International Journal of
Mathematical Modelling & Computations
Vol. 03, No. 01, 2013, 11- 23

Spline Collocation Method for Solving Boundary Value Problems

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Received: 7 January 2013; Accepted: 12 March 2013.

Abstract. The spline collocation method is used to approximate solutions of boundary value problems. The convergence analysis is given and the method is shown to have second-order convergence. A numerical illustration is given to show the pertinent features of the technique.

Keywords: Boundary-value problems, Collocation method, Spline interpolant,

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1. Introduction

Boundary value problems (BVPs) are mathematical models for several physical phenomena. For example, when an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability sets in as ordinary convection, the ordinary differential equation is sixth order. When the instability sets in as overstability, it is modeled by an eighth-order ordinary differential equation [4]. If an infinite horizontal layer of fluid is heated from below, with the supposition that a uniform magnetic field is also applied across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, instability sets in. When instability sets in as ordinary convection, it is modeled by tenth-order boundary value problem. When instability sets in as overstability, it is modeled by twelfth-order boundary value problem [4].

We consider in this paper the numerical approximation for the boundary value problems of the form

$$y^{(r)}(x) = f(x, y(x), y'(x), \dots, y^{(r-1)}(x)), \qquad x \in [a, b], \tag{1}$$

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$$y^{(m)}(a) = \alpha_m,\ m = 0, \ldots, r_2 - 1,\ y^{(m)}(b) = \beta_m,\ m = 0, \ldots, r_1 - 1, \qquad (2)$$

where $f = f(x, z_0, \ldots, z_{r-1})$ is a real valued function on $\mathbb{R}^{r+1}$ sufficiently smooth, $\alpha_i(i = 0, \ldots, r_{2-1})$, and $\beta_i(i = 0, \ldots, r_{1-1})$ are real finite constants, with

$$r_2 = \lfloor \frac{r}{2} \rfloor = \max\{n \in \mathbb{N},\ n \leqslant \frac{r}{2}\}, \quad r_1 = \lceil \frac{r}{2} \rceil = \min\{n \in \mathbb{N},\ n \geqslant \frac{r}{2}\}.$$

Theorems which list the condition for the existence and uniqueness of solution of such problems are thoroughly discussed in a book by Agarwal [1]. In general it is not possible to obtain the analytical solution of (1)-(2) for arbitrary choices of f. Consequently, we usually resort to some numerical methods for obtaining an approximate solution of (1)-(2).

A variety of numerical methods are available in the literature to solve boundary-value problems. These methods include finite-difference methods [5], orthogonal spline collocation methods [3], sinc-Galerkin methods [6].

In [7], we have developed two methods for the solution of special linear and nonlinear fifth order boundary value problem, respectively. The first one uses spline interpolants and the second is based on spline quasi-interpolants which are constructed from sextic splines. In [8], septic spline collocation method based on spline interpolants was proposed for solving the general sixth-order boundary-value problems. These methods have extended for the solution of some linear boundary value problems [10].

In the present paper, a spline collocation method using a spline interpolant which satisfies the same boundary conditions, is developed and analyzed for approximating solutions of boundary value problems. There is proved to be second order convergent.

This paper is organized as follows: Section 2 is devoted to spline collocation method for linear/nonlinear BVPs using spline interpolant. Next, we prove that this method is second order convergent and we derive the error bound of the spline solution. Finally, we give in Section 3 some numerical examples that illustrate the theoretical results and the convergence of the developed method.

2. Collocation Method Using a Spline Interpolant

2.1 *Spline Interpolant*

Collocation method is often presented as a generalization of interpolation. More specifically, if the differential operator is reduced to identity operator, the collocation method is reduced to interpolation. Moreover, the order of convergence of the collocation method is often related to that of the interpolant in the same approximation space.

In this section, we define a spline interpolant S of degree $r+1$ satisfying boundary conditions (2) with optimal approximation order. To do this we consider the uniform grid partition

$$\Delta = \{a = x_{-r-1} = \ldots = x_0 < x_1 < \ldots < x_{n-1} < x_n = \ldots = x_{n+r+1} = b\},$$

of the interval $I = [a, b]$, where $x_i = a + ih,\ 0 \leqslant i \leqslant n$, and $h = (b-a)/n$. Let $B_i,\ i = -r-1, \ldots, n-1$, be the B-splines of degree $r+1$ associated with Δ. It is well known that these B-splines form a basis for the space

$$S^r_{r+1}(I,\Delta) = s \in \mathcal{C}^r(I): \ s|_{[x_i,x_{i+1}]} \text{ is a polynomial of degree} = r+1.$$

Theorem 2.1 *Let y be the exact solution of the problem (1) with boundary conditions (2), then there exists a unique spline interpolant of y $S \in S^r_{r+1}(I,\Delta)$ satisfying*

$$S^{(m)}(x_0) = y^{(m)}(x_0) = \alpha_m, \ m = 0, \dots, r_2 - 1, \tag{3}$$

$$S(t_i) = y(t_i), \ i = 1, \dots, n+1 \tag{4}$$

$$S^{(m)}(x_n) = y^{(m)}(x_n) = \beta_m, \ m = 0, \dots, r_1 - 1, \tag{5}$$

where $t_i = (x_i + x_{i-1})/2, \ i = 1, \dots, n, \ t_{n+1} = x_{n-1}$.

Proof Let $S = \sum_{j=-r-1}^{n-1} c_j B_j$ be a spline in $S^r_{r+1}([a,b],\tau)$ that satisfies the conditions (4)-(5). Since

$$\frac{(-\omega)^{r+2-\nu}}{(r+2-\nu)!} = \sum_{j=-r-1}^{n-1} \frac{(-D)^{\nu-1}\psi_j(\omega)}{(r+1)!} B_j, \quad \nu = 1, \dots, r+2,$$

$\psi_j(\omega) = (x_{j+1} - \omega)\dots(x_{j+r+1} - \omega)$, and D is the derivative operator, we have $c_j = \sum_{\nu=1}^{j+r+2} \frac{1}{(r+1)!}(-D)^{r+2-\nu}\psi_j(a) y^{(\nu-1)}(a)$ for $j = -r-1, \dots, -r-2+r_2$, and $c_j = \sum_{\nu=1}^{n-j} \frac{1}{(r+1)!}(-D)^{r+2-\nu}\psi_j(b) y^{(\nu-1)}(b)$ for $j = n - r_1, \dots, n-1$.

The other coefficients $c_j, \ j = -r+r_2-1, \dots, n-r_1-1$, are obtained as a unique solution of a linear system introduced in ([9] Theorem 1). ■

Now, we give explicitly the coefficients $c_j, \ j = -r-1, \dots, -r-2+r_2$ and $c_j, \ j = n-r_1, \dots, n-1$, for some values of r which we will use in Section 3.

For $r = 8$, we have

$$c_{-9} = y(a), \ c_{-8} = y(a) + \frac{h}{9}y'(a), \ c_{-7} = y(a) + \frac{h}{3}y'(a) + \frac{h^2}{36}y''(a),$$

$$c_{-6} = y(a) + \frac{2h}{3}y'(a) + \frac{11h^2}{72}y''(a) + \frac{h^3}{84}y^{(3)}(a),$$

$$c_{n-4} = y(b) - \frac{2h}{3}y'(b) + \frac{11h^2}{72}y''(b) - \frac{h^3}{84}y^{(3)}(b),$$

$$c_{n-3} = y(b) - \frac{h}{3}y'(b) + \frac{h^2}{36}y''(b), \ c_{n-2} = y(b) - \frac{h}{9}y'(b), \ c_{n-1} = y(b).$$

For $r = 10$, we have

$$c_{-11} = y(a),\ c_{-10} = y(a) + \frac{h}{11}y'(a),\ c_{-9} = y(a) + \frac{3h}{11}y'(a) + \frac{h^2}{55}y''(a),$$
$$c_{-8} = y(a) + \frac{6h}{11}y'(a) + \frac{h^2}{10}y''(a) + \frac{h^3}{165}y^{(3)}(a),$$
$$c_{-7} = y(a) + \frac{10h}{11}y'(a) + \frac{7h^2}{22}y''(a) + \frac{5h^3}{99}y^{(3)}(a) + \frac{h^4}{330}y^{(4)}(a),$$
$$c_{n-5} = y(b) - \frac{10h}{11}y'(b) + \frac{7h^2}{22}y''(b) - \frac{5h^3}{99}y^{(3)}(b) + \frac{h^4}{330}y^{(4)}(b),$$
$$c_{n-4} = y(b) - \frac{6h}{11}y'(b) + \frac{h^2}{10}y''(b) - \frac{h^3}{165}y^{(3)}(b),$$
$$c_{n-3} = y(b) - \frac{3h}{11}y'(b) + \frac{h^2}{55}y''(b),\ c_{n-2} = y(b) - \frac{h}{11}y'(b),\ c_{n-1} = y(b).$$

For $r = 12$, we have

$$c_{-13} = y(a),\ c_{-12} = y(a) + \frac{h}{13}y'(a),\ c_{-11} = y(a) + \frac{3h}{13}y'(a) + \frac{h^2}{78}y''(a),$$
$$c_{-10} = y(a) + \frac{6h}{13}y'(a) + \frac{11h^2}{156}y''(a) + \frac{h^3}{286}y^{(3)}(a),$$
$$c_{-9} = y(a) + \frac{10h}{13}y'(a) + \frac{35h^2}{156}y''(a) + \frac{25h^3}{858}y^{(3)}(a) + \frac{h^4}{715}y^{(4)}(a),$$
$$c_{-8} = y(a) + \frac{15h}{13}y'(a) + \frac{85h^2}{156}y''(a) + \frac{75h^3}{572}y^{(3)}(a) + \frac{137h^4}{8580}y^{(4)}(a) + \frac{h^5}{1287}y^{(5)}(a),$$
$$c_{n-6} = y(b) - \frac{15h}{13}y'(b) + \frac{85h^2}{156}y''(b) - \frac{75h^3}{572}y^{(3)}(b) + \frac{137h^4}{8580}y^{(4)}(b) - \frac{h^5}{1287}y^{(5)}(b),$$
$$c_{n-5} = y(b) - \frac{10h}{13}y'(b) + \frac{35h^2}{156}y''(b) - \frac{25h^3}{858}y^{(3)}(b) + \frac{h^4}{715}y^{(4)}(b),$$
$$c_{n-4} = y(b) - \frac{6h}{13}y'(b) + \frac{11h^2}{156}y''(b) - \frac{h^3}{286}y^{(3)}(b),$$
$$c_{n-3} = y(b) - \frac{3h}{13}y'(b) + \frac{h^2}{78}y''(b),\ c_{n-2} = y(b) - \frac{h}{13}y'(b),\ c_{n-1} = y(b).$$

2.2 *Spline Collocation Method*

It is well known, see [2], that the interpolation with splines of degree d gives $O(h^{d+1})$ uniform norm errors for the interpolant and $O(h^{d+1-s})$ errors for the sth derivative of the interpolant. Thus, for any y . $\mathcal{C}^{r+2}([a, b])$ we have

$$\|D^s(y - S)\|_\infty = O(h^{r+2-s}), \quad \text{for } s = 0, \ldots, r. \tag{6}$$

We suppose that exact solution of the BVPs (1) and (2) is of class $\mathcal{C}^{r+2}([a, b])$. Since the interpolatory spline S satisfies (6), it follows from (1) that

$$S^{(r)}(t_i) = f\left(t_i, S(t_i) + O(h^{r+2}), \ldots, S^{(r-1)}(t_i) + O(h^3)\right) + O(h^2),\ i = 1, \ldots, n+1. \tag{7}$$

Then, the spline collocation method presented in this section consists in finding a spline

$$\widetilde{S}(x) = \mu(x) + \sum_{j=-r+r_2-1}^{n-1-r_1} \widetilde{c}_j B_j(x), \tag{8}$$

which satisfies

$$\widetilde{S}^{(r)}(t_i) = f\left(t_i, \widetilde{S}(t_i), \ldots, \widetilde{S}^{(r-1)}(t_i)\right),\ i = 1, \ldots, n+1, \tag{9}$$

where

$$\mu(x) = \sum_{j=-r-1}^{-r-2+r_2} c_j B_j(x) + \sum_{j=n-r_1}^{n-1} c_j B_j(x).$$

2.3 *Spline Solution of Linear BVPs*

In the case of linear boundary-value problem, $f(x, y, \ldots, y^{(r-1)})$ has the form

$$f\left(x, y(x), ..., y^{(r-1)}(x)\right) = -\sum_{k=0}^{r-1} p_k(x) y^{(k)}(x) + g(x),$$

where p_k, $0 \leqslant k \leqslant r-1$, and g are given continuous functions defined in the bounded interval $[a, b]$. Taking

$$C = [c_{-r+r_2-1}, \ldots, c_{n-r_1-1}]^T \text{ and } \widetilde{C} = [\widetilde{c}_{-r+r_2-1}, \ldots, \widetilde{c}_{n-r_1-1}]^T,$$

then, (7) and (9) can be written respectively in the matrix forms

$$\left(A_r^h + \sum_{k=0}^{r-1} A_k^h P_k\right) C = G + e, \tag{10}$$

$$\left(A_r^h + \sum_{k=0}^{r-1} A_k^h P_k\right) \widetilde{C} = G, \tag{11}$$

where $G = [g_1, g_2, \ldots, g_{n+1}]^T$, $e_i = O(h^2)$, $i = 1, 2, \ldots, n+1$ and A_k^h, P_k are the following $(n+1) \times (n+1)$ matrices

$$P_k = diag(p_k(t_i)),\ i = 1, \ldots, n+1,$$
$$A_k^h = \left(a_{i,j}^k(h)\right)_{1 \leqslant i,j \leqslant n+1},\ \text{where } a_{i,j}^k(h) = B_{-r+r_2-2+j}^{(k)}(t_i),$$

and

$$g_i = g(t_i) - \mu^{(r)}(t_i) - \sum_{k=0}^{r-1} p_k(t_i) \mu^{(k)}(t_i),\ i = 1, \ldots, n+1.$$

Let M_j, $j = -r-1, \ldots, n-1$, be the B-splines of degree $r+1$ associated with the uniform partition
$X_n = \{0 = x_{-r-1} = \ldots = x_0 < x_1 = 1 < \ldots < x_{n-1} = n-1 < x_n = \ldots = x_{n+r+1} = n\}$, and defined by
$B_j(x) = M_j\left(\frac{x-a}{h}\right), \quad \forall x \in [a,b]$. Therefore, $B_j^{(k)}(t_i) = \frac{1}{h^k} M_j^{(k)}\left(\frac{t_i-a}{h}\right)$.
If we put

$$A_k = \left(a_{i,j}^k\right)_{1 \leqslant i,j \leqslant n+1}, \text{ where } a_{i,j}^k = M_{-r+r_2-2+j}^{(k)}\left(\frac{t_i-a}{h}\right),$$

then,

$$A_k^h = \frac{1}{h^k} A_k, \; k = 0, \ldots, r, \tag{12}$$

with the coefficients of the matrix A_k, $k = 0, \ldots, r$ are independent of h. Indeed, since $\frac{t_i-a}{h} = i - 1 + \frac{1}{2}$, $i = 1, \ldots, n$, the coefficients $M_{-r+r_2-2+j}^{(k)}\left(\frac{t_i-a}{h}\right)$, $j = 1, \ldots, r-r_2+1$, are independent of h. On the other hand, as $M_l = M_0(\cdot - l)$, $l = 0, \ldots, n-r-2$, we deduce that $M_{-r+r_2-2+j}^{(k)}\left(\frac{t_i-a}{h}\right)$, $j = r-r_2+2, \ldots, n-r_2$, are independent of h. In the same way, we can see that $M_{-r+r_2-2+j}^{(k)}\left(\frac{t_i-a}{h}\right)$, $j = n-r_2+1, \ldots, n+1$, are independent of h. In addition, we have proved that the matrix A_r is invertible, for more details see [9].

Remark 1 By using Mathematica, we can compute explicitly the matrix A_k. As an example, we give the matrix A_k, $k = 0, 1, 2$, for $r = 2$.

$$A_0 = \begin{pmatrix} \frac{19}{32} & \frac{25}{96} & \frac{1}{48} & & & & & \\ \frac{1}{32} & \frac{15}{32} & \frac{23}{48} & \frac{1}{48} & & & & \\ 0 & \frac{1}{48} & \frac{23}{48} & \frac{23}{48} & \frac{1}{48} & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \frac{1}{48} & \frac{23}{48} & \frac{23}{48} & \frac{1}{48} & 0 \\ & & & & \frac{1}{48} & \frac{23}{48} & \frac{15}{32} & \frac{1}{32} \\ & & & & & \frac{1}{48} & \frac{25}{96} & \frac{19}{32} \\ & & & & & \frac{1}{6} & \frac{7}{12} & \frac{1}{4} \end{pmatrix}, \quad A_1 = \begin{pmatrix} -\frac{3}{16} & \frac{13}{16} & \frac{1}{8} & & & & & \\ -\frac{13}{16} & -\frac{9}{16} & \frac{5}{8} & \frac{1}{8} & & & & \\ 0 & -\frac{1}{8} & -\frac{5}{8} & \frac{5}{8} & \frac{1}{8} & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -\frac{1}{8} & -\frac{5}{8} & \frac{5}{8} & \frac{1}{8} & 0 \\ & & & & & -\frac{1}{8} & -\frac{13}{16} & \frac{3}{16} \\ & & & & & -\frac{1}{8} & -\frac{13}{16} & \frac{3}{16} \\ & & & & & -\frac{1}{2} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -\frac{15}{4} & \frac{1}{4} & \frac{1}{2} & & & & & \\ \frac{3}{4} & -\frac{3}{4} & -\frac{1}{2} & \frac{1}{2} & & & & \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ & & & & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{4} & \frac{3}{4} \\ & & & & & \frac{1}{2} & \frac{1}{4} & -\frac{15}{4} \\ & & & & & 1 & -\frac{5}{2} & \frac{3}{2} \end{pmatrix}$$

According to (12), the systems (10) and (11) can be written in the following forms

$$\left(A_r + \sum_{k=0}^{r-1} h^{r-k} A_k P_k\right) C = h^r (G + e), \tag{13}$$

$$\left(A_r + \sum_{k=0}^{r-1} h^{r-k} A_k P_k\right) \widetilde{C} = h^r G, \tag{14}$$

Proposition 2.2 *If* $\|A_r^{-1}\|_\infty \left(\sum_{k=0}^{r-1} h^{(r-k)} \|A_k\|_\infty \|P_k\|_\infty\right) < 1$, *then there exists a unique spline* $\widetilde{S}$ *that approximates the exact solution* y *of problem* (1) *with boundary conditions* (2).

Proof For a small real h such that

$$\|A_r^{-1}\|_\infty \left(\sum_{k=0}^{r-1} h^{(r-k)} \|A_k\|_\infty \|P_k\|_\infty\right) < 1,$$

the matrix

$$\left(I + A_r^{-1}\left(\sum_{k=0}^{r-1} h^{(r-k)} A_k P_k\right)\right)^{-1}, \quad \text{exists,}$$

and $\left\|\left(I + A_r^{-1}\left(\sum_{k=0}^{r-1} h^{(r-k)} A_k P_k\right)\right)^{-1}\right\|_\infty < \frac{1}{1-\|A_r^{-1}\|_\infty\left(\sum_{k=0}^{r-1} h^{(r-k)}\|A_k\|_\infty\|P_k\|_\infty\right)}$.
From (14), we get

$$\widetilde{C} = h^r \left(I + A_r^{-1}\left(\sum_{k=0}^{r-1} h^{(r-k)} A_k P_k\right)\right)^{-1} A_r^{-1} G.$$

■

Proposition 2.3 *If we choose the real* h *such that* $\|A_r^{-1}\|_\infty \left(\sum_{k=0}^{r-1} h^{(r-k)} \|A_k\|_\infty \|P_k\|_\infty\right) < \frac{1}{2}$, *then there exists a constant* K *which depends only of the functions* p_k *and* g *such that*

$$\|C - \widetilde{C}\|_\infty \leqslant K h^2. \tag{15}$$

Proof Assume that $\|A_r^{-1}\|_\infty \left(\sum_{k=0}^{r-1} h^{(r-k)} \|A_k\|_\infty \|P_k\|_\infty\right) < \frac{1}{2}$, from (13) and (14), we have

$$C - \widetilde{C} = h^r \left(I + A_r^{-1}\left(\sum_{k=0}^{r-1} h^{(r-k)} A_k P_k\right)\right)^{-1} A_r^{-1} e.$$

Since $e = O(h^2)$, there exists a constant K_1 such that $\|e\|_\infty \leqslant K_1 h^2$. Consequently,

$$\|C - \widetilde{C}\|_\infty \leqslant K_1 \frac{h^r \|A_r^{-1}\|_\infty}{1 - h^r \|A_r^{-1}\|_\infty \left(\sum_{k=0}^{r-1} h^{(-k)} \|A_k\|_\infty \|P_k\|_\infty\right)} h^2.$$

Using $0 < h \leqslant (b-a)$, we deduce that

$$\|C - \widetilde{C}\|_\infty \leqslant \frac{K_1}{\left(\sum_{k=0}^{r-1}(b-a)^{(-k)}\|A_k\|_\infty\|P_k\|_\infty\right)}h^2.$$

■

Now, we are in position to prove the main theorem of this section.

THEOREM 2.4 *The spline approximation $\widetilde{S}$ converges quadratically to the exact solution y of the BVPs defined by* (1) *and* (2)*, i.e.,* $\|y - \widetilde{S}\|_\infty = O(h^2)$.

Proof According to (6), there exists a constant Λ_0 such that

$$\|y - S\|_\infty \leqslant \Lambda_0 h^{r+2}.$$

On the other hand we have

$$S(x) - \widetilde{S}(x) = \sum_{j=-r+r_2-1}^{n-1-r_1}(c_j - \widetilde{c}_j)B_j(x).$$

Therefore, by using (15), we get

$$|S(x) - \widetilde{S}(x)| \leqslant \|C - \widetilde{C}\|_\infty \sum_{j=-r+r_2-1}^{n-1-r_1} B_j \leqslant \|C - \widetilde{C}\|_\infty \leqslant Kh^2.$$

As $\|y - \widetilde{S}\|_\infty \leqslant \|y - S\|_\infty + \|S - \widetilde{S}\|_\infty$ we deduce the stated result. ■

2.4 *Spline Solution of Nonlinear BVPs*

In this section, we assume that f is a nonlinear function and satisfies the Lipschitz condition

$$|f(x, y_0, \ldots, y_{r-1}) - f(x, z_0, \ldots, z_{r-1})| \leqslant L\sum_{i=0}^{r-1}|y_i - z_i|, \tag{16}$$

for all $x_i n$ [a, b] and all $y_i, z_i \in \mathsf{R}$, $i = 0, \ldots, r-1$, where $L > 0$ is the Lipschitz constant.

Taking $C = [c_{-r+r_2-1}, \ldots, c_{n-r_1-1}]^T$ and $\widetilde{C} = [\widetilde{c}_{-r+r_2-1}, \ldots, \widetilde{c}_{n-r_1-1}]^T$, then (7) and (9) can be written respectively in the matrix forms

$$A_r C = h^r(F - M) + e, \tag{17}$$

$$A_r \widetilde{C} = h^r(F_{\widetilde{C}} - M), \tag{18}$$

where

$$F = \begin{pmatrix} f\left(t_1, y(t_1), \dots, y^{(r-1)}(t_1)\right) \\ \vdots \\ \vdots \\ f\left(t_{n+1}, y(t_{n+1}), \dots, y^{(r-1)}(t_{n+1})\right) \end{pmatrix}, \; F_{\widetilde{C}} = \begin{pmatrix} f\left(t_1, \widetilde{S}(t_1), \dots, \widetilde{S}^{(r-1)}(t_1)\right) \\ \vdots \\ \vdots \\ f\left(t_{n+1}, \widetilde{S}(t_{n+1}), \dots, \widetilde{S}^{(r-1)}(t_{n+1})\right) \end{pmatrix}$$

$M = [\mu^{(r)}(t_1), \dots, \mu^{(r)}(t_{n+1})]^T$, and $e_i = O(h^{r+1}),\ i = 1, 2, \dots, n+1$.

Proposition 2.5 *If* $\|A_r^{-1}\|_\infty \left(\sum_{k=0}^{r-1} h^{r-k}\|A_k\|_\infty\right) < 1$, *then there exists a unique spline that approximates the exact solution* y *of problem* (1) *with boundary conditions* (2).

Proof From (17), we get

$$\widetilde{C} = h^r A_r^{-1}(F_{\widetilde{C}} - M), \tag{19}$$

setting $Z = \widetilde{C}$, we derive the following

$$\varphi(Z) = h^r A_r^{-1}(F_Z - M) = Z, \tag{20}$$

where $Z = [z_1, \dots, z_{n+1}]^T$ and

$$F_Z = \begin{pmatrix} f\left(t_1, S_Z(t_1), \dots, S_Z^{(r-1)}(t_1)\right) \\ \vdots \\ \vdots \\ f\left(t_{n+1}, S_Z(t_{n+1}), \dots, S_Z^{(r-1)}(t_{n+1})\right) \end{pmatrix}$$

with $S^{(k)}Z = \mu^{(k)} + \displaystyle\sum_{j=-r+r_2-1}^{n-1-r_1} z_{j+r-r_2+2} B_j^{(k)},\ k = 0, \dots, r-1.$

We will prove that the function $\varphi(Z)$ has a unique fixed point, i.e., the equation (19) has a unique solution. Let $Z_1, Z_2 \in \mathsf{R}^{n+1}$. Using (20), we get

$$\|\varphi(Z_1) - \varphi(Z_2)\|_\infty \leqslant h^r \|A_r^{-1}\|_\infty \|F_{Z_1} - F_{Z_2}\|_\infty, \tag{21}$$

Since f satisfies conditions (16), it follows that

$$\begin{aligned} |f(t_i, S_{Z_1}(t_i), \dots, S_{Z_1}^{(r-1)}(t_i)) &- f(t_i, S_{Z_2}(t_i), \dots, S_{Z_2}^{(r-1)}(t_i))| \\ &\leqslant L \sum_{k=0}^{r-1} |S_{Z_1}^{(k)}(t_i) - S_{Z_2}^{(k)}(t_i)| \\ &\leqslant L \left(\sum_{k=0}^{r-1} h^{-k}\|A_k\|_\infty\right) \|Z_1 - Z_2\|_\infty. \end{aligned}$$

Consequently,

$$\|F_{Z_1} - F_{Z_2}\|_\infty \leqslant L\left(\sum_{k=0}^{r-1} h^{-k}\|A_k\|_\infty\right)\|Z_1 - Z_2\|_\infty.$$

From (21), we deduce that

$$\|\varphi(Z_1) - \varphi(Z_2)\|_\infty \leqslant L\|A_r^{-1}\|_\infty \left(\sum_{k=0}^{r-1} h^{r-k}\|A_k\|_\infty\right)\|Z_1 - Z_2\|_\infty.$$

Thus, if we assume that $L\|A_r^{-1}\|_\infty\left(\sum_{k=0}^{r-1} h^{r-k}\|A_k\|_\infty\right) < 1$, φ is a strong contraction mapping. ■

From equations (17) and (18), we get

$$A_r(C - \widetilde{C}) = h^r(F - F_{\widetilde{C}}) + e. \tag{22}$$

PROPOSITION 2.6 *If* $L\|A_r^{-1}\|_\infty\left(\sum_{k=0}^{r-1} h^{r-k}\|A_k\|_\infty\right) \leqslant \frac{1}{2}$, *then there exists a constant* K_2 *which depends only of the function* f *such that*

$$\|C - \widetilde{C}\|_\infty \leqslant K_2 h^2. \tag{23}$$

Proof From (22), we have

$$(C - \widetilde{C}) = h^r A_r^{-1}(F - F_{\widetilde{C}}) + A_r^{-1}e.$$

Since $e = O(h^{r+2})$, there exists a constant K_3 such that $\|e\|_\infty \leqslant K_3 h^{r+2}$.
Consequently,

$$\|C - \widetilde{C}\|_\infty \leqslant h^r\|A_r^{-1}\|_\infty\|F - F_{\widetilde{C}}\|_\infty + K_3\|A_r^{-1}\|_\infty h^{r+2}. \tag{24}$$

On the other hand we have

$$\begin{aligned}
&|f(t_i, y(t_i), ..., y^{(r-1)}(t_i)) - f(t_i, \widetilde{S}(t_i), ..., \widetilde{S}^{(r-1)}(t_i))| \\
&\qquad \leqslant L\sum_{k=0}^{r-1} |y^{(k)}(t_i) - \widetilde{S}^{(k)}(t_i)| \\
&\qquad \leqslant L\sum_{k=0}^{r-1}\left(|y^{(k)}(t_i) - S^{(k)}(t_i)| + |S^{(k)}(t_i) - \widetilde{S}^{(k)}(t_i)|\right).
\end{aligned}$$

According to (6), there exists the constants Λ_k such that

$$\|y^{(k)} - S^{(k)}\|_\infty \leqslant \Lambda_k h^{r+2-k}\|y^{(r+2)}\|_\infty, \quad \text{for } k = 0, \ldots, r.$$

Moreover, as

$$S^{(k)}(t_i) - \widetilde{S}^{(k)}(t_i) = \sum_{j=-r+r_2-1}^{n-1-r_1} (c_j - \widetilde{c}_j) B_j^{(k)}(t_i),$$

we have

$$|S^{(k)}(t_i) - \widetilde{S}^{(k)}(t_i)| \leqslant \frac{1}{h^k}\|A_k\|_\infty \|C - \widetilde{C}\|_\infty$$

Thus,

$$\|F - F_{\widetilde{C}}\|_\infty \leqslant L\|C - \widetilde{C}\|_\infty \left(\sum_{k=0}^{r-1} h^{-k}\|A_k\|_\infty\right) + L\left(\sum_{k=0}^{r-1}\Lambda_k h^{r+2-k}\right)\|y^{(r+2)}\|_\infty.$$

From (24), we deduce that

$$\left[1 - Lh^r\|A_r^{-1}\|_\infty \left(\sum_{k=0}^{r-1} h^{-k}\|A_k\|_\infty\right)\right]\|C - \widetilde{C}\|_\infty$$
$$\leqslant h^r\|A_r^{-1}\|_\infty \left[\left(\sum_{k=0}^{r-1}\Lambda_k h^{r+2-k}\right)\|y^{(r+2)}\|_\infty + K_3 h^2\right].$$

Using the inequality $L\|A_r^{-1}\|_\infty \left(\sum_{k=0}^{r-1} h^{r-k}\|A_k\|_\infty\right) \leqslant \frac{1}{2}$, we get

$$\|C - \widetilde{C}\|_\infty \leqslant \frac{L\left(\sum_{k=0}^{r-1}\Lambda_k h^{r-k}\right)\|y^{(r+2)}\|_\infty + K_3}{L\left(\sum_{k=0}^{r-1}\Lambda_k (b-a)^{-k}\|A_k\|_\infty\right)}.$$

■

Now, we are in position to prove the main theorem of this section.

THEOREM 2.7 *The spline approximation $\widetilde{S}$ converges quadratically to the exact solution y of the BVPs defined by* (1) *and* (2)*, i.e.,* $\|y - \widetilde{S}\|_\infty = O(h^2)$.

Proof According to (6), there exists a constant Λ_0 such that

$$\|y - S\|_\infty \leqslant \Lambda_0 h^{r+2}.$$

On the other hand we have

$$S(x) - \widetilde{S}(x) = \sum_{j=-r+r_2-1}^{n-1-r_1} (c_j - \widetilde{c}_j)B_j(x).$$

Therefore, by using (23), we get

$$|S(x) - \widetilde{S}(x)| \leqslant \|C - \widetilde{C}\|_\infty \sum_{j=-r+r_2-1}^{n-1-r_1} B_j \leqslant \|C - \widetilde{C}\|_\infty \leqslant K_2 h^2.$$

As $\|y - \widetilde{S}\|_\infty \leqslant \|y - S\|_\infty + \|S - \widetilde{S}\|_\infty$, we deduce the stated result. ■

3. Test Examples and Numerical Results

To illustrate the previous theory, we solve the following problems.

Example 3.1 Consider the following boundary value problem discussed in [5] using finite-difference methods.

$$\begin{cases} y^{(8)}(x) = 7!exp(-8y(x)) - 2(7!)(1+x)^{-8}, \quad x \in [0, e^{1/2} - 1], \\ y(0) = 0, y'(0) = 1, y''(0) = -1, y^{(3)}(0) = 2, y(e^{1/2} - 1) = 1/2, \\ y'(e1/2 - 1) = \frac{1}{\sqrt{e}}, y''(e^{1/2} - 1) = -1/e, y^{(3)}(e^{1/2} - 1) = 2/e^{3/2}, \end{cases} \tag{25}$$

which has the exact solution $y(x) = ln(1 + x)$.

The results are summarized in Table 1. In the first and second columns, we give the maximum absolute errors computed at various points of the interval $[0, 1]$, for the problem (25) and the convergence orders for the spline collocation method (SCM) presented in Section 2, respectively. The table shows that the errors for SCM are better than those given by the finite-difference methods discussed in [5].

Table 1. Error norms for Problem (25)

n	Max. Abs. Errors using SCM	Order	Max. Abs. Errors [5]
8	$2.3858e-008$	1.9730	$1.76e-006$
16	$6.0772e-009$	2.1414	$1.12e-007$
32	$1.377e-009$	2.1055	$7.10e-009$
64	$3.2010e-010$	2.004	$4.44e-010$
128	$8.0002e-011$		

Example 3.2 Consider the following boundary value problem discussed in [5] using finite-difference methods.

$$\begin{cases} y^{(10)}(x) = 9!exp(-10y(x)) - 2(9!)(1+x)^{-10}, \quad x \in [0, e^{1/2} - 1], \\ y(0) = 0, y'(0) = 1, y''(0) = -1, y^{(3)}(0) = 2, y^{(4)}(0) = -6, \\ y(e^{1/2} - 1) = 1/2, y'(e^{1/2} - 1) = \frac{1}{\sqrt{e}}, y''(e^{1/2} - 1) = -1/e \\ y^{(3)}(e^{1/2} - 1) = 2/e^{3/2}, y^{(4)}(e^{1/2} - 1) = -6/e^2, \end{cases} \tag{26}$$

which has the exact solution $y(x) = ln(1 + x)$.

A comparison of the maximum errors (in absolute values) for the problem (26) and the convergence order for SCM are summarized in Table 2. The table shows that the errors for SCM are better than those given by the finite- difference methods discussed in [5].

Table 2. Error norms for Problem (26)

n	Max. Abs. Errors using SCM	Order	Max. Abs. Errors [5]
8	$1.7351e-009$	2.4128	$1.09e-005$
16	$3.2584e-010$	2.0543	$7.01e-007$
32	$7.8451e-011$	2.0224	$4.42e-008$
64	$1.9310e-011$	2.0100	$2.72e-009$
128	$4.7941e-012$		

Example 3.3 Consider the following boundary value problem

$$\begin{cases} y^{(12)}(x) - y(x) = -12(2xcos(x) + 11sin(x)), \quad x \in [-1, 1], \\ y(-1) = 0, y'(-1) = 2sin(1), y''(0) = 0, y^{(3)}(-1) = 6(cos(1) - sin(1)), \\ y^{(4)}(-1) = 8cos(1) + 12sin(1), y^{(5)}(-1) = -20cos(1) + 10sin(1), \\ y(1) = 0, y'(1) = 2sin(1), y''(1) = 4cos(1) + 2sin(1), y^{(3)}(1) = 6(cos(1) - sin(1)), \\ y^{(4)}(1) = -8cos(1) - 12sin(1), y^{(5)}(1) = -20cos(1) + 10sin(1), \end{cases} \tag{27}$$

the exact solution of the above system is $y(x) = (x^2 - 1)sin(x)$.
In Table 3 we give the maximum absolute errors computed at various points of the interval $[a, b]$, for the problem (27), and the convergence order for SCM. A comparison of the maximum errors (in absolute values) for the problem (27) is summarized in Table 4. Moreover, the methods developed in this paper are better than the non-polynomial spline method given in [11]. Indeed, the maximum absolute errors by using the SCM and non-polynomial spline method are respectively $4.4008e\ -010$ and $4.69e\ -005$ for $h = 1/16$.

Table 3. Error norms for the problem (27) using SCM.

n	Max. Abs. Errors using SCM	Order
8	$1.7783e\ -09$	2.0147
16	$4.4008e\ -010$	2.0003
32	$1.1000e\ -010$	2.0137
64	$2.7241e\ -011$	

Table 4. Comparison of numerical results for problem (27), with $n = 22$.

Siddiqi and Twizell [11]	Siddiqi and Twizell	Method presented
$x \in [x_6, x_{n-6}]$	Otherwise	
$1.366e\ -004$	$1.044e\ +024$	$3.4154e\ -010$

4. Conclusion

Spline collocation method based on a spline interpolant is developed for the approximate solution of some general BVPs. The method is also proved to be second order convergent. It has been observed that the relative errors (in absolute values), are better than those given by other collocation and finite-difference methods. So, its extension to singularly perturbed two-point boundary-value problems, using spline collocation on non-uniform partitions, is under progress.

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