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Application of the Singular Boundary Value Problem for Investigation of Piston Dynamics Under Polytropic Expansion Process

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Abstract. In this paper a mathematical simulation of a simplified internal combustion engine is presented. To contribute engine kinematics and its geometry, simple relations are derived for constrained motions. The equation of motion for the piston forms a singular boundary value problem. The uniqueness of the solution was studied in the Banach space. For solving governing equations an iterative numerical algorithm was used and the numerical method has shown very fast convergency. With this simulation the thermodynamics of expansion process is coupled with piston dynamics. Simulating an engine working at constant torque and polytropic expansions has shown high frequency piston vibration under certain conditions.

Keywords: Banach Space, Internal Combustion Engine, Polytropic Expansion, Singular Boundary Value Problem.

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1. Introduction

Modeling of an internal combustion engine encounters with important mechanical sciences where various processes of thermodynamics, heat transfer, fluid mechanics and dynamics interact on each other. In this view modeling of all of processes simultaneously, becomes too complicated [1]. In this paper a simple mathematical model is presented to simulate the effects of combustion force on the piston movement in a four stroke internal combustion engine. It is assumed that the engine

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works at a constant torque while the crank-shaft angular speed is time-dependent [5]. Finding the piston positions from T.D.C to B.D.C forms a non-homogeneous boundary value problem [3]. It is assumed that the combustion of fuel in the cylinder chamber of the engine is nearly a polytropic expansion process [9], [7]. The main goal is to investigate the equation of motion of piston under polytropic expansion of combustion gases. From the kinetics of rigid body motion, the weight of crank-shaft and connecting rod has influence on the piston movement. To omit the dynamic effects of the any mechanical linkage, they are assumed to be weightless [8]. In this way the piston dynamics is studied behind the kinematics of other parts of the internal combustion engine. The output angular velocity of crank-shaft is found from a simple kinematical study of the engine.

Nomenclature

A_p Piston area (m^2)
$B.D.C$ Bottom dead center
F Force (N)
G Green's function
K Arbitrary constant for the boundary value problem
L_1 Connecting-rod length (m)
L_2 Crank-shaft length (m)
m Exponential index for defining M_{fuel}
$\bar{M}$ Time-averaged crank-shaft torque ($N.m$)
M_{fuel} Total mass of fuel in the cylinder (kg)
M_p Piston mass
n Exponential index of polytropic process
p Static pressure (pa)
s Catalyzer variable (s)
t Time (s)
$T.D.C$ Top dead center
T Time of piston movement from T.D.C to B.D.C
V Gas-filled cylinder volume (m^3)
x Piston distance from B.D.C (m)
X Piston position from T.D.C (m)
X_A Piston stroke (m)
X_0 Distance of cylinder-head to T.D.C (m)

Greek symbols

α Mass-force proportionality coefficient
β Mass-volume proportionality coefficient
θ Crank-shaft angular position (rad)
ω Angular velocity of crank-shaft (rad/s)
$\bar{\omega}$ Time-averaged angular velocity of crank-shaft (rad/s)
ξ Fixed point of a contraction transformation
ψ Total mass of fuel in the cylinder (kg)

2. Thermodynamic Process and the Gas Pressure Force

The pressure force exerted from expanding gases on the piston is

$$F_g = PA_p \tag{1}$$

It is assumed that the pressure to be proportional to unburned mass of fuel:

$$F_g(t) \propto M_{fuel}(t) = \psi(X(t)) \tag{2}$$

Differentiation of Equation (2) gives the rate of change of the gas pressure:

$$\dot{P} = \dot{F}_g/A_p = \alpha\, \dot{X}\, \psi'/A_p \tag{3}$$

The polytropic expansion in the combustion stroke is written as:

$$pV^n = cte \tag{4}$$

Differentiation of Equation (4) gives the pressure force in the polytropic expansion:

$$F_g = -\dot{p}V/n\dot{X} \tag{5}$$

Replacing $\dot{p}$ and $V = A_p(X + X_0)$ in Equation (5) gives the pressure force as a function of piston position:

$$F_g = -\alpha(X(t) + X_0)\psi'/n \tag{6}$$

In the present simulation, it is assumed that the amount of unburned fuel inside the cylinder to be maximum at T.D.C and minimum at B.D.C. for this purpose the following relation is employed:

$$M_{fuel}(t) = \psi(X(t)) \propto V^{-m} \tag{7}$$

Applying a constant, β, converts the proportionality in Equation(7) to equality:

$$\psi(X(t)) = \beta(A_p(X + X_0))^{-m} \tag{8}$$

Differentiating of Equation (8) with respect to $X(t)$ leads to:

$$\psi' = -m\beta(A_p(X + X_0))^{-(m+1)} \tag{9}$$

Replacing Equation(9) into Equation (6) gives the pressure force as a function of piston position:

$$F_g = (m/n)\ \alpha\beta(A_p(X + X_0))^{-m} \tag{10}$$

Equation (10) makes the boundary value problem to be singular.

3. Applying Kinematics and Virtual Work Principle

Using the principle of virtual work the vertical force exerted on the (weightless) connecting rod is related to the torque of the (weightless) crank-shaft [6]:

$$F_c(t) = \bar{M}\,\omega(t)/\dot{X}(t) \tag{11}$$

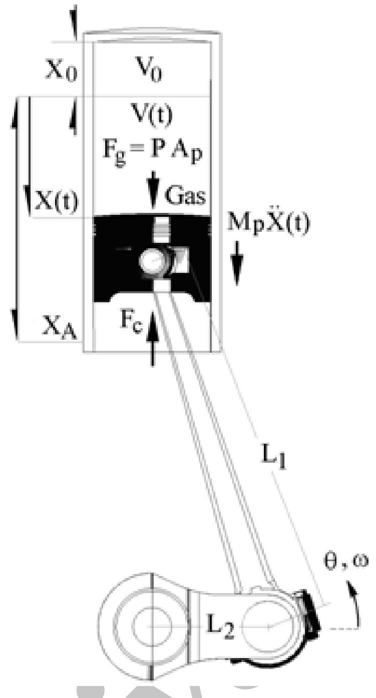

Figure 1. Engine specifications and model parameters.

The mean torque of the crank-shaft, $\bar{M}$, is the time-averaged value of torque that transforms from expanding gas into crank-shaft.

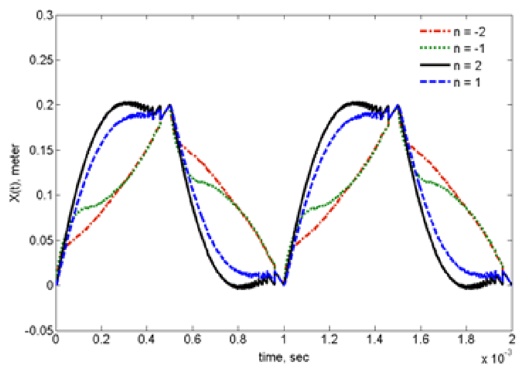

Figure 2. Effects of the polytropic expansion process with exponent, n, on the piston position when engine works at constant torque.

$$\bar{M} = \frac{\bar{\omega}}{2\pi} \int_0^{\frac{2\pi}{\bar{\omega}}} \frac{\dot{X}(t)}{\omega\,(t)}\, F_g(t)\, dt \tag{12}$$

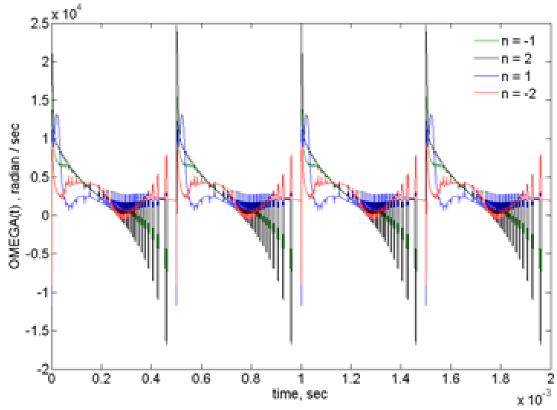

Figure 3. Effects of the polytropic expansion process with exponent, n, on the crank-shaft angular velocity when engine works at constant torque.

Considering the engine's assembly shown in Figure 1, the constrained motion of piston is given by:

$$X(t) = L_1 + L_2 - \left(\sqrt{L_1^2 - L_2^2 \sin^2\theta'} + L_2 \cos\theta'\right) \tag{13}$$

In Equation (13), $\theta' = \theta - \pi/2$. Differentiating of Equation (13) with respect to time and rearrangement gives:

$$\omega(t) = \frac{\dot{X}(t)\Big/ L_2 \sin\theta'}{\left(1 + L_2 \cos\theta' \Big/ \sqrt{L_1^2 - L_2^2 \sin^2\theta'}\right)} \tag{14}$$

Eliminating θ' using the relation between $X(t)$and $\theta'(t)$, gives $\omega(t)$:

$$\omega(t) = \frac{\dot{X}(t)(x^2 + (L_1^2 - L_2^2)^2)}{x\sqrt{2x^2(L_1^2 + L_2^2) - (x^4 + (L_1^2 - L_2^2)^2)}} \tag{15}$$

In which $x(t) = L_1 + L_2 - X(t)$ with a shift of $L_1 - L_2$ from B.D.C gives the reversed position of the piston. The angular position of crank-shaft is:

$$\theta'(t) = \cos^{-1}\frac{x^2 - (L_1^2 - L_2^2)}{2x\, L_2} \tag{16}$$

4. Dynamic Equation for Piston Movement

The equation of motion for piston in a constrained one degree of freedom motion is found easily from the CityplaceNewton's second law:

$$\sum F_{external\,,p} = M_p \ddot{X}(t) \tag{17}$$

Adding and subtracting the term$K^2X(t)$ to both side of Equation (17) doesn't change the solution results. Thus the dynamic equation of motion for piston is written as:

$$\ddot{X}(t) + K^2X(t) = K^2X(t) + \frac{\sum F_{external\,,p}}{M_p} \tag{18}$$

5. Solution of the Boundary Value Problem

The differential equation of Equation (18) with the boundary conditions $X(0) = 0$ and $X(T) = X_A$ represents a singular boundary value problem [2]. The unique solution of this boundary value problem can be obtained in terms of the Green's function [4]. Using the method of variation of parameters, the Green's function becomes:

$$G(t,s) = \begin{cases} \frac{\left(1-\cos\left(K(t-\frac{\pi}{\bar{\omega}})\right)\right)\sin(K\,s)}{-2\,K\,\sin(\frac{K}{2}(\frac{\pi}{\bar{\omega}}+s))\,\sin(\frac{K}{2}(\frac{\pi}{\bar{\omega}}-s))} & 0 \le t \le s \le \frac{\pi}{\bar{\omega}} \\ \frac{\left(1-\cos\left(K(s-\frac{\pi}{\bar{\omega}})\right)\right)\sin(K\,t)}{-2\,K\,\sin(\frac{K}{2}(\frac{\pi}{\bar{\omega}}+s))\,\sin(\frac{K}{2}(\frac{\pi}{\bar{\omega}}-s))} & 0 \le s \le t \le \frac{\pi}{\bar{\omega}} \end{cases} \tag{19}$$

Solution of the boundary value problem is a simple integral:

$$\begin{aligned} X(t) = & \int_0^t \frac{\left(1-\cos\left(K(t-\frac{\pi}{\bar{\omega}})\right)\right)\sin(K\,s)}{-2\,K\,\sin(\frac{K}{2}(\frac{\pi}{\bar{\omega}}+s))\,\sin(\frac{K}{2}(\frac{\pi}{\bar{\omega}}-s))} f(s)\;ds \\ & \int_t^{\frac{\pi}{\bar{\omega}}} \frac{\left(1-\cos\left(K(s-\frac{\pi}{\bar{\omega}})\right)\right)\sin(K\,t)}{-2\,K\,\sin(\frac{K}{2}(\frac{\pi}{\bar{\omega}}+s))\,\sin(\frac{K}{2}(\frac{\pi}{\bar{\omega}}-s))} f(s)\;ds \\ & + X_A\left(\sin(K\,t)/\sin(\frac{K\pi}{\bar{\omega}})\right) \end{aligned} \tag{20}$$

In Equation (20) the function $f(s)$ is:

$$f(s) = K^2X(s) + \frac{F_g(s) - F_c(s)}{M_p} \tag{21}$$

6. Uniqueness of the Solution

The bounds of $X(t)$ that give a unique solution for Equation (20) is determined with definition of a contraction transformation in the Banach space. The fixed point of a contraction transformation, ξ, is defined by:

$$T(\xi) = \xi \tag{22}$$

The transformation T, is defined by replacing $X(t)$ with $T(X)$ in the left hand side of Equation (20). The unique solution of Equation (20) is the fixed point of $T(X)$. The transformation, T, transforms each points of the Banach space to the Banach space. It is easily shown that, the condition for T, to be a contraction transformation, is that the lower bound of X be a non-negative number. The symmetric and reciprocal motions of piston introduce an upper bound equal to piston stroke for the piston position. For the given boundary value problem the upper and lower bounds to reach a unique solution are as follows:

$$X_{Lower} < 0 \leq X(t) \leq X_A < X_A + |X_{Lower}| \tag{23}$$

7. The numerical Method

The uniqueness of the solution of the boundary value problem, represented by Equation (18), is mathematically proved. An iterative numerical method in MATLAB is used to solve the boundary value problem behind other governing equations. In the presented numerical algorithm, Equations (10-18) are solved together to obtain unknown variables (Equations (13), (14) and (17) are auxiliary). The initial guess for $\omega(t)$ is $\bar{\omega}$ and a simple sinusoidal motion is employed for initial guess of piston velocity:

$$\dot{X}(t) = 2L_2\, \bar{\omega} \sin(\bar{\omega}\, t) \tag{24}$$

The initial guess for other variables is zero. The numerical solution results are independent of all initial
guesses. The piston velocity in each step of the iterative solution is calculated from a simple finite difference formula while equal time intervals are employed.

8. Results and Discussion

The simulation parameters and engine's specifications are selected as follows: The piston mass is 0.1 kg. The piston area is 0.2 m^2. Connecting rod arm length is 0.15 m and crank-shaft arm length is 0.1 m. the piston stroke is twice the crank-shaft arm length. Distance from cylinder head to T.D.C is one-tenth of piston stroke. In the present simulation, α is in the range of 10 to 1000 while$\beta$is in the range of 0.001 to 0.1, so the product of them varies in the range of 0.01 to 100. The polytropic exponent n varies in the range of -2 to 2. Time discretization was done for 200 equally spaced intervals. The numerical solution converges very fast, with less than 15 iterations, while satisfying a reasonable convergency criterion.

8.1 *Effects of the Polytropic Expansion Process*

It is assumed that the engine works at a constant torque. With this assumption the power index n of the polytropic expansion process has strong affects on the piston movement and the crank-shaft angular speed oscillates in time with high frequency. Figure 2 illustrates that positive and negative value of n exhibit different behaviors for the piston movement. According to the results presented in Figures 3, increasing the absolute value of the exponent index n, make the crank-shaft angular velocity oscillates at excessive large amplitudes. In these conditions, referring to Figure 4, oscillations in crank-shaft angular position, are well bounded. The crank-shaft rotates just $\pi/2$ radians in one expansion stroke.

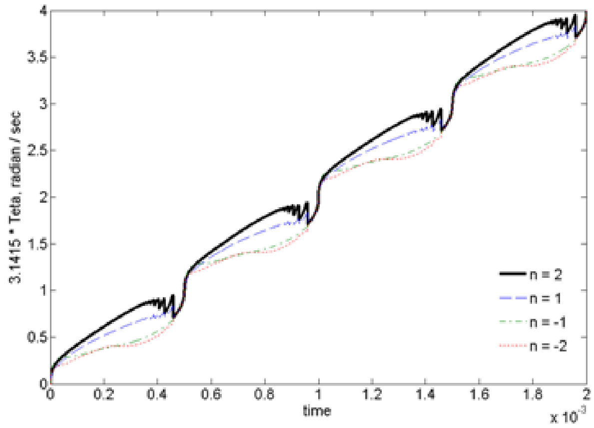

Figure 4. Effects of the polytropic expansion process with exponent, n, on the crank-shaft angular position when engine works at constant torque.

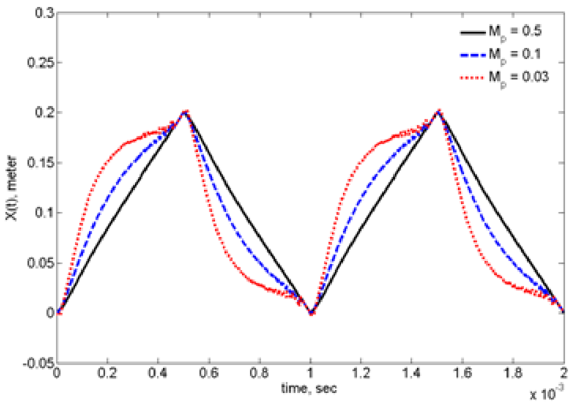

Figure 5. Effects of variations of piston mass on the piston position.

8.2 *Effects of Piston Mass*

Increasing the piston mass generates more uniform velocity for the piston, as shown in Figure 5, and smoother angular velocity of crank-shaft, that is shown with at thick curve in Figure 6.

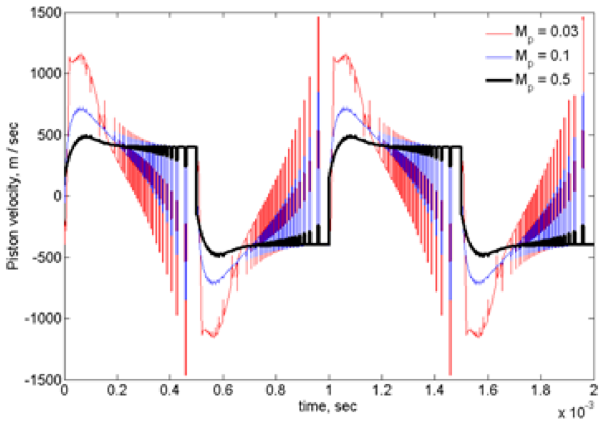

Figure 6. Effects of variations of piston mass on the piston velocity.

9. Conclusions

In this paper the effects of polytropic expansion in an internal combustion engine was studied with a simple mathematical model based on the solution of a boundary value problem. The conditions for a unique solution is determined with application of a contraction transformation in the Banach space: If the lower bound of the piston position be a non-negative number, then the condition for having a unique solution is satisfied. The (polytropic) expansion process can influence on piston vibrations when the engine works at a constant torque. Increasing the piston mass reduces velocity fluctuations of the piston and enforce an oscillating angular velocity to the crank-shaft. Solution of the presented boundary value problem is independent of the constant K. The iterative solution converges very fast and this is the main advantage of the presented mathematical model.

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