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Application of the Sinc Approximation to the Solution of Bratu's Problem

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Abstract. In this work, we study the performance of the sinc-Collocation method for solving Bratu's problem. For different choices of step size, we consider the maximum absolute errors in the solutions at sinc grid points and tabulated in tables. The comparison of the obtained results verified that this method converges to the exact solution rapidly and with $O(\exp(-c\sqrt{n}))$ accuracy where c is independent of n.

Keywords: Sinc function; Sinc-Collocation; Sinc-Galerkin; Boundary value problems; Liouville-Bratu-Gelfand problem .

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1. Introduction

The classical Liouville-Bratu-Gelfand problem is concerned with positive solutions of the equation:

$$\begin{cases} \Delta u(x) + \lambda e^{u(x)} = 0, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where parameter λ denotes the reaction term, x is a spatial variable, Ω and $\partial\Omega$ denote the spatial domain and its boundary, respectively. Equation (1) contains the exponent term e^u and thus has very strong nonlinearity. It comes from the theory of combustion, and is used as a model for the thermal reaction process such as that when a combustible medium is placed in a vessel whose walls are maintained at a fixed temperature [1–3, 6, 13, 14]. Also the Liouville-Bratu-Gelfand equation (1)

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represents the steady state of diffusion and transfer of heat conduction [10]. There exists steady-state heat transfer for small λ. As λ is greater than a critical value, the reaction will lead to explosion so that equation (1) has no solutions. We state the critical λ as λ_c. Equation (1) has one or two solutions when $\lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where the critical value λ_c satisfies the equation $1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh(\frac{\theta}{4})$. It was evaluated in [3] that the critical value λ_c is given by $\lambda = 3.513830719$.
It is hard to solve the Liouville-Bratu-Gelfand problem in a general domain Ω. The classical Gelfand problem is with radially symmetric domain, i.e. $u = u(r)$. In this case, one has

$$\begin{cases} u''(x) + \frac{N-1}{r}u'(x) + \lambda e^{u(x)} = 0, \quad r \in (0,1], \quad N = 1,2,3,\ldots \\ u(0) = u(1) = 0. \end{cases}$$

where $N = 1, 2,$ and 3 correspond to the infinite slab, infinite circular cylinder, and sphere, respectively. When $N = 1$, problem (1) is equivalent to the problem:

$$\begin{cases} u''(x) + \lambda e^{u(x)} = 0, \quad 0 \leqslant x \leqslant 1, \\ u(0) = u(1) = 0. \end{cases} \tag{2}$$

where λ is a physical parameter. The Bratu's problem in one-dimensional planar coordinates (2) has analytical solution in the following form:

$$u(x) = -2\log[\frac{\cosh[0.5(x-0.5)\theta]}{\cosh(\frac{\theta}{4})})]$$

where θ is the solution of $\theta = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$.
Many authors trying to solve problem (2) by analytical and numerical methods, B-spline method has been used in [4], the finite difference methods have been used by [3, 14], Adomian decomposition methods have been proposed by [5, 11, 20], Laplace transform decomposition method has been applied by [19] and homotopy analysis method has been developed in [12], weighted residual method in [1], differential transformation method in [7]and also variational methods in [8, 9], multigrid-based methods in [15] and variational iteration method(VIM)has been used in [2] and recently in our work [16] we applied sinc-Galerkin method.
In this paper, we applied sinc-Collocation method for numerical solution of Bratu's problem (2). The sinc-Collocation methods were developed and analyzed thoroughly by Professor Stenger in [17, 18]. The method converges to the exact solution rapidly and with $O(\exp(-c\sqrt{n}))$ accuracy where c is independent of n. The main advantage of the sinc methods is that it is particularly suited for solving all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous coefficients; for such problems the resulting system of algebraic equations can always be written down explicitly, and is relatively small in size. Furthermore, the computer programs based on sinc methods are usually considerably shorter than the corresponding ones based on classical methods of approximation.

2. Function Preliminaries

The goal of this section is to recall notations, definitions and theorems of the sinc function, state some known results, and derive useful formulas that are important for this paper. These are thoroughly discussed by Professor Stenger in [18].

The sinc function is defined on the whole real line by

$$S(k,h)(x) = sinc(x) = \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0. \end{cases} \tag{3}$$

For $h > 0$, the translated sinc function with evenly spaced nodes are given as

$$S(k,h)(x) = sinc\left(\frac{x-kh}{h}\right), \quad k = 0, \pm 1, \pm 2, \dots \tag{4}$$

If f is defined on the real line, then for $h > 0$ the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) sinc(\frac{x-hk}{h}), \tag{5}$$

Is called the Whittaker cardinal expansion of f whenever this series converges. To construct approximations on the interval $(0,1)$, which are used in this paper, consider the conformal maps

$$\phi(z) = \ln(\frac{z}{1-z}). \tag{6}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{ z = x + iy : \left| \arg(\frac{z}{1-z}) \right| < d \leqslant \frac{\pi}{2} \right\},$$

onto the infinite strip

$$D_d = \left\{ \varsigma = \xi + i\eta : |\eta| < d \leqslant \frac{\pi}{2} \right\}.$$

The composition

$$S_j(x) = S(j,h) \circ \phi(x) = sinc\left(\frac{\phi(x) - jh}{h}\right), \tag{7}$$

define the basis element for problem (2) on the interval $(0,1)$. The "mesh size" h is the mesh size in D_d for the uniform grids $\{kh\}, -\infty < k < \infty$. The sinc points $z_k \in (0,1)$ in D_E will be denoted by x_k because they are real. The inverse image of uniform grids are

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}}, \tag{8}$$

where

$$h = \sqrt{\frac{\pi d}{\alpha M}}, \quad 0 \leqslant \alpha \leqslant 1 \tag{9}$$

and

$$N \equiv \left[\left| \frac{\alpha}{\beta} M + 1 \right|\right] \tag{10}$$

where $[x]$ is the integer part of x.

Lemma 2.1 *([18]) Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (6). Then*

$$\delta_{jk}^{(0)} = [S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 1, \ \mathit{if}\ j=k, \\ 0, \ \mathit{if}\ j\neq k, \end{cases} \tag{11}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 0, & \mathit{if}\ j=k, \\ \frac{(-1)^{k-j}}{k-j}, & \mathit{if}\ j\neq k, \end{cases} \tag{12}$$

$$\delta_{jk}^{(2)} = h^2\frac{d}{d\phi}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & \mathit{if}\ j=k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & \mathit{if}\ j\neq k. \end{cases} \tag{13}$$

3. The sinc-collocation method

The interpolation formula on the interval $(0,1)$ takes the form

$$f(x) \approx \sum_{j=-M}^{N} f_j \ S(j,h)\circ\phi(x). \tag{14}$$

The n-th derivative of the function f at points x_j can be approximated using a finite number of terms as

$$f^{(n)}(x) \approx \sum_{j=-M}^{N} f_j \frac{d^n}{dx^n}[S(j,h)\circ\phi(x)]. \tag{15}$$

Setting

$$\frac{d^i}{d\phi^i}[S(k,h)\circ\phi(x)] = S_k^{(i)}(x), \quad 0\leqslant i\leqslant 2,$$

and noting that

$$\frac{d}{dx}[S(k,h)\circ\phi(x)] = S_k^{(1)}(x)\phi'(x),$$

$$\frac{d^2}{dx^2}[S(k,h)\circ\phi(x)] = S_k^{(2)}(x)[\phi'(x)]^2 + S_k^{(1)}(x)[\phi''(x)],$$

and

$$\delta_{kj}^{(n)} = h^n\frac{d^n}{d\phi^n}[S(k,h)\circ\phi(x)]_{x=x_j}.$$

We assume that $u(x)$, the solution of (2) is approximated by the finite expansion of sinc basis functions

$$u_m(x) = \sum_{k=-M}^{N} c_k \; S(k,h) \circ \phi(x), \quad m = M + N + 1. \tag{16}$$

If we replace each terms of (2) with its corresponding approximation given by the right-hand side of (15) and (14) we have

$$\sum_{k=-M}^{N} \frac{d^2}{dx^2}[S(k,h) \circ \phi(x)]c_k + \lambda e^{\sum_{k=-M}^{N} c_k [S(k,h)\circ\phi(x)]} = 0. \tag{17}$$

By substituting $x = x_j = \phi^{-1}(jh)$ in (17) and applying the Collocation to it, the discrete sinc-Collocation system for the determination of the unknown coefficients is given by

$$\sum_{k=-M}^{N} [\frac{1}{h^2}\delta_{jk}^{(2)}[\phi'(x_j)]^2 - \frac{1}{h}\delta_{jk}^{(1)}\phi''(x_j)]c_k + \lambda e^{c_j} = 0, \quad j = -M, \ldots, N. \tag{18}$$

To obtain a matrix representation of the equations in (18), denote the $m \times m(m = M + N + 1)$ Toeplitz matrices

$$I^{(i)} \equiv [\delta_{jk}^{(i)}], \quad i = 0, 1, 2$$

whose jk-th entry is given by (11), (12), and (13).

Let $D(g(x_j))$ denote the $m \times m$ diagonal matrix with

$$D(g(x))_{ij} = \begin{cases} g(x_i), \text{ if } i = j, \\ \;\; 0, \quad \text{if } j \neq k. \end{cases} \tag{19}$$

Let $C1$ be the m-vector with k-th component given by c_k and let $C2$ be the m-vector with j-th component given by e^{c_j}. Then the non-linear system in (18) takes the matrix form

$$\Lambda C1 + \lambda C2 = 0, \tag{20}$$

where

$$\Lambda = \frac{1}{h^2} I^{(2)} D((\phi')^2) - \frac{1}{h} I^{(1)} D(\phi'').$$

Now we have a non-linear system of m equations of the m unknown coefficients. The system (20) may be easily solved by a variety of methods in the approximate sinc-Collocation solution $u_m(x)$ of $u(x)$.

4. Numerical results

In order to illustrate the performance of the sinc-Collocation method in solving Bratu's problem (2) and justify the accuracy and efficiency of the method, we consider problem (2) with various λ.

Table 1. Observed maximum absolute errors.

M	$\lambda=1$	$\lambda=2$	$\lambda=3.51$	$\lambda=-2$	$\lambda=-\pi^2$
10	3.02×10^{-4}	8.60×10^{-4}	3.43×10^{-2}	4.20×10^{-4}	1.52×10^{-3}
25	6.46×10^{-6}	1.84×10^{-5}	9.83×10^{-4}	8.90×10^{-6}	3.21×10^{-5}
50	7.30×10^{-8}	2.07×10^{-7}	1.12×10^{-5}	9.99×10^{-8}	3.61×10^{-7}
75	2.24×10^{-9}	6.40×10^{-9}	3.45×10^{-7}	3.10×10^{-9}	1.11×10^{-8}
100	1.18×10^{-10}	3.36×10^{-10}	1.81×10^{-8}	1.8×10^{-10}	5.8×10^{-10}
130	5.36×10^{-12}	1.52×10^{-11}	8.23×10^{-10}	7.3×10^{-12}	2.6×10^{-11}

Table 2. Numerical errors for $\lambda = 1$.

x	Our method	Method in[16]	Decomposition in[12]
0.1	2.77×10^{-4}	2.76×10^{-4}	2.68×10^{-3}
0.2	2.88×10^{-4}	2.92×10^{-4}	1.52×10^{-4}
0.4	3.01×10^{-4}	3.02×10^{-4}	2.20×10^{-3}
0.5	3.03×10^{-4}	3.08×10^{-4}	3.01×10^{-3}
0.6	3.01×10^{-4}	3.02×10^{-4}	2.20×10^{-3}
0.7	2.98×10^{-4}	2.96×10^{-4}	1.52×10^{-3}
0.8	2.88×10^{-4}	2.92×10^{-4}	2.02×10^{-3}
0.9	2.77×10^{-4}	2.76×10^{-4}	2.68×10^{-3}

Table 3. Numerical errors for $\lambda = 2$.

x	Our method	Method in[16]	Decomposition in[12]	Laplas in[13]
0.1	6.88×10^{-4}	6.85×10^{-4}	1.52×10^{-2}	2.13×10^{-3}
0.2	7.58×10^{-4}	7.73×10^{-4}	1.47×10^{-2}	4.21×10^{-3}
0.3	8.21×10^{-4}	8.18×10^{-4}	5.89×10^{-3}	6.19×10^{-3}
0.4	8.49×10^{-4}	8.54×10^{-4}	3.25×10^{-3}	8.00×10^{-3}
0.5	8.60×10^{-4}	8.77×10^{-4}	6.98×10^{-3}	9.60×10^{-3}
0.6	8.49×10^{-4}	8.54×10^{-4}	3.25×10^{-3}	1.09×10^{-3}
0.7	8.21×10^{-4}	8.18×10^{-4}	5.89×10^{-3}	1.19×10^{-3}
0.8	7.58×10^{-4}	7.73×10^{-4}	1.47×10^{-2}	1.24×10^{-3}
0.9	6.88×10^{-4}	6.85×10^{-4}	1.52×10^{-2}	1.09×10^{-3}

All experiments were performed in Mathematica 7.0.
In all tables, the maximum absolute errors over the set of sinc grid points

$$S=\{x_{-M},x_{-M+1},\ldots,x_N\};$$

$$x_k=\frac{e^{kh}}{e^{kh}+1},\quad ,k=-M,\ldots,N$$

is taken as

$$\parallel E_C(h)\parallel=\max_{-M\leqslant k\leqslant N}\left|u_{\text{exact solution}}(x_k)-u_{m,\text{sinc-Collocation}}(x_k)\right|,$$

In our presented method, we take $d=\frac{\pi}{2}$, $\alpha=\beta=1$ and by using (9) we can obtain h.
In Table 1, we applied our method for $N = 10, 25, 50, 75, 100$ and 130 with various $\lambda = 1, 2, 3.51, -2$ and $\lambda = -\pi^2$. The maximum absolute errors in solutions of problem (2) are compared with sinc-Galerkin, Decomposition, Laplas and B-spline methods for $N = 10$ and tabulated in tables 2, 3, 4, 5 and 6 which show that our results have more accuracy.

Table 4. Numerical errors for $\lambda = 3.51$.

x	Our method	Method in[16]	B-spline in[4]
0.1	1.13×10^{-2}	1.14×10^{-2}	3.84×10^{-2}
0.2	2.00×10^{-2}	2.04×10^{-2}	7.48×10^{-2}
0.3	2.76×10^{-2}	2.80×10^{-2}	1.06×10^{-1}
0.4	3.25×10^{-2}	3.31×10^{-2}	1.27×10^{-1}
0.5	3.43×10^{-2}	3.50×10^{-2}	1.35×10^{-1}
0.6	3.25×10^{-2}	3.31×10^{-2}	1.27×10^{-1}
0.7	2.76×10^{-2}	2.80×10^{-2}	1.06×10^{-1}
0.8	2.00×10^{-2}	2.04×10^{-2}	7.48×10^{-2}
0.9	1.13×10^{-2}	1.14×10^{-2}	3.84×10^{-2}

Table 5. Numerical errors for $\lambda = -2$.

x	Our method	Method in[16]
0.1	3.88×10^{-4}	3.86×10^{-4}
0.2	3.66×10^{-4}	3.71×10^{-4}
0.3	3.53×10^{-4}	3.51×10^{-4}
0.4	3.44×10^{-4}	3.45×10^{-4}
0.5	3.41×10^{-4}	3.46×10^{-4}
0.6	3.44×10^{-4}	3.45×10^{-4}
0.7	3.53×10^{-4}	3.51×10^{-4}
0.8	3.66×10^{-4}	3.71×10^{-4}
0.9	3.88×10^{-4}	3.86×10^{-4}

Table 6. Numerical errors for $\lambda = -\pi^2$.

x	Our method	Method in[16]
0.1	1.22×10^{-3}	1.22×10^{-3}
0.2	1.04×10^{-3}	1.05×10^{-3}
0.3	9.27×10^{-4}	9.22×10^{-4}
0.4	8.61×10^{-4}	8.62×10^{-4}
0.5	8.41×10^{-4}	8.50×10^{-4}
0.6	8.61×10^{-4}	8.62×10^{-4}
0.7	9.27×10^{-4}	9.22×10^{-4}
0.8	1.04×10^{-3}	1.05×10^{-3}
0.9	1.22×10^{-3}	1.22×10^{-3}

5. Conclusion

The sinc-collocation method has been considered for the numerical solution of Bratu's problem which is a second-order nonlinear boundary-value problem. The numerical results verified that the present method is an applicable technique and approximates the solution very well. The new approach converges to the exact solution rapidly and with $O(\exp(-c\sqrt{n}))$ accuracy.

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