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B-spline Collocation Approach for Solution of Klein-Gordon Equation

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Abstract. We develope a numerical method based on B-spline collocation method to solve linear Klein-Gordon equation. The proposed scheme is unconditionally stable. The results of numerical experiments have been compared with the exact solution to show the efficiency of the method computationally. Easy and economical implementation is the strength of this approach.

Keywords: Klein-Gordon equation, Collocation, Cubic B-spline method, Stability.

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1. Introduction

We consider the following linear Klein-Gordon equation with $a \geqslant 0$,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + au = h(x,t), \quad a \leqslant x \leqslant b, \quad t \geqslant 0 \ , \tag{1}$$

with the initial conditions,

$$u(x,0) = f_0(x), \qquad a \leqslant x \leqslant b, \tag{2}$$

$$\frac{\partial u(x,0)}{\partial t} = f_1(x), \quad a \leqslant x \leqslant b, \tag{3}$$

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and subject to the boundary conditions,

$$u(a,t) = g_0(t), \quad u(b,t) = g_1(t), \qquad t \geqslant 0 \ , \tag{4}$$

where a and c are constants and $h(x,t)$ is called the source term. It is interesting to point here that if $a = 0$, equation (1) becomes the inhomogeneous wave equation. The Klein-Gordon equation is considered one of the most important mathemathical models in quantum field theory. The equation appears in relativistic physics and is used to describe dispersive wave phenomena in general [15].
The Klein-Gordon equation has been extensively studied by using traditional methods such as finite difference method, finite element method, or collocation method. The methods investigated the concepts of existance, uniqueness of the solution and the weak solution as well.

B-spline functions have some attractive properties. Due to the being piecewise polynomial, they can be integrated and differentiated easily. Since they have compact support, numerical methods in which B-spline functions are used as a basic function lead to matrix system including band matrices. Such systems can be handled and solved with low computational cost [2]. Spline solutions of partial differential equations are suggested in many studies [1, 3–5, 7, 10, 12, 13, 16, 17].

This paper is arranged as follows: In section 2, we present a finite-difference approximation to discretize the equation (1) in time variable. In section 3, we apply cubic B-spline collocation method to solve the problem in space direction. The stability analysis of the method is given in section 4. In section 5, numerical experiments are conducted to demonstrate the efficiency of the proposed method computationally.

2. Temporal Discretization

We consider a uniform mesh Δ with the grid points $\lambda_{j,n}$ to discretize the region $\Omega = [a,b] \times (t_0, T]$. Each $\lambda_{j,n}$ is the vertices of the grid points (x_j, t_n), where $x_j = a + jh$, $j = 0, 1, 2, \ldots, N$ and $t_n = t_0 + nk$, $n = 0, 1, 2, \ldots$ and h and k are mesh sizes in the space and time direction respectively.

At first we discretize the problem in time variable using the following finite difference approximation with uniform step size k.

$$u_{tt}^n \cong \frac{u^{n+1} - 2u^n + u^{n-1}}{k^2}, \tag{5}$$

$$u^n \cong \frac{u^{n+1} + u^n}{2}, \tag{6}$$

and

$$u_{xx}^n \cong \frac{u_{xx}^{n+1} + u_{xx}^n}{2}. \tag{7}$$

Substituting the above approximations into equation (1), we have,

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{k^2} + a\frac{u_j^{n+1} + u_j^n}{2} = c^2 \frac{(u_{xx})_j^{n+1} + (u_{xx})_j^n}{2} + h(x_j, t_n), \tag{8}$$

thus, we obtain,

$$(1+\frac{ak^2}{2})u_j^{n+1}-\frac{c^2k^2}{2}\,(u_{xx})_j^{n+1}=\frac{c^2k^2}{2}\,(u_{xx})_j^{n}+k^2h(x_j,t_n)+(2-\frac{ak^2}{2})u_j^n-u_j^{n-1}, \quad (9)$$

after some simplifications, the above equation can be written in the following form,

$$(1+\frac{ak^2}{2})u^*-\frac{c^2k^2}{2}(u_{xx})^*=r(x), \quad (10)$$

where,

$$r(x)=\frac{c^2k^2}{2}\,(u_{xx})^n+k^2h(x,t_n)+(2-\frac{ak^2}{2})u^n-u^{n-1}, \quad (11)$$

with the boundary conditions,

$$u^*(a)=g_0(t_n), \qquad u^*(b)=g_1(t_n). \quad (12)$$

In order to start any computations using the above formula we need the values of u at the nodal points at the zero and first time levels.

To compute u^1 we may use the initial conditions $u(x,t_0)=f_0(x)$ and $u_t(x,t_0)=f_1(x)$. Using Taylor series for u at $t=t_0+k$ following [14] we have,

$$u^1=u^0+ku_t^0+\frac{k^2}{2!}\,u_{tt}^0+O(k^3), \quad (13)$$

u^0 and u_t^0 are known from initial conditions exactly thus we need to compute the term u_{tt}^0. By using Eq. (1) we obtain,

$$u_{tt}^0=[c^2u_{xx}+h(x,t)-au]_{t=0}. \quad (14)$$

Now substituting (14) and initial conditions into (13) we can obtain an approximation for u at $t=t_0+k$,

$$u^1=f_0(x)+kf_1(x)+\frac{k^2}{2!}[c^2u_{xx}+h(x,t)-au]_{t=0}+O(k^3).$$

3. B-spline Collocation Method

In this section we use the B-spline collocation method to solve equation (1) with the boundary conditions (12). Let $\Delta^*=\{a=x_0<x_1<...<x_N=b\}$ be the partition in $[a,b]$. We define the cubic B-spline for $j=-1,0,...,N+1$ by the

Table 1. Values of $B_j(x)$ and its derivatives at the nodal points.

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$B_j(x)$	0	1	4	1	0
$B'_j(x)$	0	$3/h$	0	$-3/h$	0
$B''_j(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

following relation in [11] as,

$$B_{3,j} = \frac{1}{h^3}\begin{cases} (x-x_{j-2})^3, & x \in [x_{j-2}, x_{j-1}], \\ h^3+3h^2(x-x_{j-1})+3h(x-x_{j-1})^2-3(x-x_{j-1})^3, & x \in [x_{j-1}, x_j], \\ h^3+3h^2(x_{j+1}-x)+3h(x_{j+1}-x)^2-3(x_{j+1}-x)^3, & x \in [x_j, x_{j+1}], \\ (x_{j+2}-x)^3, & x \in [x_{j+1}, x_{j+2}], \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Our numerical treatment for solving equation (1) using the collocation method with cubic B-splines is to find an approximate solution $\hat{S}(x)$ to exact solution $u(x,t)$
in the form,

$$\hat{S}(x) = \sum_{j=-1}^{N+1} \hat{c_j}(t) B_j(x), \tag{16}$$

where $\hat{c_j}(t)$ are unknown time dependent parameters to be determined from the boundary conditions and collocation of the differential equation.

Using approximate function (16) and cubic B-spline (15), the approximate values at the knots of $\hat{S}(x)$ and its derivatives are determined in terms of the time dependent parameters $\hat{c_j}(t)$ as,

$$\hat{S}(x) = \hat{c}_{j-1} + 4\hat{c}_j + \hat{c}_{j+1}, \tag{17}$$

$$h\hat{S}'(x) = 3(\hat{c}_{j+1} - \hat{c}_{j-1}), \tag{18}$$

$$h^2\hat{S}''(x) = 6(\hat{c}_{j-1} - 2\hat{c}_j + \hat{c}_{j+1}). \tag{19}$$

Let $\hat{S}(x)$ satisfies the equation (10) plus the boundary conditions, thus we have,

$$L\hat{S}(x_j) = r(x_j), \qquad 0 \leqslant j \leqslant N, \tag{20}$$

$$\hat{S}(x_0) = g_0(t_n), \quad \hat{S}(x_n) = g_1(t_n),$$

where $Lu^* = (1+\frac{ak^2}{2})u^* - \frac{c^2k^2}{2}(u_{xx})^*$. Substituting (16) into (20) and using (17) and (19), we have,

$$(1+\frac{ak^2}{2})(\hat{c}_{j-1}+4\hat{c}_j+\hat{c}_{j+1}) - \frac{3c^2k^2}{h^2}(\hat{c}_{j-1}-2\hat{c}_j+\hat{c}_{j+1}) = r_j, \quad 1 \leqslant j \leqslant N-1, \tag{21}$$

and after simplification it leads to the following system of linear equations,

$$(1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2})\hat{c}_{j-1}+(4+2ak^2+\frac{6c^2k^2}{h^2})\hat{c}_j+(1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2})\hat{c}_{j+1}=r_j,\ \ 1\leqslant j\leqslant N-1. \tag{22}$$

To obtain a unique solution for $\hat{C}=(\hat{c}_{-1},\hat{c}_0,...,\hat{c}_{N+1})$, we need to use the boundary conditions. Using the first boundary condition we have,

$$u(a,t_n)=\hat{S}(a)=g_0(t_n)=\hat{c}_{-1}+4\hat{c}_0+\hat{c}_1, \tag{23}$$

by eliminating $\hat{c}_{-1}$ from the above equation and the equation (22) for $j=0$ we have,

$$\frac{18c^2k^2}{h^2}\hat{c}_0=r_0-(1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2})g_0(t_n). \tag{24}$$

Similarly, using the boundary condition,

$$u(b,t_n)=\hat{S}(b)=g_1(t_n)=\hat{c}_{N-1}+4\hat{c}_N+\hat{c}_{N+1}, \tag{25}$$

and eliminating $\hat{c}_{N+1}$ from the above equation and equation (22) for $j=N$ we have,

$$\frac{18c^2k^2}{h^2}\hat{c}_N=r_N-(1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2})g_1(t_n). \tag{26}$$

Associating (24) and (26) with (22) we obtain a linear $(N+1)\times(N+1)$ system of equations in the following form,

$$A\hat{C}=\hat{B}, \tag{27}$$

where,

$$A=\begin{pmatrix} \frac{18c^2k^2}{h^2} & 0 & 0 & & \\ (1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2}) & (4+2ak^2+\frac{6c^2k^2}{h^2}) & (1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2}) & & \\ & \ddots & \ddots & \ddots & \\ & & (1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2}) & (4+2ak^2+\frac{6c^2k^2}{h^2}) & (1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2}) \\ & & 0 & 0 & \frac{18c^2k^2}{h^2} \end{pmatrix}, \tag{28}$$

$$\hat{B}=\begin{pmatrix} r_0-(1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2})g_0(t_n) \\ r_1 \\ \vdots \\ r_{N-1} \\ r_N-(1+\frac{ak^2}{2}-\frac{3c^2k^2}{h^2})g_1(t_n) \end{pmatrix}, \qquad \hat{C}=\begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \vdots \\ \hat{c}_N \end{pmatrix}. \tag{29}$$

Since $a \geqslant 0$, it is easily seen that A is strictly diagonally dominant, and hence nonsingular by Gershgorin's theorem (see [11] and the reference there in). Since A is nonsingular, we can solve (27) for $(\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_N)$ and substitute into the boundary equations (23) and (25) to obtain $\hat{c}_{-1}$ and $\hat{c}_{N+1}$. Hence the method of collocation applied to equation (1) using a basis of cubic B-splines has a unique solution given by (16).

4. Stability Analysis

Following [8, 9] we stablished the Von Neumann stability analysis of the proposed method. For stability analysis we should consider the homogeneous part of equation (9) as follows,

$$(1+\frac{ak^2}{2})u_j^{n+1} - \frac{c^2k^2}{2}\,(u_{xx})_j^{n+1} - \frac{c^2k^2}{2}\,(u_{xx})_j^{n} - (2-\frac{ak^2}{2})u_j^n + u_j^{n-1} = 0, \quad (30)$$

using the properties of B-spline functions which is specified in relations (17)-(19) and also using equation (30) we can obtain,

$$r_1\hat{c}_{j-1}^{n+1} + r_2\hat{c}_j^{n+1} + r_1\hat{c}_{j+1}^{n+1} + r_3\hat{c}_{j-1}^{n} + r_4\hat{c}_j^{n} + r_3\hat{c}_{j+1}^{n} + \hat{c}_{j-1}^{n-1} + 4\hat{c}_j^{n-1} + \hat{c}_{j+1}^{n-1} = 0, \quad (31)$$

where,

$$\begin{aligned} r_1 &= 1 + \frac{ak^2}{2} - \frac{3c^2k^2}{h^2}, \qquad (32)\\ r_2 &= 4 + 2ak^2 + \frac{6c^2k^2}{h^2},\\ r_3 &= \frac{ak^2}{2} - 2 - \frac{3c^2k^2}{h^2},\\ r_4 &= 2ak^2 - 8 + \frac{6c^2k^2}{h^2}. \end{aligned}$$

Now, it is necessary to assume that the solution of the scheme (31) at the mesh point (x_j, t_n) may be written as $\hat{c}_j^n = \xi^n \exp(i\theta j)$, where ξ is, in general, complex, θ is real, and $i = \sqrt{-1}$. Thus using $\hat{c}_j^n = \xi^n \exp(i\theta j)$ in (31), we obtain the characteristic equation,

$$P(\xi) \equiv A\xi^2 + B\xi + C = 0, \quad (33)$$

where,

$$p = 3ak^2 - 2ak^2\sin^2\frac{\theta}{2} + \frac{12c^2k^2}{h^2}\,\sin^2\frac{\theta}{2},$$

$$q = 6 - 4\sin^2\frac{\theta}{2},$$

$$A = p+q, B = p-2q, C = q.$$

Following [6] using Routh Horwitz criterion and transformation $\xi = (1+z)/(1-z)$ in equation (32), we get,

$$(1 - z^2)P(\frac{1+z}{1-z}) = (A - B + C)z^2 + 2(A - C)z + (A + B + C). \qquad (34)$$

The necessary and sufficient condition for $|\xi| < 1$ is that $A-B+C > 0$, $A-C > 0$, $A+B+C > 0$. The conditions $A+B+C > 0$ and $A-C > 0$ are satisfied for $a \geqslant 0$ and for all θ except $(\theta, a) = (0, 0)$ or $(2\pi, 0)$. We can treat this case separately. The condition $A - B + C > 0$ is always satisfied.

For $\theta = 0$ or 2π and $a = 0$, from equation (33), we have,

$$\xi^2 - 2\xi + 1 = 0. \qquad (35)$$

In this case also, $|\xi| \leqslant 1$.

Thus our scheme is unconditionally stable.

5. Numerical Examples

To illustrate the efficiency and applicability of our present method computationally, we consider three examples of linear Klein-Gordon equation, which their exact solutions are known.

Example 5.1: We consider equation(1) with $a = 1$, $c = 1$ and the initial conditions,

$$\begin{cases} u(x, 0) = 0, & 0 \leqslant x \leqslant 1, \\ u_t(x, 0) = \cosh(x), & 0 \leqslant x \leqslant 1. \end{cases} \qquad (36)$$

The right hand side function is $f(x, t) = 0$. The analytical solution is $tcosh(x)$.We extract the boundary functions from the exact solution. The L_∞, L_2-errors and Root-Mean-Square (RMS) of errors are obtained in Table 2 for t=1, 2, 3, 4 and 5, with the step sizes of $k = 0.001$, $h = 0.01$.

Example 5.2: In this example, we consider the linear Klein-Gordon equation (1) with $a = 1$ and $c = 1$ in the interval $0 \leqslant x \leqslant 2$. Subject to the initial conditions,

$$\begin{cases} u(x, 0) = 0, & 0 \leqslant x \leqslant 2, \\ u_t(x, 0) = x, & 0 \leqslant x \leqslant 2. \end{cases} \qquad (37)$$

The right hand side function is $f(x, t) = 0$. The exact solution is $u(x, t) = x\sin(t)$. We extract the boundary functions from the exact solution. The L_∞ and L_2-errors and Root-Mean-Square (RMS) of errors are obtained in Table 3 for t=1, 2, 3, 4 and 5 and values of $k = 0.001$, $h = 0.02$.

Example 5.3: We consider the linear Klein-Gordon equation (1) with $a = 1$, $c = 1$ and $f(x, t) = 2sin(x)$ in the interval $0 \leqslant x \leqslant \pi$. Subject to the initial

Table 2. Errors in the solution of Example 4.1.

t	L_2-error	L_∞-error	RMS
1	$2.2459e-005$	$2.4487e-006$	$2.2348e-006$
2	$4.1227e-005$	$4.8691e-006$	$4.1022e-006$
3	$5.5311e-005$	$7.2048e-006$	$5.5037e-006$
4	$6.3805e-005$	$9.4362e-006$	$6.3488e-006$
5	$6.5394e-005$	$1.1171e-005$	$6.5070e-006$

Table 3. Errors in the solution of Example 4.2.

t	L_2-error	L_∞-error	RMS
1	$2.7542e-005$	$4.7337e-006$	$2.7405e-006$
2	$1.0614e-004$	$1.7842e-005$	$1.6561e-005$
3	$2.8222e-004$	$3.7773e-005$	$2.2709e-005$
4	$3.8543e-004$	$6.2847e-005$	$3.8352e-005$
5	$5.6880e-004$	$9.1424e-005$	$5.6597e-005$

Table 4. Errors in the solution of Example 4.3.

t	L_2-error	L_∞-error	RMS
0.2	$7.9077e-005$	$9.2072e-006$	$6.4352e-006$
0.4	$1.6000e-004$	$1.8682e-005$	$1.3021e-005$
0.6	$2.3849e-004$	$2.7927e-005$	$1.9408e-005$
0.8	$3.1450e-004$	$3.6935e-005$	$2.5594e-005$
1	$3.8798e-004$	$4.5698e-005$	$3.1573e-005$

conditions,

$$\begin{cases} u(x,0) = \sin(x), & 0 \leqslant x \leqslant \pi, \\ u_t(x,0) = 1, & 0 \leqslant x \leqslant \pi. \end{cases} \tag{38}$$

The analytical solution is given as $u(x,t) = \sin(x) + \sin(t)$.

We extract the boundary functions from analytical solution.

The L_∞, L_2-errors and Root-Mean-Square (RMS) of errors are tabulated in Table 4 for t=0.2, 0.4, 0.6, 0.8 and 1, with the step sizes of $k = 0.001$ and $h = 0.02$.

6. Conclusion

For the linear Klein-Gordon equation, a two-level spline-difference scheme was discussed in this work. This method is based on B-spline collocation. Finite difference approximations for the time derivatives and spline for the spatial derivative are used. During the computation, we found that the proposed difference scheme is stable for the Klein-Gordon equation with $a \geqslant 0$. To examine the accuracy and efficiency of the proposed algorithm, we gave three examples. These computational results show that our proposed algorithm is effective and accurate.

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