{t}})$ *is a feasible solution of Model (6).* ■

THEOREM 3.3 $d^* \in [0, 1]$.

Proof

Assume that, in optimal solution of Model (5) or (6) corresponding to one j*, say* $j = p$*, we have* $t^*_{j \neq p} = 0$ *and* $t^*_p = 1$ *hence from the first type of constraints for* $j = p$, $\mathbf{u}^* y_p - u_o - \mathbf{w}^* x_p - 1 + d^* \leqslant 0$ $(\mathbf{u}^* y_p + v_o - \mathbf{w}^* x_p - 1 + d^* \leqslant 0)$ *on the other hand from second type of constraint*

$$-\mathbf{u}^* y_p + u_o + \mathbf{w}^* x_p \leqslant 0 \Rightarrow \mathbf{u}^* y_p - u_o + \mathbf{w}^* x_p \geqslant 0$$

$$(-\mathbf{u}^* y_p - v_o + \mathbf{w}^* x_p \leqslant 0 \Rightarrow \mathbf{u}^* y_p + v_o + \mathbf{w}^* x_p \geqslant 0)$$

Consequently, it is shown $d^* \leqslant 1$.

In addition, hence is a feasible point of Model (5) or (6), we have $d^* \geqslant 0$. *Therefore it can be concluded* $d^* \in [0, 1]$.

■

Indeed, the new proposed BCC-MILP model is an extension of Foroughi (2011) for VRS situation in output-oriented and input-oriented view. The output (input)-oriented linear programming is constructed to determine a DMU's potential output (input) given its inputs (outputs) if it operated efficiently as DMUs along the best practice frontier .In next section, the usefulness of this model is shown by a real data set.

4. Illustrative Examples

In this section we illustrate the performance of the proposed approach. We first use the data set that is taken by Shang & Sueyoshi [13]. It consists of Twelve flexible manufacturing systems (FMSs) are evaluated in terms of two inputs: annual operating and depreciation cost measured in units of $100,000 as input 1, and floor space requirements of each specific system measured in thousands of square feet as input 2; and four outputs: improvements in qualitative benefits, work in process (WIP), average number of tardy jobs, and average yield. The data with the BCC efficiency scores are recorded in Table 1. We can see that ten DMUs are efficient, and two DMUs, DMU_8 and DMU_12, inefficient.

To find the most efficient FMS are applied Model (4) for the data of Table 1 which the optimal value is: $\varepsilon^* = 0.006113$. There also exists a polynomial time algorithm, Epsilon algorithm, which introduced by Amin and Toloo [2]. Applying this algorithm resulted in same value as received from solving Model (5). Using this value, DMU_4 is identified as most BCC-efficient DMU by the both proposed

Table 1.

DMUs	Data of FMSs						BCC-Efficiency	BCC-Efficiency
	Inputs		Outputs				Input-oriented	Output-oriented
	Input1	Input2	Output1	Output2	Output3	Output4		
1	17.02	5	42	45.3	14.2	30.1	1.00	1.00
2	16.46	4.5	39	40.1	13	29.8	1.00	1.00
3	11.76	6	26	39.6	13.8	24.5	1.00	1.00
4	10.52	4	22	36	11.3	25	1.00	1.00
5	9.5	3.8	21	34.2	12	20.4	1.00	1.00
6	4.79	5.4	10	20.1	5	16.5	1.00	1.00
7	6.21	6.2	14	26.5	7	19.7	1.00	1.00
8	11.12	6	25	35.9	9	24.7	0.99	1.00
9	3.67	8	4	17.4	0.1	18.1	1.00	1.00
10	8.93	7	16	34.3	6.5	20.6	1.00	1.00
11	17.74	7.1	43	45.6	14	31.1	1.00	1.00
12	14.85	6.2	27	38.7	13.8	25.4	0.89	1.00

mix integer output and input-oriented models. Table 2 records the results of the approach proposed in the present paper and Foroughi's Model which should be noted that the optimal solution of Model (3) is indicated DMU_5 is most CCR-efficient unit.

Table 2.

Variables	Model (5)	Model (6)	Model(7)
v_1^*	0.2198	0.22	0.0485
v_2^*	0.1234	0.123	0.0197
u_1^*	0.0060	0.006	0.0061
u_2^*	0.0060	0.006	0.0061
u_3^*	0.0465	0.047	0.0061
u_4^*	0.1517	0.152	0.0061
u_o^*	1.861	—–	——
v_o^*	—–	-1.861	——
t_j^*	$t^*_{j=4}=1, t^*_{j\neq 4}=0$	$t^*_{j=4}=1, t^*_{j\neq 4}=0$	$t^*_{j=5}=1, t^*_{j\neq 5}=0$

5. Conclusions

Ranking of DMUs is a very significant topic in DEA research. Many methods, each with its own strategy or logic in ranking the DMUs have been proposed; some studies have been develop to generate common weights for fully ranking DMUs and identifying the unique DMU as the best efficient unit ranking efficient DMUs and selecting the best DMU. In this paper, we presented alternative mix integer models for selecting BCC-efficient DMU. Finding the most BCC-efficient DMU with the proposed method is performed based on common set of weight. It was shown that the proposed model is always feasible. Moreover, the model has finite optimal solution and can determine the most BCC-efficient DMU by solving only one problem, which shows the advantages of the model in comparing with the traditional DEA ranking model such as super efficiency and cross efficiency which solves n problems and may not have finite optimal solution in some cases.

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