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A Single Server Bernoulli Vacation Queue with two Type of Services and with Restricted Admissibility

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Abstract. A single server queue with Bernoulli vacation has been considered. In addition the admission to queue is based on a Bernoulli process and the server gives two type of services. For this model the probability generating function for the number of customers in the queue at different servers state are obtained using supplementary variable technique. Some performance measures are calculated. Some particular cases are obtained and numerical examples are also presented.

Keywords: Bernoulli, vacation-Bernoulli process-Supplementary, variable, technique-Performance measures.

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1. Introduction

The queueing literature exposes many useful results for vacation queues. In a queue, after serving a customer(s) the server becomes unavailable for a random period of time, called vacation period. Such queues are called vacation queues. Various modifications have been defined on the number of vacation periods and the time points at which the vacation starts or ends. One such model is Bernoulli vacation model. In Bernoulli vacation model at each service completion epoch the decision to take a vacation depends on a Bernoulli distribution. This type of vacation policy was first introduced by Keilson and Servi (1986). Subsequently Keilson and Servi (1987), Ramaswami and Servi (1988), Doshi (1986, 1990), Takagi (1991), Kalyanaraman and Renganathan(1996) and Kalyanaraman and Pazhani Bala Muguran (2008) among others have studied this and models of similar nature. In 1991, Takagi made elaborate study on vacation models in his book on Queueing Analysis

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and in 2006, Tian and Zhang extensively studied various vacation systems in their book entitled Vacation Queueing Models: Theory and Applications. Neuts (1984) considered an $M/G/1$ queue with restriction on the number of customers to be admitted during a service period or with restriction on the time period at which the customers are admitted. Madan and Dayyeh (2002) investigated a bulk queue with restricted admissibility of batches and with Bernoulli schedule server vacation. Anabosi and Madan (2003) have analyzed a single server queue with two types of service under Bernoulli schedule server vacation. In their paper, the server provides two types of heterogeneous exponential services and the customer may choose either type service and with single vacation policy. In this article, a single server infinite capacity Poisson arrival queue with two type of services, with Bernoulli vacation and with restriction on admission of arrivals has been studied. The corresponding mathematical model has been defined in section 2 and the governing differential difference equations, boundary conditions and the normalizing condition are given in section 3. For this model the probability generating functions of the number of customers in the queue when the server provides i^{th} type of service $i = 1, 2$, the probability generating function of the number of customers in the queue when the server is on vacation and the probability generating function of the number of customers in queue irrespective of the server state are derived in section 4. Also performance measures related to this queueing model are derived from these probability generating functions and are given in section 5. In section 6, some particular models are analyzed by taking specific values to the parameters. In the last section, a numerical study has been carried out.

2. The Model

The arrival follows Poisson with rate $\lambda(> 0)$ and a single server provides two type of services, respectively called type 1 service and type 2 service. The service time distributions are general and the distribution functions are respectively $B_1(x)$ and B_2). It is assumed that, when the service is about to start the customer may choose type i^{th} service $(i = 1, 2)$ with probability $p_i \quad (p_1 + p_2 = 1)$. As soon as the service of a customer is completed the server may go for a vacation of random length with probability $q \quad (0 \leqslant q < 1)$ or may continue to serve the next customer, if any, probability $(1-q)$. If there are no customers in the queue, at the completion of service, the server remains in the system without taking further vacation. The vacation period follows a general distribution with distribution function $V(y)$. Further it is assumed that not all the arriving customers are allowed to join the system at all times. Let $r0 \leqslant r \leqslant 1$ be the probability that an arriving customer will be allowed to join the system while the server is busy and let $p \quad 0 \leqslant p \leqslant 1$ be the probability that an arriving customer will be allowed to join the system while the server is on vacation. For the analysis the supplementary variable elapsed service time (elapsed vacation time) has been introduced. Let $\mu_i(x)dx$ be the conditional probability of completion of the i^{th} type of service during the interval $(x, x + dx]$ given that the elapsed service time is x, so that $\mu_i(x) = \frac{b_i}{1-B_i(x)}, i = 1, 2$ and let $\gamma(x)dx$ be the conditional probability of completion of the vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x, so that $\gamma(x) = \frac{v(x)}{1-V(x)}$. The following notations are introduced to define the model mathematically.

$P_n^{(i)}(x, t) = Pr$ { at time t, there are n customers in the queue excluding one receiving the i^{th} type of service and the elapsed service time is x } $, i = 1, 2$.

$V_n(x, t) = Pr$ { at time t, the server is on vacation with elapsed vacation time is x and the number of customers in the queue is n }

$Q(t) = Pr\{$at time t, there are no customers in the system and the server is idle$\}$
Let $P_n^{(i)}(x)$, $V_n(x)$ and Q denote the corresponding steady state probabilities. The probability generating functions for the probabilities $\{P_n^{(i)}(x)\}$, $\{V_n(x)\}$ are respectively defined as,

$$P^(i)(x,z) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x), i = 1, 2.$$

$$V(x,z) = \sum_{n=0}^{\infty} z^n V_n(x).$$

3. The Governing Equations

The forward Kolmogorov equations related to the model defined in the proceeding section are

$$\frac{d}{dx}P_0^{(1)}(x) + (\lambda + \mu_1(x))P_0^{(1)}(x) = \lambda(1-r)P_0^{(1)}(x) \tag{1}$$

$$\frac{d}{dx}P_n^{(1)}(x) + (\lambda + \mu_1(x))P_n^{(1)}(x) = \lambda(1-r)P_n^{(1)}(x) + r\lambda P_{n-1}^{(1)}(x), n = 1, 2, \cdots \tag{2}$$

$$\frac{d}{dx}P_0^{(2)}(x) + (\lambda + \mu_2(x))P_0^{(2)}(x) = \lambda(1-r)P_0^{(2)}(x) \tag{3}$$

$$\frac{d}{dx}P_n^{(2)}(x) + (\lambda + \mu_2(x))P_n^{(2)}(x) = \lambda(1-r)P_n^{(2)}(x) + r\lambda P_{n-1}^{(2)}(x), n = 1, 2, \cdots \tag{4}$$

$$\frac{d}{dx}V_0 + (\lambda + \gamma(x))V_0(x) = \lambda(1-p)V_0(x) \tag{5}$$

$$\frac{d}{dx}V_n + (\lambda + \gamma(x))V_n(x) = \lambda(1-p)V_n(x) + p\lambda V_{n-1}(x), n = 1, 2, \cdots \tag{6}$$

$$\lambda r Q = (1-q)\int_0^{\infty} \mu_1(x)P_0^{(1)}(x)dx + (1-q)\int_0^{\infty} \mu_2(x)P_0^{(2)}(x)dx + \int_0^{\infty} \gamma(x)V_0(x)dx \tag{7}$$

The boundary conditions are

$$\begin{aligned} P_0^{(1)}(0) = {} & r\lambda Q p_1 + (1-q)p_1 \textstyle\int_0^{\infty} \mu_1(x)P_1^{(1)}(x)dx + (1-q)p_1 \int_0^{\infty} \mu_2(x)P_1^{(2)}(x)dx \\ & + p_1 \textstyle\int_0^{\infty} \gamma(x)V_1(x)dx \end{aligned} \tag{8}$$

$$P_0^{(1)}(0) = (1-q)p_1 \int_0^\infty \mu_1(x) P_{n+1}^{(1)}(x)dx + (1-q)p_1 \int_0^\infty \mu_2(x) P_{n+1}^{(2)}(x)dx \\ +p_1 \int_0^\infty \gamma(x) V_{n+1}(x)dx, n = 1, 2, \cdots \tag{9}$$

$$P_0^{(2)}(0) = r\lambda Q p_2 + (1-q)p_2 \int_0^\infty \mu_1(x) P_1^{(1)}(x)dx + (1-q)p_2 \int_0^\infty \mu_2(x) P_1^{(2)}(x)dx \\ +p_2 \int_0^\infty \gamma(x) V_1(x)dx \tag{10}$$

$$P_0^{(2)}(0) = (1-q)p_2 \int_0^\infty \mu_1(x) P_{n+1}^{(1)}(x)dx + (1-q)p_1 \int_0^\infty \mu_2(x) P_{n+1}^{(2)}(x)dx \\ +p_2 \int_0^\infty \gamma(x) V_{n+1}(x)dx, n = 1, 2, \cdots \tag{11}$$

$$V_n(0) = q \int_0^\infty \mu_1(x) P_n^{(1)}(x)dx + q \int_0^\infty \mu_2(x) P_n^{(2)}(x)dx, n = 0, 1, 2, \cdots \tag{12}$$

and the normalization condition is

$$Q + \sum_{n=0}^{\infty} \int_{n=0}^{\infty} [P_n^{(1)}(x) + P_n^{(2)}(x) + V_n(x)]dx = 1$$

4. The Analysis

Multiplying equations 2, 4 and 6 by z^n and summing from $n = 1$ to ∞ and then adding (1), (3) and (5) with the corresponding equations, we get

$$\frac{\frac{d}{dx} P^{(1)}(x,z)}{P^{(1)}(x,z)} = -T - \mu_1(x) \quad \text{where} T = \lambda r(1-z) \tag{13}$$

$$\frac{\frac{d}{dx} P^{(2)}(x,z)}{P^{(2)}(x,z)} = -T - \mu_2(x) \tag{14}$$

$$\frac{\frac{d}{dx} V(x,z)}{V(x,z)} = -R - \gamma(x) \quad \text{where} R = \lambda p(1-z) \tag{15}$$

Integration of the above equations leads to

$$P^{(1)}(x,z) = A(1 - B_1(x))e^{-Tx} \tag{16}$$

$$P^{(2)}(x,z) = A_0(1 - B_2(x))e^{-Tx} \tag{17}$$

$$V(x,z) = A_1(1 - V(x))e^{-Bx} \tag{18}$$

Taking $x = 0$ in equations (16), (17) and (18), one can find the constants A, A_0 and A_1 as

$$A = P^{(1)}(0, z), \tag{19}$$

$$A_0 = P^{(2)}(0, z)\text{and}, \tag{20}$$

$$A_1 = V(0, z), \tag{21}$$

Using equations (19), (20) and (21) respectively in equations (16), (17) and (18) , we get

$$P^{(1)}(x, z) = P^{(1)}(0, z)(1 - B_1(x))e^{-Tx} \tag{22}$$

$$P^{(2)}(x, z) = P^{(2)}(0, z)(1 - B_2(x))e^{-Tx} \tag{23}$$

$$V(x, z) = V(0, z)(1 - V(x))e^{-Rx} \tag{24}$$

Applying a similar manipulations on equations (8), (9), (10) and (11) as in the case of (1) -(6) and using equations (7), (22), (23) and (24) , we get

$$|z-(1-q)P_1B_1^*(T)|p^{(1)}(0,z) = r\lambda p_1(z-1)Q+(1-q)p_1B_2^*(T)P^{(2)}(0,z)+p_1V^*(R)V(0,z) \tag{25}$$

$$|z-(1-q)P_2B_2^*(T)|p^{(2)}(0,z) = r\lambda p_2(z-1)Q+(1-q)p_2B_1^*(T)P^{(1)}(0,z)+p_2V^*(R)V(0,z) \tag{26}$$

Using equation (26) on equation (25), we get

$$|z-(1-q)(P_1B_1^*(T)+p_2B_2^*(T)|P^{(1)}(0,z) = p_1V^*(R)V(0,z)+\lambda rp_1(z-1)Q, \tag{27}$$

Using equation (25) on equation (26), we get

$$|z-(1-q)(P_1B_1^*(T)+p_2B_2^*(T)|P^{(2)}(0,z) = p_2V^*(R)V(0,z)+\lambda rp_2(z-1)Q, \tag{28}$$

Applying similar manipulations on equation (12) as in the case of (8)- (11) and using equations (27) and (28)

$$V(0, z) = \frac{\lambda rq(z-1)(p_1B_1^*(T) + p_2B_2^*(T))Q}{z - (1 - q + qV^*(R))(p_1B_1^*(T) + p_2B_2^*(T))} \tag{29}$$

Using equation (29) on equations (27) and (28), we get

$$P^{(1)}(0,z) = \frac{\lambda r(z-1)p_1 Q}{z-(1-q+qV^*(R))(p_1B_1^*(T)+p_2B_2^*(T))} \tag{30}$$

$$P^{(2)}(0,z) = \frac{\lambda r(z-1)p_2 Q}{z-(1-q+qV^*(R))(p_1B_1^*(T)+p_2B_2^*(T))} \tag{31}$$

$$\begin{aligned} V(z) &= \textstyle\int_0^\infty V(x,z)dx \\ &= V(0,z)\frac{(1-V^*(R))}{R} \end{aligned} \tag{32}$$

$$\begin{aligned} P^{(1)}(z) &= \textstyle\int_0^\infty P^{(1)}(x,z)dx \\ &= P^{(1)}(0,z)\frac{(1-B_1^*(T))}{T} \end{aligned} \tag{33}$$

$$\begin{aligned} P^{(2)}(z) &= \textstyle\int_0^\infty P^{(2)}(x,z)dx \\ &= P^{(2)}(0,z)\frac{(1-B_2^*(T))}{T} \end{aligned} \tag{34}$$

The unknown idle probability Q is obtained using the normalizing condition $Q + P^{(1)}(1) + P^{(2)}(1)V(1) = 1$ as

$$Q = \frac{1+\lambda pqV^{*^{(1)}}(0)+\lambda r(p_1B_1^*(0)+p_2B_2^*(0))}{1+\lambda q(p-r)V^{*^{(1)}}(0)} \tag{35}$$

Equations (32), (33) and (34) together with equation (35) are respectively, the probability generating functions of the number of customers in the queue when the server is on vacation, server is serving type 1 service and serving type 2 service respectively. Here $Q > 0$ guarantees the existence of the probability generating functions in equations (32), (33) and (34) and therefore the stability condition for the system is

$$\frac{\lambda rqV^{*^{(1)}}(0)+\lambda r(p_1B_1^*(0)+p_2B_2^*(0))}{\lambda q(r-p)V^{*^{(1)}}(0)-1} < 1.$$

The probability generating function that the number of customers in the queue irrespective of the server state is

$$\begin{aligned} U(z) &= Q + P^{(1)}(z) + P^{(2)}(z) + V(z) \\ &= \tfrac{[R(z-1)+q(p_1B_1^*(T)+p_2B_2^*(T))(R-T)(1-v^*(R))]Q}{R[z-(1-q+qV^*(R))(p_1B_1^*(T)+p_2B_2^*(T))]} \end{aligned}$$

5. The Performance Measures

Using straightforward calculations the following performance measures have been obtained. (i) Mean number of customers in the queue

$$L_q = U'(1) = \frac{\lambda^2 r(C_2 + C_3 - C_4)}{2C_1(1 + \lambda q(p - r)V^{*^{(1)}}(0))}$$

where

$$C_1 = 1 + \lambda pqV^{*^{(1)}}(0) + \lambda r(p_1B_1^{*^{(1)}}(0) + p_2B_2^{*^{(1)}}(0))$$

$$C_2 = pqV^{*^{(2)}}(0) + 2rqDV^{*^{(1)}}(0)(p_1B_1^{*^{(1)}}(0) + p_2B_2^{*^{(1)}}(0))$$

$$D = 1 - \lambda(p - r)(p_1B_1^{*^{(1)}}(0) + p_2B_2^{*^{(1)}}(0))$$

$$C_3 = r(p_1B_1^{*^{(2)}}(0) + p_2B_2^{*^{(2)}}(0))(1 + \lambda q(p - r))V^{*^{(1)}}(0))$$

$$C_4 = \lambda pq(p - r)V^{*^{(2)}}(0))(p_1B_1^{*^{(1)}}(0) + p_2B_2^{*^{(1)}}(0))$$

(ii) The variance of the number of customers in the queue

$$V_{Lq} = U''(1) + U'(1) - (U'(1))^2$$
$$= \frac{2\lambda rC_0(1+\lambda q(p-r)V^{*^{(1)}}(0)) - 3\lambda^4 r^2(C_2+C_3-C_4)}{12C_1^2(1+\lambda q(p-r)V^{*^{(1)}}(0))^2}$$

where

$$U''(1) = \frac{\lambda^2 r[3C_5(C_2+C_3-C_4) - 2C_1(C_6+C_7-C_8)]}{6C_1^2(1+\lambda q(p-r)V^{*^{(1)}}(0))^2}$$
$$C_0 = 3(C_5 + C_1)(C_2 + C_3 - C_4) - 2C_1(C_6 + C_7 - C_8)$$
$$C_5 = 2\lambda^2 pq(p - r)V^{*^{(1)}}(0))(p_1B_1^{*^{(1)}}(0) + p_2B_2^{*^{(1)}}(0)) + \lambda^2P^2qV^{*^{(2)}}(0)$$
$$+\lambda^2 r^2(p_1B_1^{*^{(2)}}(0) + p_2B_2^{*^{(2)}}(0))$$
$$C_6 = \lambda r^2(p_1B_1^{*^{(3)}}(0) + p_2B_2^{*^{(3)}}(0))(1 + \lambda q(p - r)V^{*^{(1)}}(0)) + \lambda p^2qV^{*^{(3)}}(0)$$
$$C_7 = 3\lambda rDD_0$$
$$D_0 = pqV^{*^{(2)}}(0)(p_1B_1^{*^{(1)}}(0) + p_2B_2^{*^{(1)}}(0)) + rqV^{*^{(1)}}(0)(p_1B_1^{*^{(2)}}(0) + p_2B_2^{*^{(2)}}(0))$$
$$C_8 = \lambda^2p^2q(p - r)V^{*^{(3)}}(0)(p_1B_1^{*^{(1)}}(0) + p_2B_2^{*^{(1)}}(0))$$

(iii) The expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda'} = \frac{\lambda(C_2 + C_3 - C_4)}{2C_1}$$

where

$$\lambda' = \text{actual arrival rate}$$
$$= \lambda r[P(1) + Q] + \lambda pV(1)$$
$$= \frac{\lambda r}{1+q\lambda(p-r)V^{*^{(1)}}(0)}$$

(iv) Variance of the waiting time in the queue

$$V_{Wq} = \frac{U''(1)}{\lambda'^2} - (\frac{U'(1)}{\lambda'})^2$$
$$= \frac{2D_1(1+\lambda Q(p-r)V^{*(1)}(0)) - 3\lambda^2 r(C_2+C_3-C_4)^2}{12C_1^2 r}$$

where $D_1 = 3C_5(C_2 + C_3 - C_4) - 2C_1(C_6 + C_7 - C_8)$
(v) Mean number of customers in the queue when the server is busy

$$L_{qb} = P'(1) = P^{(1)'}(1) + P^{(2)'}(1)$$
$$= \frac{\lambda^2 r D_2}{2C_1(1+\lambda q(p-r))V^{*(1)}(0))}$$

where

$$D_2 = r(p_1B_1^{*(2)}(0) + p_2B_2^{*(2)}(0))(1 + \lambda pqV^{*(1)}(0)) - (p_1B_1^{*(1)}(0) + p_2B_2^{*(1)}(0))D_3$$
$$D_3 = \lambda p^2 qV^{*(2)}(0) + 2\lambda rpqV^{*(1)}(0)(p_1B_1^{*(1)}(0) + p_2B_2^{*(1)}(0))$$

(vi) Mean number of customers in the queue when the server is on vacation

$$L_{qv} = V'(1) = \frac{\lambda^2 r(D_4D_5 - \lambda qr^2V^{*(1)}(0)(p_1B_1^{*(2)}(0) + p_2B_2^{*(2)}(0)))}{2C_1(1 + \lambda q(p-r)V^{*(1)}(0))}$$

where

$$D_4 = 2qrV^{*(1)}(0)(p_1B_1^{*(2)}(0) + p_2B_2^{*(2)}(0)) + qpV^{*(2)}(0)$$
$$D_5 = 1 + \lambda r(p_1B_1^{*(1)}(0) + p_2B_2^{*(1)}(0))$$

(vii) Utilization factor= the fraction of time that the server is busy

$$\rho = 1 - V(1) - Q$$
$$= \frac{\lambda r(p_1B_1^{*(1)}(0) + p_1B_2^{*(1)}(0))}{\lambda q(r-q)V^{*(1)}(0) - 1}$$

$(viii)$ Mean number of customers in the system

$$L = L_q + \rho$$
$$= \frac{\lambda^2 r(C_2+C_3-C_4) - 2\lambda rC_1(p_1B_1^{*(1)}(0) + p_1B_2^{*(2)}(0))}{2C_1(1+\lambda q(p-r)V^{*(1)}(0))}$$

(ix) Mean response time= Mean time a customers spends in the system

$$M = \frac{L}{\lambda'}$$
$$= \frac{\lambda(C_2+C_3-C_4) - 2C_1(p_1B_1^{*(1)}(0) + p_2B_2^{*(1)}(0))}{2C_1}$$

6. Some Particular Models

In this section some particular models are derived by taking known distributions to service time and vacation time. The service times (both type 1 and type 2) are negative exponential with parameters μ_1 for type 1 and μ_2 for type 2. The arrival process is Poisson with parameter λ. We consider three different models by assigning different distributions to the vacation times. For model 1 $(M - 1)$, the vacation times are negative exponential, for model 2 $(M - 2)$ it is hyper exponential, for model 3 ($M-3$), it is Erlang-k distribution where as for model 4 ($M-4$) it is negative exponential. For all this models using the formulas in sections 4 and 5 the results for the idle probability, the probability generating function for the number of customers in the queue irrespective of server state, the mean number of customers in the queue, the variance of the number of customers in the queue, the expected waiting time in the queue, the variance of the waiting time in the queue, the mean number of customers in the queue when the server is busy, the mean number of customers in the queue when the server is on vacation the utilization factor, the mean number of customers in the system and the mean response time are obtained.

Model (1): In this model the vacation time distribution is negative exponential with parameter θ.

$$Q = \frac{\mu_1\mu_2(\theta-\lambda pq)-\lambda r\theta(p_1\mu_2+p_2\mu_2\mu_1)}{\mu_1\mu_2(\theta-\lambda q(p-r))}$$

$$U(z) = \frac{\{(R+\theta)(T+\mu_1)(T+\mu_2)(z-1)+q(R-T)[T(p_1\mu_1+p_2)+\mu_1\mu_2]\}Q}{z(R+\theta)(T+\mu_1)(T+\mu_2)-[\theta+R(1-q)][T(p_1\mu_1+p_2\mu_2)+\mu_1\mu_2]}$$

$$L_q = U'(1) = \frac{\lambda^2 rU_4}{\mu_1\mu_2 U_2(\theta-\lambda q(p-r))}$$

$$V_{Lq} = U''(1)+U'(1)-[U'(1)]^2$$

$$= \frac{\lambda^2 r[(\theta-\lambda q(p-r))[2(U_3U_4+U_2U_5)+\theta\mu_1\mu_2U_2U_4]-\lambda^2 r\theta U_4^2]}{\mu_1^2\mu_2^2U_2^2\theta(\theta-\lambda q(p-r))^2}$$

$$W_q = \frac{L_q}{\lambda'} = \frac{\lambda U_4}{\mu_1\mu_2U_2\theta}$$

$$V_{W_q} = \frac{U''(1)}{\lambda'^2} - (\frac{U'(1)}{\lambda'})^2$$

$$= \frac{2(\theta-\lambda q(p-r))(U_3U_4+U_2U_5)-\lambda^2 r\theta U_4^2}{\mu_1^2\mu_2^2U_2^2\theta^3 r}$$

$$L_{qb} = \frac{\lambda^2 r\{r\theta(p_1\mu_2^2+p_2\mu_1^2)(\theta\theta-\lambda pq)+\lambda pqU_0(p_1\mu_2+p_2\mu_1)\}}{\mu_1\mu_2U_2(\theta-\lambda q(p-r))}$$

$$L_{qv} = \frac{\lambda^2 r\{qU_0(\mu_1\mu_2-\lambda r(p_1\mu_2+p_2\mu_1))+\lambda qr^2\theta(p_1\mu_2^2+p_2\mu_1^2)\}}{\mu_1\mu_2U_2(\theta-\lambda q(p-r))}$$

$$\rho = \frac{\lambda r\theta(p_1\theta\mu_2+p_2\mu\mu_1)}{\mu_1\mu_2(\theta-\lambda q(p-r))}$$

$$L = L_q+\rho = \frac{\lambda^2 rU_4+\theta U_2\lambda r(p_1\mu_2+p_2\mu_1)}{\mu_1\mu_2U_2(\theta-\lambda q(p-r))}$$

$$M = \frac{\lambda U_4+\theta U_2(p_1\mu_2+p_2\mu_1)}{\mu_1\mu_2U_2\theta}$$

where

$U''(1) = \frac{2\lambda^2 r(U_3U_4+U_2U_5)}{\mu_1^2\mu_2^2U_2^2\theta(\theta-\lambda q(p-r))}$

$U_0 = p\mu_1\mu_2 + r\theta(p_1\mu_2 + p_2\mu_1)$

$U_1 = \mu_1\mu_2 + \lambda(p-r)(p_1\mu_2 + p_2\mu_1)$

$U_2 = \mu_1\mu_2(\theta - \lambda qp) - \lambda r\theta(p_1\mu_2 + p_2\mu_1)$

$U_3 = \lambda^2 pq\mu_1\mu_2U_0 + \lambda^2r^2\theta^2(p_1\mu_2^2 + p_2\mu_1^2)$

$U_4 = qU_0U_1 + r\theta(p_1\mu_2^2 + p_2\mu_1^2)(\theta - \lambda q(p-r))$

$U_5 = qU_1U_6 + r^2\theta^2(p_1\mu_2^3 + p_2\mu_1^3)(\theta - \lambda q(p-r))$

$U_6 = \lambda p^2\mu_1^2\mu_2^2 + \lambda r\theta p\mu_1\mu_2(p_1\mu_2 + p_2\mu_1) + \lambda\theta^2r^2(p_1\mu_2^2 + p_2\mu_1^2)$

$\lambda' = \frac{\lambda r\theta}{\theta-\lambda q(p-r)}$

Model (2): In this model the vacation time distribution is Hyper exponential with parameters $q_1, q_2, (q_1+q_2=1), \theta_1$ and θ_2.

$Q = \frac{\mu_1\mu_2[\theta_1\theta_2-\lambda pq(q_1\theta_2+q_2\theta_1)]-\lambda r\theta_1\theta_2(p_1\mu_2+p_2\mu_2}{\mu_1\mu_2[\theta_1\theta 2-\lambda q(p-r)(q_1\theta_2+q_2\theta_1)]}$

$U(z) = \frac{R(Z-1)A_9+q(R-T)A_{11}[(R+\theta_1)(R+\theta_2)-A_{10}]}{R\{ZA_9-[(1-q)(R+\theta_1)+(R+\theta_2)+qA_{10}A_{11}]\}}Q$

$L_q = U'(1) = \frac{\lambda^2r[qU_1A_2+r\theta_1\theta_2A_3(p_1\mu_2^2+p_2\mu_1^2)]}{\mu_1\mu_2A_3A_4}$

$V_{Lq} = U''(1) + U'(1) - [U'(1)]^2$

$= \frac{\lambda^2r\{2A_3A_{12}+\theta_1\theta_2A_8A_{13}\}}{\theta_1\theta_2A_3^2A_4^2\mu_1^2\mu_2^2}$

$W_q = \frac{L_q}{\lambda'} = \frac{\lambda[qU_1A_2+r\theta_1\theta 2A_3(p_1\mu_2^2+p_2\mu_1^2)]}{\mu_1\mu_2A_4\theta_1\theta_2}$

$V_{W_q} = \frac{U''(1)}{\lambda'^2} - (\frac{U'(1)}{\lambda'})^2$

$= \frac{2A_3A_{12}-\lambda^2r\theta_1\theta_2A_{13}^2}{r\mu_1^2\mu_2^2A_4^2\theta_1^3\theta_2^3}$

$L_{qb} = \frac{\lambda^2r\{r\theta_1\theta_2(p_1\mu_2^2+p_2\mu_1^2)[\theta_1\theta_2-\lambda pq(q_1\theta_2+q_2\theta_1)]+\lambda pqA_2(p_1\mu_2+p_2\mu_1)\}}{\mu_1\mu_2A_3A_4}$

$L_{qv} = \frac{\lambda^2r\{qA_2[\mu_1\mu_2-\lambda r(p_1\mu_2+p_2\mu_1)]+\lambda r^2q\theta_1\theta_2(q_1\theta_1+q_2\theta_2)(p_1\mu_2^2+p_2\mu_1^2)\}}{\mu_1\mu_2A_3A_4}$

$\rho = \frac{\lambda r\theta_1\theta_2(p_1\mu_2+p_2\mu_1)}{\mu_1\mu_2A_3}$

$L = L_q + \rho$
$= \frac{\lambda^2rqU_1A_2+\lambda r\theta_1\theta_2[\lambda rA_3(p_1\mu_2^2+p_2\mu_1^2)+A_4(p_1\mu_2+p_2\mu_1)]}{\mu_1\mu_2A_3A_4}$

$M = \frac{\lambda qU_1A_2+\theta_1\theta_2[\lambda rA_3(p_1\mu_2^2+p_2\mu_1^2)+A_4(p_1\mu_2+p_2\mu_1)]}{\mu_1\mu_2A_4\theta_1\theta_2}$

where

$U''(1) = \frac{2\lambda^2 r A_{12}}{\theta_1\theta_2 A_3 A_4^2 \mu_1^2 \mu_2^2}$

$A_2 = p\mu_1\mu_2(q_1\theta_1^2 + q_2\theta_1^2) + r\theta_1\theta_2(q_1\theta_1 + q_2\theta_1)(p_1\mu_2 + p_2\mu_1)$

$A_3 = \theta_1\theta_2 - \lambda q(p - r)(q_1\theta_2 + q_2\theta_1)$

$A_4 = \mu_1\mu_2[\theta_1\theta_2 - \lambda qp(q_1\theta_2 + q_2\theta_1)] - \lambda r\theta_1\theta_2(p_1\mu_2 + p_2\mu_1)$

$A_5 = \lambda p^2\mu_1^2\mu_2^2(q_1\theta_2^3 + q_2\theta_1^3) + \lambda pr\theta_1\theta_2\mu_1\mu_2(q_1\theta_2^2 + q_2\theta_1^2)(p_1\mu_2 + p_2\mu_1)$
$+\lambda r^2\theta_1^2\theta_2^2(q_1\theta_2 + q_2\theta_1)(p_1\mu_2^2 + p_2\mu_1^2)$

$A_6 = \lambda^2 pq\mu_1^2 A_2 + \lambda^2 r^2\theta_1^2\theta_2^2(p_1\mu_2^2 + p_2\mu_1^2)$

$A_7 = A_6(p_1\mu_2^2 + p_2\mu_1^2) + \lambda r\theta_1\theta_2 A_4(p_1\mu_2^3 + p_2\mu_1^3)$

$A_8 = \mu_1\mu_2 A_3 A_4 - \lambda^2 rqU_1 A_2 - \lambda^2 r^2\theta_1\theta_2 A_3(p_1\mu_2^2 + p_2\mu_1^2)$

$A_9 = (R + \theta_1)(R + \theta_2)(T + \mu_1)(T + \mu_2)$

$A_{10} = R(q_1\theta_2 + q_2\theta_1) + \theta_1\theta_2$

$A_{11} = T(p_1\theta_2 + p_2\theta_1) + \mu_1\mu_2$

$A_{12} = qU_1(A_2A_6 + A_5A_4) + r\theta_1\theta_2 A_3 A_7$

$A_{13} = qU_1U_2 + r\theta_1\theta_2 A_3(p_1\mu_2^2 + p_2\mu_1^2)$

$\lambda' = \frac{\lambda r\theta_1\theta_2}{A_3}$

Model (3): In this model the vacation time distribution is Erlang-k with parameter θ.

$$Q = \frac{\mu_1\mu_2(\theta-\lambda pq)-\lambda r\theta(p_1\mu_2+p_2\mu_2\mu_1)}{\mu_1\mu_2(\theta-\lambda q(p-r))}$$

$$U(z) = \frac{[p(z-1)(T+\mu_1)(T+\mu_2)(k\theta+R)^k+q(p-r)l((k\theta+R)^k-(k\theta)^k)]Q}{p[z(T+\mu_1)(T+\mu_2)(k\theta+R)^k-l((1-q)(k\theta+R)^k+q(k\theta)^k)]}$$

$$L_q = U'(1) = \frac{\lambda^2 r\{qU_1Y_0+2kr\theta(\theta-\lambda q(p-r))(p_1\mu_2^2+p_2\mu_2^2)\}}{2kU_2\mu_1\mu_2(\theta-\lambda q(p-r))}$$

$$V_{Lq} = U''(1)+U'(1)-[U'(1)]^2$$
$$= \frac{2\lambda^2 r\theta(\theta-\lambda q(p-r))Y_7-3\lambda^4r^2\theta(qU_1Y_0+2kr\theta(\theta-\lambda q(p-r))(p_1\mu_2^2+p_2\mu_1^2)^2}{12k^2U_2^2\mu_1^2+\mu_2^2\theta(\theta-\lambda q(p-r))^2}$$

$$W_q = \frac{L_q}{\lambda'} = \frac{\lambda\{qU_1Y_0+2kr\theta(\theta-\lambda q(p-r))(p_1\mu_2^2+p_2\mu_1^2)\}}{2kU_2\mu_1\mu_2\theta}$$

$$V_{W_q} = \frac{U''(1)}{\lambda'^2} - \left(\frac{U'(1)}{\lambda'}\right)^2$$
$$= \frac{2(\theta-\lambda q(p-r))Y_{10}-3\lambda^2 r\theta(qU_1Y_0+2kr\theta(\theta-\lambda q(p-r))(p_1\mu_2^2+p_2\mu_1^2))^2}{12rk^2\theta^3\mu_2^2\mu_1^2U_2^2}$$

$$L_{qb} = \frac{\lambda^2 r\{2kr\theta(\theta-\lambda pq)(p_1\mu_2^2+p_2\mu_1^2)+\lambda pqY_0(p_1\mu_2+p_2\mu_1)\}}{2kU_2\mu_{1\mu}\mu_2(\theta-\lambda q(p-r))}$$

$$L_{qv} = \frac{\lambda^2 r\{qY_0(\mu_1\mu_2-\lambda r(p_1\mu_2+p_2\mu_1))+2\lambda kr^2q\theta(p_1\mu_2^2+p_2\mu_1^2)\}}{2kU_2\mu_1\mu_2U_2(\theta-\lambda q(p-r))}$$

$$\rho = \frac{\lambda r\theta(p_1\theta\mu_2+p_2\mu\mu_1)}{\mu_1\mu_2(\theta-\lambda q(p-r))}$$

$$L = L_q+\rho$$
$$= \frac{\lambda^2 r\{qU_1Y_0+2kr\theta(\theta-\lambda q(p-r))(p_1\mu_2^2+p_2\mu_1^2)\}+2kU_2\lambda r\theta(p_1\mu_2+p_2\mu_1)}{2kU_2\mu_1\mu_2(\theta-\lambda q(p-r))}$$

$$M = \frac{L}{\lambda'} = \frac{\lambda qU_1Y_0+2\lambda kr\theta(\theta-\lambda q(p-r))(p_1\mu_2^2+p_2\mu_1^2)+2kU_2\theta(p_1\mu_2+p_2\mu_1)}{2kU_2\mu_1\mu_2\theta}$$

where

$$U''(1) = \frac{\lambda^2 rY_3}{6k^2U_2^2\mu_1^2\mu_2^2\theta[\theta-\lambda q(p-r)]}$$

$$l = \mu_1\mu_2+T(p_1\mu_2+p_2\mu_1)$$

$$Y_0 = (k+1)p\mu_1\mu_2+2kr\theta(p_1\mu_2+p_2\mu_1)$$

$$Y_1 = p^2q(k+1)\mu_1^2\mu_2^2+2kr^2\theta^2(p_1\mu_2^2+p_2\mu_1^2)+2pqrk\theta\mu_1\mu_2)$$

$$Y_2 = -\lambda p^2\mu_1^2\mu_2^2(k+1)(k+2)-3k(k+1)\lambda pr\theta(p_1\mu_2+p_2\mu_1-6\lambda k^2r^2\theta^2(p_1\mu_2^2+p_2\mu_1^2))$$

$$Y_3 = \lambda Y_1(p_1\mu_2^2+p_2\mu_1^2)+2U_2kr\theta(p_1\mu_2^3+p_2\mu_1^3)$$

$$Y_7 = qU_1U_8+6kr\theta Y_9(\theta-\lambda q(p-r))$$

$$Y_8 = 3Y_0(\lambda^2Y_1+k\theta U_2\mu_1\mu_2)-2U_2Y_2$$

$$Y_9 = \lambda Y_3+k\theta U_2\mu_1\mu_2(p_1\mu_2^2+p_2\mu_1^2)$$

$$Y_{10} = qU_1(3\lambda^2Y_0Y_1-2U_2Y_2)+6\lambda kr\theta Y_3(\theta-\lambda q(p-r))$$

$$\lambda' = \frac{\lambda r\theta}{\theta-\lambda q(p-r)}.$$

Model (4): For $M/M/1$ queue with compulsory server vacation, $p = r = q = 1, p_1 = 1, p_2 = 0, \mu_1 = \mu$.

$$Q = \frac{\theta\mu-\lambda(\theta+\mu)}{\theta\mu}$$

$$U(z) = \frac{[\theta+\lambda(1-z)][\mu+\lambda(1-z)](z-1)Q}{z[\theta+\lambda(1-z)][\mu+\lambda(1-z)]-\theta\mu}$$

$$L_q = U'(1) = \frac{\lambda^2[\mu^2+\theta^2+\theta\mu]}{\theta\mu[\theta\mu-\lambda(\theta+\mu)]}$$

$$V_{Lq} = U''(1) + U'(1) - [U'(1)]^2$$
$$= \frac{\lambda^2\{\lambda^2[\mu^2+\theta^2+\theta\mu]^2+[\theta\mu-\lambda(\theta+\mu)][2\lambda\mu^3+\theta(2\lambda+\mu)(\mu^2+\theta^2+\theta\mu)]}{\theta^2\mu^2[\theta\mu-\lambda(\theta+\mu)]^2}$$

$$W_q = \frac{L_q}{\lambda'} = \frac{\lambda[(\mu^2+\theta^2+\theta\mu)]}{\theta\mu[\theta\mu-\lambda(\theta+\mu)]}$$

$$V_{W_q} = \frac{U''(1)}{\lambda'^2} - (\frac{U'(1)}{\lambda'})^2$$
$$= \frac{\lambda^2[\mu^2+\theta^2+\theta\mu]+2\lambda[\theta\mu-\lambda(\theta+\mu)][\theta^3+\mu^3+\theta\mu(\theta+\mu)]}{\theta^2\mu^2[\theta\mu-\lambda(\theta+\mu)]^2}$$

$$L_{qb} = \frac{\lambda^2(\theta^2+\lambda\mu)}{\theta\mu[\theta\mu-\lambda(\theta+\mu)]}$$

$$L_{qv} = \frac{\lambda^2(\theta-\lambda+\mu)}{\theta[\theta\mu-\lambda(\theta+\mu)]}$$

$$\rho = \frac{\lambda}{\mu}$$

$$L = L_q + \rho$$
$$= \frac{\lambda^2[\mu^2+\theta^2+\theta\mu]}{\theta\mu[\theta\mu-\lambda(\theta+\mu)]} + \frac{\lambda}{\mu}$$

$$M = \frac{L}{g}$$
$$= \frac{\lambda[\mu^2+\theta^2+\theta\mu]}{\theta\mu[\theta\mu-\lambda(\theta+\mu)]} + \frac{l}{\mu}$$

where

$$U''(1) = \frac{2\lambda^2\{\lambda^2[\mu^2+\theta^2+\theta\mu]^2+\lambda[\theta\mu-\lambda(\theta+\mu)][\theta^3+\mu^3+\theta\mu(\theta+\mu)]\}}{\theta^2\mu^2[\theta\mu-\lambda(\theta+\mu)]^2}$$

$$\lambda' = \lambda.$$

7. The Numerical Study

In this section, some numerical examples are given to show the effect of the probability q on L_q,V_{Lq},L_{qb}, L_{qv}, L,W_q, V_{Wq}, ρ and M for the model analyzed in this paper with taking some particular distributions to service times and vacation time. Model with negative exponential service times and vacation time is called model 1 $(M-1)$, model with negative exponential service times and hyper exponential vacation time is called model 2 $(M-2)$ and model with negative exponential service times and Erlangian vacation time is called model 3 $(M-3)$. Here the parameters $r = 0.3, p = 0.7, p_1 = 0.4, p_2 = 0.6, k = 5, \lambda = 2.0, \mu_1 = 2.5, \mu_2 = 1.5, q_1 = 0.3, q_2 = 0.7, \theta = 3.0, \theta_1 = 5.0$ and $\theta_2 = 4.0$ are fixed. Figures $1-5$ represent the functions of L_q, L_{qb}, L_{qv}, L, W_q, with respect to the probability q (for model 1, 2 and 3). All functions are found to be increasing functions with small variations. Table 6 presents the variance values of L_q and W_q with respect to probability q (for model 1, 2 and 3). Table 7 presents the values of ρ and M with respect to probability q (for model 1, 2 and 3). The table values shows that as q increases the variances also increase but after certain point the variation is too large. Where as in the case of ρ and M the variations are steadily increasing

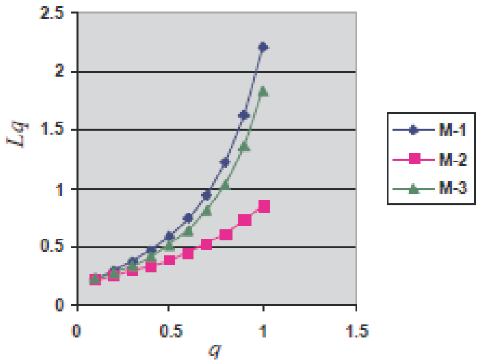

Figure 1. q versus mean number of customers in the queue

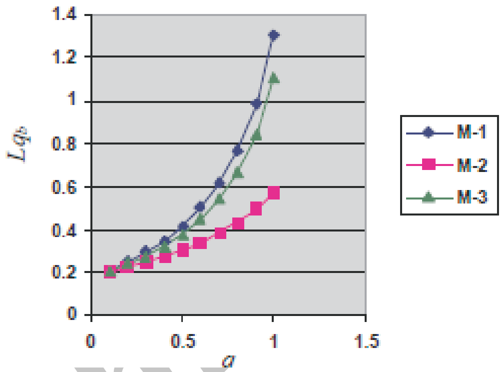

Figure 2. q versus mean number of customers in the queue when the server is busy

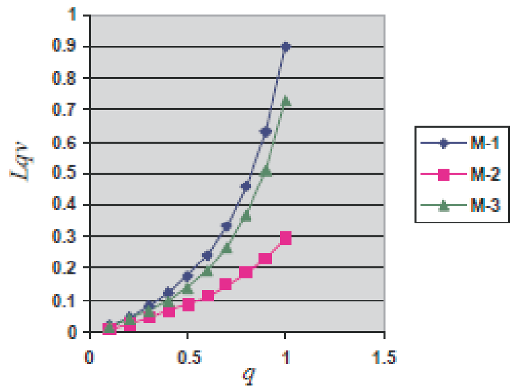

Figure 3. q versus mean number of customers in the queue when the server is on vacation

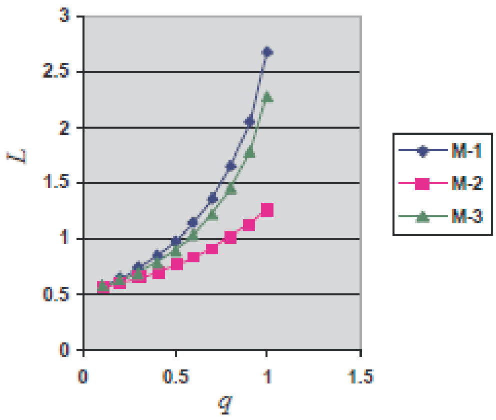

Figure 4. q versus mean number of customers in the system

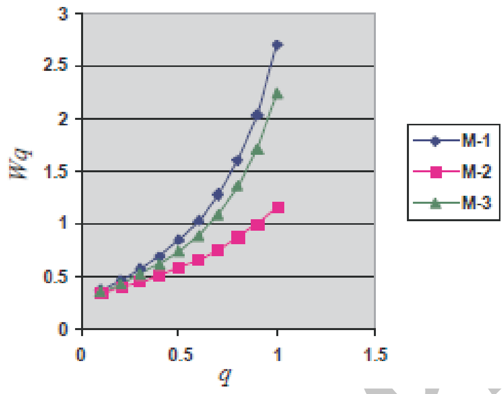

Figure 5. q versus expected waiting time in the queue

Table 1. Variance of the number customers in the queue, variance of the waiting time in the queue of $M-1$, $M-2$ and $M-3$

q	V_{Lq}			V_{Wq}		
	$M-1$	$M-2$	$M-3$	$M-1$	$M-2$	$M-3$
0.1	0.4852	0.4222	0.4483	0.6676	0.5657	0.5941
0.2	0.6679	0.5153	0.5819	0.9292	0.6919	0.7648
0.3	0.9076	0.6267	0.7553	1.2596	0.8395	0.9806
0.4	1.2278	0.7610	0.9847	1.6855	1.0135	1.2590
0.5	1.6655	0.9244	1.2954	2.2485	1.2208	1.6274
0.6	2.2812	1.1255	1.7285	3.0160	1.4707	2.1299
0.7	3.1792	1.3758	2.3549	4.1037	1.7759	2.8424
0.8	4.5530	1.6920	3.3053	5.7246	2.1545	3.9049
0.9	6.7945	2.0984	4.8433	8.3086	2.6328	5.5990
1.0	10.8025	2.6315	7.5717	12.8364	3.2503	8.5682

8. Conclusion

In this paper we considered a single server queue with Bernoulli vacation. The customers are admitted to queue using a Bernoulli process and the single server provides two type of services. Using supplementary variable technique the probability generating functions of number of customers in the queue at different server states are obtained. Some performance measures are calculated from the probability generating functions. Further we performed numerical analysis by assuming particular values to the parameters.

Table 2. Utilization factor and mean response time of $M-1$, $M-2$ and $M-3$

q	ρ		M		
	$M-1/M-2$	$M-3$	$M-1$	$M-2$	$M-3$
0.1	0.3452	0.3424	0.9356	0.9045	0.9210
0.2	0.3549	0.3491	1.0249	0.9552	0.9933
0.3	0.3652	0.3561	1.1301	1.0118	1.0786
0.4	0.3761	0.3633	1.2560	1.0754	1.1805
0.5	0.3877	0.3709	1.4090	1.1474	1.3044
0.6	0.4000	0.3787	1.5993	1.2296	1.4586
0.7	0.4131	0.3869	1.8423	1.3242	1.6553
0.8	0.4271	0.3955	2.1632	1.4344	1.9153
0.9	0.4421	0.4044	2.6070	1.5642	2.2746
1.0	0.4582	0.4138	3.2606	1.7196	2.8040

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