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Hierarchical Computation of Hermite Spherical Interpolant

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Abstract. In this paper, we propose to extend the hierarchical bivariate Hermite Interpolant to the spherical case. Let T be an arbitrary spherical triangle of the unit sphere S and let u be a function defined over the triangle T. For $k \in \mathbb{N}$, we consider a Hermite spherical Interpolant problem H_k defined by some data scheme $\mathcal{D}_k(u)$ and which admits a unique solution p_k in the space $B_{n_k}(T)$ of homogeneous Bernstein-Bézier polynomials of degree $n_k = 2k$ (resp. $n_k = 2k+1$) defined on T. We discuss the case when the data scheme $\mathcal{D}_r(u)$ are nested, i.e., $\mathcal{D}_{r-1}(u) \subset \mathcal{D}_r(u)$ for all $1 \leqslant r \leqslant k$. This, give a recursive formulae to compute the polynomial p_k. Moreover, this decomposition give a new basis for the space $B_{n_k}(T)$, which are the hierarchical structure. The method is illustrated by a simple numerical example.

Keywords: Spherical splines, Hermite interpolation, Recursive computation.

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1. Introduction

As is known, methods for building the classical univariate or bivariate Hermite spline interpolants needs the Hermite fundamental functions. In the absence of a recursive formula to calculate these basic functions, the calculation of the Hermite interpolant become difficult and complicated. To avoid this complexity Mazroui et al (see [5] and [6]) have proposed a simple, practical and useful method for calculating the Hermite interpolant recursively. More precisely, the Hermite interpolant p_k can be decomposed in the form $p_k = p_0 + q_1 + \ldots + q_k$, where, p_0 is the polynomial interpolating the set $\mathcal{D}_0(u)$ and q_r, $1 \leqslant r \leqslant k$, are particular splines.

In practice, since this decomposition make the calculation of Hermite interpolant p_k simple it can be used in the following applications, computing integrals, smoothing curves and compressing data. For more details see [5] and [6].

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Since this method is simple it is natural to extend it to several variables. One obvious way to do this is to use the tensor product. With regard to this extension, it was described in [9] (see also [7, 10]) a recursive construction for tensor product Hermite interpolants. In [8], it was proposed a method allowing to build recursively bivariate Hermite spline interpolants of class $\mathcal{C}^k$ on $\mathbb{R}^2$. Recently, we are proposed in [3] a recursive method for the construction of a Hermite spherical spline interpolant of class $\mathcal{C}^k$ and degree $4k+1$ on S. In this work, we deal with a new method allowing us to build recursively Hermite spherical interpolants on spherical triangles T. In this case the degree $n_k = 2k$ (resp. $n_k = 2k+1$) and the data scheme $\mathcal{D}_k(u)$ constricting of values and derivatives of u at vertices V_i and points interior B_j of triangle T. But in [3], $n_k = 4k+1$ and in addition to these data, we add values and derivatives of u at center points arcs $M_{i,j}$.

Let T be a spherical triangle with vertices V_1, V_2 and V_3, and for convenience, let $V_4 = V_1$ and $V_5 = V_2$.

To define some useful derivatives associated with T, let $g_{i,j}$ be a tangent vector to S at v_i contained in the plane passing through v_i, v_j, and the origin, not parallel with v_i, $i, j = 1, 2, 3$, $i \neq j$, and for convenience, $g_{i,0} = g_{i,3}$ and $g_{i,4} = g_{i,1}$. In addition, let μ_j, ν_j be independent unit vectors lying in the tangent plane of S at $B_j \in T$, where μ_j, ν_j are two non parallel directions.

Let u be a regular function defined on T. For each $k \in \mathbb{N}$, there exists a unique homogenous Bernstein-Bézier polynomial p_k in the space $\mathbb{B}_{n_k}(T)$ that interpolates a given data set $\mathcal{D}_k(u)$. In general, $\mathcal{D}_k(u)$ is formed by the values and the derivatives of u at the vertices V_i, $1 \leqslant i \leqslant 3$, and at other points B_j, $1 \leqslant j \leqslant d_k$, inside T and/or on the edges of T. More specifically, the set $\mathcal{D}_k(u)$ can be written in the form

$$\mathcal{D}_k(u) = \{D^\alpha u(V_i), D^{\gamma_j} u(B_j);\ |\alpha| \leqslant \rho_k,\ \gamma_j \in I_k,\ 1 \leqslant i \leqslant 3\}, \tag{1}$$

where $I_k = \{\gamma_j \in \mathbb{N} \times \mathbb{N},\ n_{j,k} \leqslant |\gamma_j| \leqslant N_{i,j},\ 1 \leqslant j \leqslant d_k,\ n_{j,k}, N_{i,j} \in \mathbb{N}\}$. The quantities $D^\alpha u(V_i)$, $\alpha = (\alpha^1, \alpha^2) \in \mathbb{N}\times\mathbb{N}$ and $|\alpha| = \alpha^1 + \alpha^2$, (resp. $D^{\gamma_j} u(B_j)$, $\gamma_j = (\gamma_j^1, \gamma_j^2) \in \mathbb{N} \times \mathbb{N}$) denote the directional derivatives of u at V_i (resp. B_j) obtained by differentiating u α^1 (resp. γ_j^1) times in the directions $g_{i,i+1}$ (resp. μ_j) and α^2 (resp. γ_j^2) times in the directions $g_{i,i-1}$ (resp. ν_j).

Let H_k be the Hermite interpolation problem in $B_{n_k}(T)$, corresponding to the data scheme $\mathcal{D}_k(u)$. Our aim is to establish a recursive formula that allows us compute step by step the polynomial p_k, solution of the problem H_k. This computation will be possible if some conditions are satisfied. Indeed, assume that the sets $\mathcal{D}_r(u)$, $0 \leqslant r \leqslant k$, are nested, i.e.,

$$\mathcal{D}_0(u) \subset \mathcal{D}_1(u) \subset \ldots \subset \mathcal{D}_{k-1}(u) \subset \mathcal{D}_k(u). \tag{2}$$

It is clear that (2) is equivalent to $n_{r-1} \leqslant n_r$, $\rho_{r-1} \leqslant \rho_r$ and $I_{r-1} \subset I_r$ for $1 \leqslant r \leqslant k$. Therefore, the polynomial p_k can be written in the form $p_k = p_0 + q_1 + \ldots + q_k$, where each q_j is a homogenous Bernstein-Bézier polynomial of degree $\leqslant n_j$ that can be determined by the data set $\mathcal{D}_j(u - p_{j-1})$. The multirsolution structure of this decomposition means that p_0 may be considered as a coarse approximation of p_k, and q_j are correction terms or detail polynomials. Moreover, this representation of p_k gives rise to a new basis for the space $B_{n_k}(T)$. We show that this basis is constituted by the last Hermite basis functions of each space $B_{n_r}(T)$, $r = 1, \ldots, k$ and it is useful in practice.

As the bivariate case, we encounter serval different Hermite interpolation problems which have unique solutions and such that their corresponding data schemes satisfy (2). As application we deal in this works with those defined by the following two data schemes

$$\mathcal{D}_k(u) = \{D^\alpha u(V_i),\ D^\gamma u(B);\ |\alpha| \leqslant k,\ |\gamma| \leqslant k-1,\ i = 1,2,3\}, \tag{3}$$

$$\mathcal{D}_k(u) = \{D^\alpha u(V_i),\ D^\gamma u(B);\ |\alpha| \leqslant k-1,\ |\gamma| \leqslant k,\ i = 1,2,3\}, \tag{4}$$

where B is an arbitrary point inside T. It is well known that there exists a unique polynomial of degree $n_k = 2k+1$ (resp. $n_k = 2k$) that interpolates $\mathcal{D}_k(u)$ given in (3) (resp. in (4)).

The paper is organized as follows. In Section 2, we give some preliminary results on homogeneous Bernstein-Bézier polynomials. Section 3 is devoted to the main results of this paper, namely, we first establish the hierarchical computation of Hermite polynomials $p_k \in B_{n_k}(T)$ when corresponding data schemes $D_k(u)$ are nested. Then, for an arbitrary data schemes, we deduce a new basis for $B_{n_k}(T)$. As an application of the above results, we describe in Section 4 the explicit decomposition of Hermite polynomial of odd or even degree that interpolate the data schemes given on (3) or (4). Finally, in Section 5 we give a numerical example.

2. Preliminary results

In this section, we present the connection between the functions defined on S and homogeneous trivariate functions, and we introduce some definitions.

A trivariate function F is said to be positively homogeneous of degree $t \in \mathbb{R}$ provided that for every real number $a > 0$,

$$F(av) = a^t F(v),\ v \in \mathbb{R}^3 \backslash \{0\}.$$

Lemma 2.1 *(see Alfeld et al. [1]) Given a function f defined on S, and let $t \in \mathbb{R}$. Then*

$$F_t(v) = \|v\|^t f\left(\frac{v}{\|v\|}\right)$$

is the unique homogenous extension of f of degree t to all of $\mathbb{R}^3\backslash\{0\}$, i.e., $F_t|_S = f$, and F_t is homogenous of degree t.

Let g be a given unit vector. Then, as in [1], we define the directional derivative D_g of f at a point $v \in S$ by

$$D_g f(v) = D_g F(v) = g^T \nabla F(v),$$

where F is some homogenous extension of f, and ∇F is the gradient of the trivariate function F.
While a polynomial of degree d has a natural homogenous extension to $\mathbb{R}^3$, a general function f on S has infinitely many different extensions. The value of its derivative may depend on which extension that we take (for more detail see [1]).

Let $\mathcal{P}_d$ be the space of trivariate polynomials of total degree at most d, and let $\mathcal{H}_d = \mathcal{P}_d|_S$ be its restriction to the sphere S. A trivariate polynomial p is called homogeneous of degree d if $p(\lambda x,\ \lambda y,\ \lambda z) = \lambda^d p(x,y,z)$ for all $\lambda \in \mathbb{R}$, and harmonic if $\Delta p = 0$, where Δ is the Laplace operator defined by $\Delta f = (D_x^2 + D_y^2 + D_z^2) f$.

Definition 2.2 (see [2]) *The linear space*

$$\mathcal{H}_d = \{p|_S \ : \ p \in \mathcal{P}_d \text{ and } p \text{ is homogeneous of degree } d \text{ and harmonic}\}$$

is called the space of spherical harmonics of exact degree d.

Let be given a spherical triangle T. The associated spherical Bernstein basis functions of degree d are defined by

$$B^d_{ijk}(v) = \frac{d!}{i!j!k!} b_1^i(v) b_2^j(v) b_3^k(v), \quad i + j + k = d,$$

where $b_1(v)$, $b_2(v)$, $b_3(v)$ are spherical barycentric coordinates of v relative to T. These $\binom{d+2}{2}$ functions are linearly independent [2], and form a basis for the space denoted, in what follows, by B_d. Each $p \in B_d$ is called a spherical Bernstein-Bézier (SBB) polynomial. It is clear that p can be written in the form $p = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}$ and it is uniquely determined by its B-coefficients c_{ijk}.

It is well known (see [2]) that B^d_{ijk} are actually linear combinations of spherical harmonics.

Proposition 2.3 (see [2]) *For all $d \geqslant 1$, we have*

$$B_d = \begin{cases} \mathcal{H}_0 \oplus \mathcal{H}_2 \ldots \oplus \mathcal{H}_{2k} & \textit{if} \quad d = 2k, \\ \mathcal{H}_1 \oplus \mathcal{H}_3 \ldots \oplus \mathcal{H}_{2k+1} & \textit{if} \quad d = 2k+1. \end{cases}$$

From the above proposition, it is simple to see that

$$B_{d-1} \not\subset B_d \quad \text{but} \quad B_{d-2} \subset B_d.$$

For the Hermite data scheme $\mathcal{D}_k(u)$ given in (1), we denote by H_k the corresponding interpolation problem. Then H_k can be formulated as follows

$$H_k \begin{cases} \text{Find } p_k \in B_{n_k}(T) \text{ such that} \\ D^\alpha p_k(V_i) = D^\alpha u(V_i), \ |\alpha| \leqslant \rho_k \ \text{ and } \ i = 1,2,3, \\ D^{\gamma_j} p_k(B_j) = D^{\gamma_j} u(B_j), \ n_{j,k} \leqslant |\gamma_j| \leqslant N_{j,k} \ \text{ and } \ 1 \leqslant j \leqslant d_k. \end{cases}$$

Definition 2.4 *We say that $\mathcal{D}_k(u)$ is a $B_{n_k}(T)$-unisolvent data scheme if the problem H_k has a unique solution $p_k \in B_{n_k}(T)$.*

In what follows, we deal with sets $\mathcal{D}_k(u)$ that are $B_{n_k}(T)$-unisolvent. Then, as $dim B_{n_k}(T) = \binom{n_k+2}{2}$, the integers ρ_k, d_k, $n_{j,k}$ and $N_{j,k}$, $1 \leqslant j \leqslant d_k$, are given so that $card(\mathcal{D}_k(u)) = \binom{n_k+2}{2}$.

Let $\mathcal{B}_k = \left\{\varphi^\alpha_{i,k}, \psi^{\gamma_j}_k, \ |\alpha| \leqslant \rho_k, \ \gamma_j \in I_k \text{ and } i = 1,2,3\right\}$ be the Hermite basis for $B_{n_k}(T)$ corresponding to the problem H_k. More precisely, $\varphi^\alpha_{i,k}$ and $\psi^{\gamma_j}_k$ are determined by the following interpolation conditions

$$\begin{cases} D^\beta \varphi^\alpha_{i,k}(V_t) = \delta_{i,t}\delta_{\alpha,\beta}, & \text{for } |\beta| \leqslant \rho_k \text{ and } 1 \leqslant i,t \leqslant 3, \\ D^{\gamma_j} \varphi^\alpha_{i,k}(B_j) = 0, & \text{for all } \gamma_j \in I_k, \\ D^\beta \psi^{\gamma_j}_k(V_t) = 0, & \text{for all } |\beta| \leqslant \rho_k \text{ and } 1 \leqslant t \leqslant 3, \\ D^{\gamma_s} \psi^{\gamma_j}_k(B_s) = \delta_{j,s}\delta_{\gamma_j,\gamma_s}, & \text{for all } \gamma_s \in I_k, \end{cases} \tag{5}$$

where δ is the Kronecker delta.

Using the basis B_k, the solution p_k of H_k can be written in the unique form

$$p_k(\lambda) = \sum_{i=1}^{3} \sum_{|\alpha|\leqslant\rho_k} D^\alpha u(V_i)\varphi_{i,k}^\alpha(\lambda) + \sum_{\gamma_j\in I_k} D^{\gamma_j} u(B_j)\psi_k^{\gamma_j}(\lambda). \tag{6}$$

3. Recursive computation of Hermite spherical interpolants

The lack of recursive formulae for computing the basis elements of $\mathcal{B}_k$ makes the use (6) rather complicated. To remedy this problem, we established a decomposition of p_k. In other works, if we assume that $\mathcal{D}_{k-1}(u) \subset \mathcal{D}_k(u)$. then by using the fact that $B_{n_{k-1}}(T) \subset B_{n_k}(T)$, we deduce that $p_k = p_{k-1} + q_k$, where p_{k-1} is the unique solution of the Hermite problem H_{k-1} and q_k is particular polynomial in $B_{n_k}(T)$.

In order to illustrate this decomposition, we need the following lemma.

Lemma 3.1 *If $I_{k-1} \subset I_k$ and $\rho_{k-1} \leqslant \rho_k$, the for $|\alpha| \leqslant \rho_{k-1}$, $\gamma_j \in I_{k-1}$ and $i = 1, 2, 3$ we have*

$$\varphi_{i,k}^\alpha = \varphi_{i,k-1}^\alpha - \overline{\varphi}_{i,k}^\alpha \text{ and } \psi_k^{\gamma_j} = \psi_{k-1}^{\gamma_j} - \overline{\psi}_k^{\gamma_j},$$

where

$$\overline{\varphi}_{i,k}^\alpha = \sum_{l=1}^{3} \sum_{|\beta|=\rho_{k-1}+1}^{\rho_k} D^\beta \varphi_{i,k-1}^\alpha(V_l)\varphi_{l,k}^\beta + \sum_{\gamma_j\in I_k\setminus I_{k-1}} D^{\gamma_j}\varphi_{i,k-1}^\alpha(B_j)\psi_k^{\gamma_j}. \tag{7}$$

$$\overline{\psi}_k^{\gamma_j} = \sum_{l=1}^{3} \sum_{|\beta|=\rho_{k-1}+1}^{\rho_k} D^\beta \psi_{k-1}^{\gamma_j}(V_l)\varphi_{l,k}^\beta + \sum_{\gamma_s\in I_k\setminus I_{k-1}} D^{\gamma_s}\psi_{k-1}^{\gamma_j}(B_s)\psi_k^{\gamma_s}. \tag{8}$$

Proof Let I_k be the Hermite interpolation operator defined for a function u by $I_k u = u_k \in B_{n_k}$. As I_k is exact on B_{n_k}, i.e., $I_k p = p$ for all $p \in B_{n_k}$, we deduce that $I_k\varphi_{i,k-1}^\alpha = \varphi_{i,k-1}^\alpha$. In other words, we have

$$\varphi_{i,k-1}^\alpha = \sum_{l=1}^{3} \sum_{|\beta|\leqslant\rho_k} D^\beta \varphi_{i,k-1}^\alpha(V_i)\varphi_{l,k}^\beta + \sum_{\gamma_j\in I_k} D^{\gamma_j}\varphi_{i,k-1}^\alpha(B_j)\psi_k^{\gamma_j}.$$

On the other hand, from (5), we deduce that for all $\alpha \leqslant \rho_{k-1}$

$$\sum_{l=1}^{3} \sum_{|\beta|\leqslant\rho_{k-1}} D^\beta \varphi_{i,k-1}^\alpha(V_i)\varphi_{l,k}^\beta = \varphi_{i,k}^\alpha \quad \text{and} \quad \sum_{\gamma_j\in I_{k-1}} D^{\gamma_j}\varphi_{i,k-1}^\alpha(B_j)\psi_k^{\gamma_j} = 0.$$

where after, we get the first equality. Using a similar technique, one can establish the other equalities. ■

Now, we give the main result of this paper.

Theorem 3.2 *Let p_k and p_{k-1} be the Hermite spherical polynomial solutions of problems H_k and H_{k-1} respectively. If $\mathcal{D}_{k-1}(u) \subset \mathcal{D}_k(u)$, then the spherical polynomial p_k can be decomposed as follows*

$$p_k = p_{k-1} + q_k, \quad \textit{where}$$

$$q_k = \sum_{i=1}^{3} \sum_{|\alpha|=\rho_{k-1}+1}^{\rho_k} D^\alpha (u - p_{k-1})(V_i)\varphi_{i,k}^\alpha + \sum_{\gamma_j \in I_k \setminus I_{k-1}} D^{\gamma_j}(u - p_{k-1})(B_j)\psi_k^{\gamma_j}.$$

Proof Recall that $\mathcal{D}_{k-1}(u) \subset \mathcal{D}_k(u)$ implies that $I_{k-1} \subset I_k$ and $\rho_{k-1} \leqslant \rho_k$. Then, the expression of the Hermite polynomial p_k given in (6) becomes

$$\begin{aligned} p_k = &\sum_{i=1}^{3} \sum_{|\alpha|\leqslant\rho_{k-1}} D^\alpha u(V_i)\varphi_{i,k}^\alpha + \sum_{\gamma_j \in I_{k-1}} D^{\gamma_j} u(B_j)\psi_k^{\gamma_j} + \\ &\sum_{i=1}^{3} \sum_{|\alpha|=\rho_{k-1}+1}^{\rho_k} D^\alpha u(V_i)\varphi_{i,k}^\alpha + \sum_{\gamma_j \in I_k \setminus I_{k-1}} D^{\gamma_j} u(B_j)\psi_k^{\gamma_j}. \end{aligned}$$

Using the expressions for $\varphi_{i,k}^\alpha, \psi_k^{\gamma_j}, \overline{\varphi}_{i,k}^\alpha$ and $\overline{\psi}_k^{\gamma_j}$, given in Lemma (3.1), we get

$$p_k = p_{k-1} + \sum_{i=1}^{3} \sum_{|\alpha|=\rho_{k-1}+1}^{\rho_k} D^\alpha (u - p_{k-1})(V_i)\varphi_{i,k}^\alpha + \sum_{\gamma_j \in I_k \setminus I_{k-1}} D^{\gamma_j}(u - p_{k-1})(B_j)\psi_k^{\gamma_j}.$$

■

Remark 1 From the above expression for q_k, we deduce that its corresponding insolvent data set is $\mathcal{D}_k(u - p_{k-1})$.

COROLLARY 3.3 *Assume that $\mathcal{D}_0(u) \subset \mathcal{D}_1(u) \subset \ldots \subset \mathcal{D}_k(u)$. Then we have the spherical polynomial p_k can be decomposed in the form*

$$p_k = p_0 + q_1 + \ldots + q_k, \tag{9}$$

where $q_s = \sum_{i=1}^{3} \sum_{|\alpha|=\rho_{s-1}+1}^{\rho_s} C_{i,s}^\alpha \varphi_{i,s}^\alpha + \sum_{\gamma_j \in I_s \setminus I_{s-1}} \widetilde{C}_s^{\gamma_j} \psi_s^{\gamma_j}$, $1 \leqslant s \leqslant k$ and p_0 is the solution of the Hermite problem H_0, and the coefficients

$$C_{i,s}^\alpha = D^\alpha (u - p_{s-1})(V_i), \qquad \widetilde{C}_s^{\gamma_j} = D^{\gamma_j}(u - p_{s-1})(B_j)$$

can be computed recursively as follows
for $s = 1$, $|\alpha| = \rho_0 + 1, \ldots, \rho_1$ and $\gamma_j \in I_1 \setminus I_0$,

$$C_{i,1}^\alpha = D^\alpha u(V_i) - D^\alpha p_0(V_i), \qquad \widetilde{C}_1^{\gamma_j} = D^{\gamma_j} u(B_j) - D^{\gamma_j} p_0(B_j),$$

and for $s \geqslant 2$, $|\alpha| = \rho_{s-1}+1, \dots, \rho_s$ *and* $\gamma_j \in I_s \backslash I_{s-1}$,

$$C_{i,s}^{\alpha} = D^{\alpha}u(V_i) - D^{\alpha}p_0(V_i) - \sum_{t=1}^{s-1}\left[\sum_{l=1}^{3}\sum_{|\beta|=\rho_{t-1}+1}^{\rho_t} C_{l,t}^{\beta} D^{\alpha}\varphi_{l,t}^{\beta}(V_i) + \sum_{\gamma_m \in I_t \backslash I_{t-1}} \widetilde{C}_t^{\gamma_m} D^{\alpha}\psi_t^{\gamma_m}(V_i)\right]$$

$$\widetilde{C}_s^{\gamma_j} = D^{\gamma_j}u(B_j) - D^{\gamma_j}p_0(B_j) - \sum_{t=1}^{s-1}\left[\sum_{l=1}^{3}\sum_{|\beta|=\rho_{t-1}+1}^{\rho_t} C_{l,t}^{\beta} D^{\gamma_j}\varphi_{l,t}^{\beta}(B_j) + \sum_{\gamma_m \in I_t \backslash I_{t-1}} \widetilde{C}_t^{\gamma_m} D^{\gamma_j}\psi_t^{\gamma_m}(B_j)\right].$$

Proof The decomposition (9) of p_k follows from Theorem 3.2. On the other hand, it is clear that for $s = 1, \dots, k$, we have

$$p_{s-1} = p_0 + \sum_{t=1}^{s-1}\left[\sum_{l=1}^{3}\sum_{|\beta|=\rho_{t-1}+1}^{\rho_t} C_{l,t}^{\beta}\varphi_{l,t}^{\beta} + \sum_{\gamma_j \in I_t \backslash I_{t-1}} \widetilde{C}_t^{\gamma_j}\psi_t^{\gamma_j}\right].$$

Then, by using the obvious equality $C_{i,s}^{\alpha} = D^{\alpha}u(V_i) - D^{\alpha}p_{s-1}(V_i)$, we deduce that

$$C_{i,s}^{\alpha} = D^{\alpha}u(V_i) - D^{\alpha}p_0(V_i) - \sum_{t=1}^{s-1}\left[\sum_{l=1}^{3}\sum_{|\beta|=\rho_{t-1}+1}^{\rho_t} C_{l,t}^{\beta} D^{\alpha}\varphi_{l,t}^{\beta}(V_i) + \sum_{\gamma_m \in I_t \backslash I_{t-1}} \widetilde{C}_t^{\gamma_m} D^{\alpha}\psi_t^{\gamma_m}(V_i)\right].$$

In same way, we can obtain the recursive formula for $\widetilde{C}_s^{\gamma_j}$. ∎

Now, if we put $\rho_{-1} = -1$ and $I_{-1} = \emptyset$, then we have the following result.

Theorem 3.4 *The family*

$$\widehat{\mathcal{B}}_k = \{\varphi_{i,s}^{\alpha}, \psi_s^{\gamma_j},\ 1 \leqslant i \leqslant 3,\ 0 \leqslant s \leqslant k,\ \rho_{s-1}+1 \leqslant |\alpha| \leqslant \rho_s \text{ and } \gamma_j \in I_s \backslash I_{s-1}\}$$

forms a basis for the space $B_{n_k}(T)$. *Moreover,* $\widehat{\mathcal{B}}_k, k \in \mathbb{N}$, *are hierarchical.*

Proof Let $p \in B_{n_k}(T)$. Since the Hermite interpolation interpolation operator I_k is exact on $B_{n_k}(T)$, we deduce that $p = I_k(p) = p_0 + q_1 + \cdots + q_k$, where p_0 is the unique solution of the Hermite problem H_0, and q_s, $1 \leqslant s \leqslant k$, are particular polynomials in $B_{n_k}(T)$ defined by

$$q_s = \sum_{i=1}^{3}\sum_{|\alpha|\leqslant\rho_0} \mu_{i,0}^{\alpha}\varphi_{i,0}^{\alpha}(\lambda) + \sum_{\gamma_j \in I_0}\sigma_0^{\gamma_j}\psi_0^{\gamma_j}(\lambda) + \sum_{s=1}^{k-1}\left[\sum_{i=1}^{3}\sum_{|\alpha|=\rho_{s-1}+1}^{\rho_s}\mu_{i,s}^{\alpha}\varphi_{i,s}^{\alpha}(\lambda) + \sum_{\gamma_j \in I_s \backslash I_{s-1}}\sigma_s^{\gamma_j}\psi_s^{\gamma_j}(\lambda)\right] = 0.$$

Using the definitions of Hermite basis functions given in (5), and starting from $s = 0$ to $s = k$ and from $|\alpha| = \rho_{s-1} + 1$ to $|\alpha| = \rho_s$, we obtain step by step $D^\alpha f(A_i) = \mu_{i,s}^\alpha = 0$ and $D^{\gamma_j} f(B_j) = \sigma_s^{\gamma_j} = 0$. Consequently, $\widehat{\mathcal{B}}_k$ is a basis for $\mathcal{B}_k$.

On the other hand, if we put $\widehat{\mathcal{B}}_0 = \mathcal{B}_0$, it is simple to check that $\widehat{\mathcal{B}}_k = \widehat{\mathcal{B}}_{k-1} \cup \overline{\widehat{\mathcal{B}}}_k$, where

$$\overline{\widehat{\mathcal{B}}}_k = \{\varphi_{i,k}^\alpha, \psi_k^{\gamma_j},\ 1 \leqslant i \leqslant 3, \quad \rho_{k-1} + 1 \leqslant |\alpha| \leqslant \rho_k \text{ and } \gamma_j \in I_k \backslash I_{k-1}\}.$$

Then we have $\widehat{\mathcal{B}}_{k-1} \subset \widehat{\mathcal{B}}_k$. ■

Remark 2

The comparaison of the two bases $\mathcal{B}_k$ and $\widehat{\mathcal{B}}_k$ of the space $B_{n_k}(T)$ leads to the following observations.

i) The hierarchical structure of the bases $\widehat{\mathcal{B}}_k$, $k \in \mathbb{N}$, can be used for several practices in numerical analysis like compressing data and surfaces.

ii) If we denote by $T_{\alpha,k}$ (resp. $\overline{T}_{\gamma_j,k}$) the number of B-coefficients of each $\varphi_{i,k}^\alpha$, $1 \leqslant i \leqslant 3$, (resp. $\psi_k^{\gamma_j}$) that are not necessarily equal to zeros, then by straightforward computation we get

$$T_{\alpha,k} = \binom{n_k+2}{2} - 2\binom{\rho_k+2}{2} - \binom{|\alpha|+1}{2} \quad \text{and} \quad \overline{T}_{\gamma_j,k} = \binom{n_k+2}{2} - 3\binom{\rho_k+2}{2}.$$

These B-coefficients are solution of linear systems of size $T_{\alpha,k}$ or $\overline{T}_{\gamma_j,k}$, that derive from Hermite interpolation problems given by (5). For the elements of $\widehat{\mathcal{B}}_k$, the number of B-coefficients is only $T_{\alpha,s}$ for $\varphi_{i,s}^\alpha$ and $\overline{T}_{\gamma_j,s}$ for $\psi_s^{\gamma_j}$, when α is such that $\rho_{s-1} + 1 \leqslant |\alpha| \leqslant \rho_s$, $\gamma_j \in I_s \backslash I_{s-1}$ and $0 \leqslant s \leqslant k$. Then the size of their corresponding systems are respectively $T_{\alpha,s}$ and $\overline{T}_{\gamma_j,s}$. However, the complexity of determining the basis $\widehat{\mathcal{B}}_k$ is far less than that of $\mathcal{B}_k$.

iii) Computation of the polynomial $p_k \in B_{n_k}(T)$ at several points: According to (ii), each basis function $\varphi_{i,k}^\alpha$ or $\psi_k^{\gamma_j}$ is determined by a large number of B-coefficients, so the computation of the polynomial p_k needs at lot of operations. As in practice this computation is required for several points T, we conclude that it is useful to use the new basis which allows us to reduce extensively the number of operations.

3.1 *Application*

In this section, we are interested in the decomposition of polynomials that arise from some unisolvent interpolation problems

Lemma 3.5 *The data set $D_k(u)$ given in (3) (resp. in (4)) uniquely determines a SBB-polynomial p_k of degree $n_k = 2k$ (resp. $n_k = 2k + 1$) solution of the problem H_k.*

Proof The proof is similar to the proof of the bivariate case (see [4]). Indeed, assume that p_k is written in its SBB-form, and the corresponding Bézier coefficients are numbered as in Figure (1). Assume that $n_k = 2k$, it is simple to verify that

$$dim\ B_{n_k}(T) = card(D_k(u)) = \binom{2k+2}{2} = (2k+1)(k+1).$$

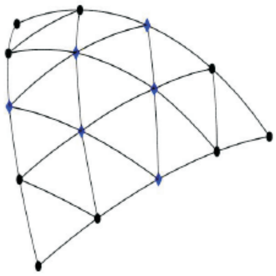


Figure 1. $k = 2,\ n_k = 4$.

Then, showing that $D_k(u)$ is a determining set for $B_{n_k}(T)$ is equivalent to show that $D_k(u)$ uniquely determines all B-coefficients of p_k. Indeed, the $\mathcal{C}^k$ smoothness at v_1 implies that the data set $\{D^\alpha u(v_1),\ |\alpha| \leqslant k-1\}$ uniquely determines the $\binom{(k-1)+2}{2} = \frac{k(k+1)}{2}$ coefficients corresponding to domain points marked with $\bullet$ closest to vertex v_1 (see Figure(1)). The situation at v_2 and v_3 is analogous. Moreover, it is easy to see that $\{D^\gamma u(B),\ |\gamma| \leqslant k\}$ uniquely determines the $\binom{k+2}{2} = \frac{(k+1)(k+2)}{2}$ coefficients corresponding to domain points marked with $\blacklozenge$ (diamond). Thus, a total of $3\frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = (2k+1)(k+1)$ coefficients are already determined, and this completes the proof. ■

Corollary 3.6 *Let $p_k \in B_{2k+1}(T)$ (resp. $\overline{p}_k \in B_{2k}(T)$) be the Hermite spherical polynomial interpolant associated to the data set (4) (resp. (3)). Then p_k and $\overline{p}_k$ can be decomposed in the form*

$$p_k = p_0 + q_1 + \ldots + q_k,$$
$$\overline{p}_k = \overline{p}_0 + \overline{q}_1 + \ldots + \overline{q}_k,$$

where p_0 is the spherical polynomial interpolating the value of u at $V_i,\ i = 1, 2, 3$ and $\overline{p}_0$ is the spherical polynomial equal to $u(B)$, while

$$q_s = \sum_{i=1}^{3} \sum_{|\alpha|=s} D^\alpha(u - p_{s-1})(V_i)\varphi_{i,s}^\alpha + \sum_{|\gamma|=s-1} D^\gamma(u - p_{s-1})(B_j)\psi_s^\gamma$$

and

$$\overline{q}_s = \sum_{i=1}^{3} \sum_{|\alpha|=s-1} D^\alpha(u - \overline{p}_{s-1})(V_i)\widetilde{\varphi}_{i,s}^\alpha + \sum_{|\gamma|=s} D^\gamma(u - \overline{p}_s)(B_j)\widetilde{\psi}_s^\gamma.$$

For each $1 \leqslant s \leqslant k$, the elements $\{\varphi_{i,s}^\alpha, \psi_s^\gamma,\ |\alpha| = s,\ |\gamma| = s-1\}$ (resp.$\{\widetilde{\varphi}_{i,s}^\alpha, \widetilde{\psi}_s^\gamma,\ |\alpha| = s-1,\ |\gamma| = s\}$) are the last Hermite basis functions for $B_{2s+1}(T)$ (resp. $B_{2s}(T)$).

Corollary 3.7 *The collection $\{\varphi_{i,s}^\alpha, \psi_s^\gamma,\ |\alpha| = s,\ |\gamma| = s-1,\ 1 \leqslant s \leqslant k$ and $1 \leqslant i \leqslant 3\}$ (resp.$\{\widetilde{\varphi}_{i,s}^\alpha, \widetilde{\psi}_s^\gamma,\ |\alpha| = s-1,\ |\gamma| = s,\ 1 \leqslant s \leqslant k$ and $1 \leqslant i \leqslant 3\}$) form a basis for $B_{2k+1}(T)$, (resp. $B_{2k}(T)$).*

Proof This result follows from Theorem 3.4 with $n_k = 2k+1$ (resp. $n_k = 2k$), taking into account that the functions $\widetilde{\varphi}_{i,0}^\alpha$ and ψ_s^γ such that $|\alpha| = |\gamma| = -1$ are omitted. ■

According to Remark *2*, we have two cas:

Case : $n_k = 2k+1$

- The explicit expression for T_{α_k} and $\overline{T}_{\gamma_k}$ are

$$T_{\alpha,k} = (k+1)^2 - \frac{|\alpha|(|\alpha|+1)}{2} \text{ and } \overline{T}_{\gamma,k} = \frac{k(k+1)}{2}.$$

- The total number of B-coefficients needed for the determination of $\mathcal{B}_{k,T}$ is given by

$$\Sigma_{2k+1} = 3\sum_{|\alpha|\leqslant k} T_{\alpha,k} + \sum_{|\gamma|\leqslant k-1} \overline{T}_{\gamma,k} = 3 + \frac{37}{4}k + \frac{89}{8}k^2 + \frac{25}{4}k^3 + \frac{11}{8}k^4,$$

- The number of B coefficients for the determination of the new basis $\widehat{\mathcal{B}}_{k,T}$ is given by

$$\sigma_{2k+1} = \sum_{s=0}^{k}\left[3\sum_{|\alpha|=s} T_{\alpha,s} + \sum_{|\gamma|=s-1} \overline{T}_{\gamma,s}\right] = 3 + \frac{47}{6}k + \frac{15}{2}k^2 + \frac{19}{6}k^3 + \frac{1}{2}k^4.$$

Case : $n_k = 2k$

- The explicit expression for T_{α_k} and $\overline{T}_{\gamma_k}$ are

$$T_{\alpha,k} = (k+1)^2 - \frac{|\alpha|(|\alpha|+1)}{2} \text{ and } \overline{T}_{\gamma,k} = \frac{(k+1)(k+2)}{2}.$$

- The total number of B-coefficients needed for the determination of $\mathcal{B}_{k,T}$ is given by

$$\Sigma_{2k} = 3\sum_{|\alpha|\leqslant k-1} T_{\alpha,k} + \sum_{|\gamma|\leqslant k} \overline{T}_{\gamma,k} = 1 + \frac{19}{4}k + \frac{65}{8}k^2 + \frac{23}{4}k^3 + \frac{11}{8}k^4,$$

- The number of B coefficients for the determination of the new basis $\widehat{\mathcal{B}}_{k,T}$ is given by

$$\sigma_{2k} = \sum_{s=0}^{k}\left[3\sum_{|\alpha|=s-1} T_{\alpha,s} + \sum_{|\gamma|=s} \overline{T}_{\gamma,s}\right] = 1 + \frac{29}{6}k + \frac{13}{2}k^2 + \frac{19}{6}k^3 + \frac{1}{2}k^4.$$

In the following table, we give Σ_{2k+1}, σ_{2k+1}, Σ_{2k} and σ_{2k}, for the first values of k.

Table 1. Σ_{2k+1}, σ_{2k+1}, Σ_{2k} and σ_{2k}, for $k = 1 \ldots 10$.

k	0	1	2	3	4	5	6	7	8	9	10
Σ_{2k+1}	3	31	138	411	970	1968	3591	6058	9621	14565	21208
σ_{2k+1}	3	22	82	220	485	938	1652	2712	4215	6270	8998
Σ_{2k}	1	21	111	355	870	1806	3346	5706	9135	13915	20361
σ_{2k}	1	16	70	200	455	896	1596	2640	4125	6160	8866

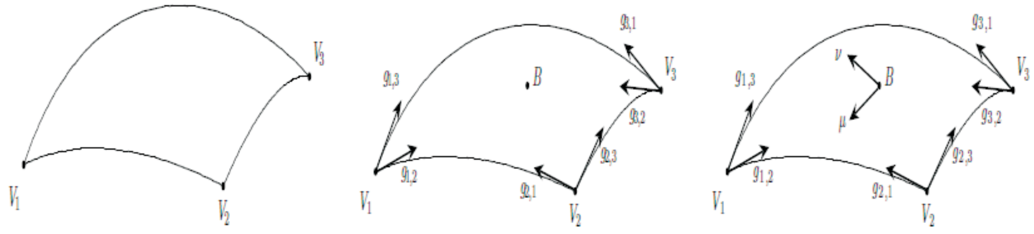


Figure 2. $\mathcal{D}_0(u) \subset \mathcal{D}_1(u) \subset \mathcal{D}_2(u)$..

In order to illustrate our results, we give in the next section some numerical examples.

4. Numerical Examples

In this section, we give an example which illustrate the theoretical results. Let T be the spherical triangle with vertices $V_1 = (0, \frac{\pi}{2})$, $V_2 = (0, \frac{\pi}{4})$, $V_3 = (\frac{3\pi}{4}, \frac{\pi}{4})$. In this example, we describe the decomposition of the Hermite interpolant p_2 to the Sphere-Like Surfaces associated at the function $f(x, y, z) = \sum_{i=1}^{3}(g_i(x, y, z))^{-\frac{1}{2}}$ where $g_i(x, y, z) = \left(\frac{x}{\alpha_i}\right)^2 + \left(\frac{y}{\alpha_{i+1}}\right)^2 + \left(\frac{z}{\alpha_{i+2}}\right)^2$ and $(\alpha_1, \ldots, \alpha_5) = (5, 1, 2, 5, 1)$.

B is the center of gravity of T(see Figure (2)). From corollary 3.7, we have $p_2 = p_0 + q_1 + q_2$. In Figure 3 we present the graphs of p_0, p_2 and the detail functions q_1 and q_2, for $n_k = 2k + 1$, in this case we have

$$\mathcal{D}_0(u) = \{u(V_i),\ 1 \leqslant i \leqslant 3\},$$

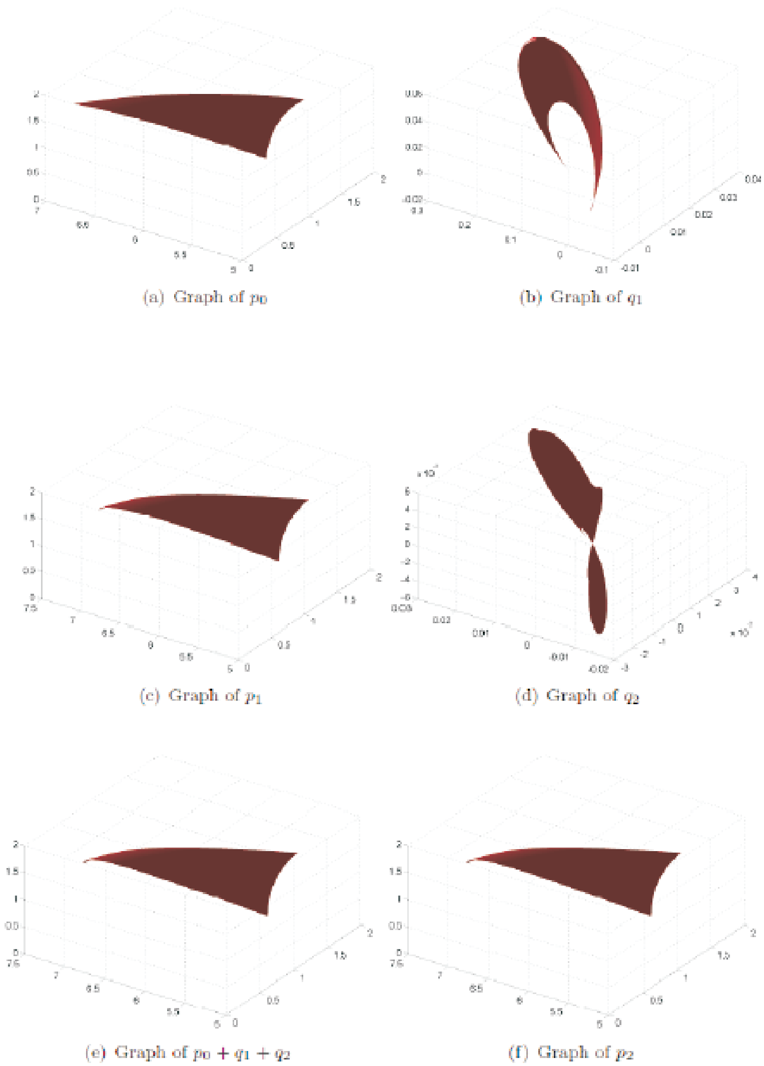
$$\mathcal{D}_1(u) = \{D^{\alpha}_{g_{i,j}} u(V_i), u(B),\ \alpha = 0, 1;\ 1 \leqslant i, j \leqslant 3;\ i \neq j\},$$

and

$$\mathcal{D}_2(u) = \{D^{\alpha}_{g_{i,j}} u(V_i), u(B), D_{\mu}u(B), D_{\nu}u(B);\ \alpha = 0, 1, 2;\ 1 \leqslant i, j \leqslant 3;\ i \neq j\}.$$

Table 2. The maximum error between f and different step of decomposition.

$\lVert f_{\lvert T} - p_0 \rVert_\infty$	$\lVert f_{\lvert T} - (p_0 + q_1) \rVert_\infty$	$\lVert f_{\lvert T} - p_1 \rVert_\infty$	$\lVert f_{\lvert T} - (p_0 + q_1 + q_2) \rVert_\infty$	$\lVert f_{\lvert T} - p_2 \rVert_\infty$
$2.5687\ 10^{-1}$	$5.0113\ 10^{-3}$	$4.9384\ 10^{-3}$	$1.5202\ 10^{-4}$	$1.4375\ 10^{-4}$



(a) Graph of p_0 (b) Graph of q_1

(c) Graph of p_1 (d) Graph of q_2

(e) Graph of $p_0 + q_1 + q_2$ (f) Graph of p_2

Figure 3. Decomposition of p_2.

Table 3. The maximum error between p_2 and different step of decomposition

$\|p_2 - p_0\|_\infty$	$\|p_2 - (p_0 + q_1)\|_\infty$	$\|p_2 - p_1\|_\infty$	$\|p_2 - (p_0 + q_1 + q_2)\|_\infty$
$2.3549\ 10^{-1}$	$2.9702\ 10^{-3}$	$2.8973\ 10^{-3}$	$1.6367\ 10^{-4}$

5. Conclusion

In this paper, we proposed to extend the hierarchical bivariate Hermite Interpolant to the spherical case. Let T be an arbitrary spherical triangle of the unit sphere S and let u be a function defined over the triangle T. For $k \in \mathbb{N}$, we considered a Hermite spherical Interpolant problem H_k defined by some data scheme $\mathcal{D}_k(u)$ and which admits a unique solution p_k in the space $B_{n_k}(T)$ of homogeneous Bernstein-Bézier polynomials of degree $n_k = 2k$ (resp. $n_k = 2k + 1$) defined on T.

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