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Solving Fractional Nonlinear Schrödinger Equations by Fractional Complex Transform Method

B. Ghazanfari [a,*] and A. G. Ghazanfari [a]

[a] *Department of Mathematics, Faculty of Science, Lorestan University, P.C. : 68137-17133, Khorramabad, Iran.*

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Abstract. In this paper, we apply fractional complex transform to convert the fractional nonlinear Schrödinger equations to the nonlinear Schrödinger equations.

Keywords: Fractional complex transform, Schrödinger equation, Jumaries derivative

Index to information contained in this paper

1. Introduction

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering. Many important phenomena in electromagnetism, acoustics, viscoelasticity, electrochemistry and material science are well described by differential equations of fractional order [8, 9]. To find the explicit solutions of linear and nonlinear fractional differential equations, many powerful methods have been used such as the variational iteration method [2, 10], homotopy perturbation method [1], and the Exp-function method [12]. The fractional complex transform was first proposed by He and Li [3]. We extend the fractional complex transform method to solve the fractional nonlinear Schrödinger equations. The fractional nonlinear Schrödinger equation

$$i\frac{\partial^\alpha \Psi(X,t)}{\partial t^\alpha} = -\frac{1}{2}\nabla^{2\beta}\Psi + \Gamma(X) + \nu|\Psi|^2\Psi, \quad X \in R^n, t > 0 \tag{1}$$

with initial condition

$$\Psi(X,0) = \Psi_0(X), \tag{2}$$

* Corresponding author. Email: bahman_ ghazanfari@yahoo.com

where Ψ is unknown function,$\Gamma(X)$ is known, ν is a real constant and $0 < \alpha, \beta \leqslant 1$ are parameters describing the order of the fractional Jumaries derivative [6, 7]. Nonlinear Schrödinger equation is one of the canonical nonlinear equations in physics, arising in various field such as nonlinear optics, plasma physics, and surface waves.

2. Fractional complex transform

Jumaries derivative [6, 7] is a modified Riemann-Liouville derivative defined as

$$D_z^\gamma f(z) = \begin{cases} \frac{1}{\Gamma(-\gamma)}\frac{d}{dz}\int_0^z (z-\tau)^{-\gamma-1}(f(\tau)-f(0))d\tau, & \gamma < 0, \\ \frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\int_0^z (z-\tau)^{-\gamma}(f(\tau)-f(0))d\tau, & 0 < \gamma < 1, \\ (f^{(\gamma-n)}(z))^{(n)}, & n \leqslant \gamma < n+1, \quad n \geqslant 1, \end{cases} \tag{3}$$

where $f(z)$ is a real continuous (but not necessarily differentiable) function.The fundamental mathematical operations and results of Jumaries derivative are given in [6, 7]. In this section, we review some of them.

$$\begin{array}{ll} D_z^\gamma c = 0, & \gamma > 0, c = constant, \\ D_z^\gamma (cf(z)) = cD_z^\gamma f(z), & \gamma > 0, c = constant, \\ D_z^\gamma z^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\gamma)} z^{\beta-\gamma}, & \beta > \gamma > 0, \\ D_z^\gamma (f(z)g(z)) = (D_z^\gamma f(z))g(z) + f(z)(D_z^\gamma g(z)), & \\ D_z^\gamma (f(z(t))) = f_z'(z)\cdot z^{(\gamma)}(t) = f_z^{(\gamma)}(z)(z_t')^\gamma. & \end{array}$$

3. Examples

The fractional complex transform [3, 4][5] can convert a fractional differential equation into its differential partner.

Example 1. Consider the fractional nonlinear Schrödinger equation

$$i\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha} = -\frac{1}{2}\frac{\partial^{2\beta}\Psi(x,t)}{\partial x^{2\beta}} - |\Psi|^2\Psi, \quad x \in \mathbb{R}, t > 0. \tag{4}$$

with initial condition

$$\Psi(x,0) = e^{ix^\beta/\Gamma(1+\beta)}, \tag{5}$$

By the fractional complex transform

$$T = p\frac{t^\alpha}{\Gamma(1+\alpha)}, \quad X = q\frac{x^\beta}{\Gamma(1+\beta)}, \tag{6}$$

where p and q are constants which are unknown to be further determined. Using Jumariers chain rule [6, 7], we have

$$\begin{aligned} \frac{\partial^\alpha \Psi}{\partial t^\alpha} &= \frac{\partial \Psi}{\partial T}\frac{\partial^\alpha T}{\partial t^\alpha} = p\frac{\Psi}{\partial T}, \\ \frac{\partial^{2\beta}\Psi}{\partial x^{2\beta}} &= \frac{\partial^2\Psi}{\partial X^2}\left(\frac{\partial^\beta X}{\partial x^\beta}\right)^2 = q^2\frac{\partial^2\Psi}{\partial X^2}. \end{aligned} \tag{7}$$

By setting $p = 1$ and $q = 1$, we have

$$i\frac{\partial \Psi}{\partial T} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial X^2} - |\Psi|^2\Psi, \quad X \in \mathbb{R}, T > 0, \tag{8}$$

with initial condition

$$\Psi(X, 0) = e^{iX}, \tag{9}$$

The exact solution is given in [1] as follows:

$$\Psi(X, T) = \cos(X + T/2) + i\sin(X + T/2) = e^{i(X+T/2).} \tag{10}$$

Hence,

$$\Psi(x,t) = \cos(\tfrac{x^\beta}{\Gamma(1+\beta)} + \tfrac{t^\alpha}{2\Gamma(1+\alpha)}) + i\sin(\tfrac{x^\beta}{\Gamma(1+\beta)} + \tfrac{t^\alpha}{2\Gamma(1+\alpha)})$$
$$= e^{i(\frac{x^\beta}{\Gamma(1+\beta)} + \frac{t^\alpha}{2\Gamma(1+\alpha)})}.$$

Example 2. Consider the fractional nonlinear Schrödinger equation

$$i\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha} = -\frac{1}{2}\frac{\partial^{2\beta}\Psi(x,t)}{\partial x^{2\beta}} + \Psi\cos^2(x^\beta/\Gamma(1+\beta)) + |\Psi|^2\Psi, \quad x \in \mathbb{R}, t > 0. \tag{11}$$

with initial condition

$$\Psi(x, 0) = \sin(x^\beta/\Gamma(1+\beta)). \tag{12}$$

By the fractional complex transform

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad X = \frac{x^\beta}{\Gamma(1+\beta)}, \tag{13}$$

We find

$$i\frac{\partial \Psi}{\partial T} = -\frac{1}{2}\frac{\partial^2 \Psi}{\partial X^2} + \Psi\cos^2(X) + |\Psi|^2\Psi, \tag{14}$$

with initial condition

$$\Psi(X, 0) = \sin(X). \tag{15}$$

The exact solution is given in [1] as follows:

$$\Psi(X, T) = \sin(X)e^{-3Ti/2}. \tag{16}$$

Hence,

$$\Psi(x,t) = \sin(\frac{x^\beta}{\Gamma(1+\beta)})e^{\frac{-3it^\alpha}{2\Gamma(1+\alpha)}}.$$

Example 3. Consider the two dimensional fractional Schrödinger equation

$$\begin{aligned} & i\frac{\partial^{\alpha}\Psi(x,y,t)}{\partial t^{\alpha}} = -\frac{1}{2}\left(\frac{\partial^{2\beta}\Psi}{\partial x^{2\beta}} + \frac{\partial^{2\beta}\Psi}{\partial y^{2\beta}}\right) \\ & +\Psi(1-\sin^2(x^{\beta}/\Gamma(1+\beta))\sin^2(y^{\beta}/(1+\beta))) \\ & +|\Psi|^2\Psi, \quad x,y\in\mathbb{R}, t>0. \end{aligned} \tag{17}$$

with initial condition

$$\Psi(x,y,0) = \sin(x^{\beta}/\Gamma(1+\beta))\sin(y^{\beta}/(1+\beta)). \tag{18}$$

By the fractional complex transform

$$T = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad X = Y = \frac{x^{\beta}}{\Gamma(1+\beta)}. \tag{19}$$

We find

$$\begin{aligned} & i\frac{\partial\Psi(X,Y,T)}{\partial T} = -\frac{1}{2}\left(\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial^2\Psi}{\partial^2 Y^2}\right) \\ & +\Psi(1-\sin^2(X)\sin^2(Y)) \\ & +|\Psi|^2\Psi, \quad x,y\in\mathbb{R}, t>0. \end{aligned} \tag{20}$$

with initial condition

$$\Psi(X,Y,0) = \sin(X)\sin(Y). \tag{21}$$

The exact solution is given in [11] as follows:

$$\Psi(X,Y,T) = \sin(X)\sin(Y)e^{-2iT}. \tag{22}$$

Hence,

$$\Psi(x,y,t) = \sin\left(\frac{x^{\beta}}{\Gamma(1+\beta)}\right)\sin\left(\frac{y^{\beta}}{\Gamma(1+\beta)}\right)e^{\frac{-2it^{\alpha}}{\Gamma(1+\alpha)}}. \tag{23}$$

4. Conclusion

The fractional complex transform is very simple and use of this method does not need the knowledge of fractional calculus.

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