On (a, c, b) Policy Queue with Change Over Times Under Bernoulli Schedule Vacation Interruption

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Abstract. This paper analyzes a renewal input working vacations queue with change over times and Bernoulli schedule vacation interruption under (a, c, b) policy. The service and vacation times are exponentially distributed. The server begins service if there are at least c units in the queue and the service takes place in batches with a minimum of size a and a maximum of size b $(a \leqslant c \leqslant b)$. The change over periods follow if there are $(c-1)$ customers at vacation completion instant or $(a-1)$ customers at service completion instant. The steady state queue length distributions at arbitrary and pre-arrival epochs are obtained. Performance measures and optimal cost policy are presented with numerical experiences. The genetic algorithm and quadratic fit search method are employed to search the optimal values of some important parameters of the system.

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1. Introduction

Working vacation models are widely used to analyze problems in the area of computer communication, manufacturing, production and transportation systems. Unlike the classical vacation models where the server completely stops service, in working vacation models service is provided during vacation at a rate usually lower than the regular service rate as introduced in [18] to analyze an $M/M/1$ queue with multiple working vacation (MWV). Later, an extension to $GI/M/1/MWV$ queue is

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carried out in [1]. Further studies related to working vacation are found in [4], [11], [6], [21], [9], etc.

The concept of vacation interruption was first introduced for an $M/M/1$ queue, [10]. Then, [12] generalized their results to a $GI/M/1$ queue with working vacations and vacation interruption. A $GI/M/1$ queue with set-up period, working vacation and vacation interruption has been studied in [24]. A working vacation queue with service interruption and multi-optional repair has been discussed in [8]. A $MAP/G/1$ queue with working vacations and vacation interruption has been analyzed in [22]. For the Bernoulli schedule vacation interruption, [23] first studied an $M/M/1$ queue with vacation interruption under the Bernoulli rule. Recently, [13] studied a $GI/Geo/1$ single working vacation queue with start-up period and Bernoulli vacation interruption using embedded Markov chain technique. Further, see [5] for a study on $M/G/1$ queue with single working vacation and Bernoulli schedule vacation interruption.

In many real-life queueing situations jobs are served with a control limit policy. For example, in some manufacturing systems it is possible to process jobs only when the number of units to be processed exceeds a specified level, and when service starts, it is profitable to continue it even when the queue size is less than the specified level but not less than a secondary limit. The optimal management policy for a single server and single or bulk service characterized by the bi-level service discipline has been discussed in [19]. An optimal control of batch arrival, bulk service queueing system with N policy has been analyzed in [20]. The infinite buffer multiple vacations queue with change over times under (a, c, d) policy has been studied in [3], where the arrivals and service times are exponentially distributed and the corresponding discrete time queue has been presented in [2].

One of the most fundamental objectives in the performance evaluation of queueing models is to search for an optimal value. Many practical optimal design problems are characterized by mixing continuous and discrete variables, discontinuous and non-convex design spaces. In such cases, the standard non linear optimization techniques will be inefficient, computationally expensive, and in most cases, find relative optimum that is closest to the starting point. Genetic algorithm (GA) is well suited for solving such problems and in most cases they find the global optimum solution with high probability. More details on GA can be found in [7], [16], [14], [15], etc.

Quadratic fit search method (QFSM) is another optimization technique which can be used when the objective function is highly complex and obtaining its derivative is a difficult task. Given a 3-point pattern, one can fit a quadratic function through corresponding functional values that has a unique optimum for the given objective function. Quadratic fit uses this approximation to improve the current 3-point pattern by replacing one of its points with approximate optimum. For the details of QFSM one may refer [17].

Motivated by the optimization problem, this paper focuses on an infinite buffer renewal input MWV queue with change over times and Bernoulli schedule vacation interruption under (a, c, b) policy. The inter-arrival time of customers and service time of batches are respectively, arbitrarily and exponentially distributed. We provide a recursive method using the supplementary variable technique to develop the steady state queue length distributions at various epochs. Various performance measures and a cost model is developed to determine the optimum service rate and vacation rate using GA and QFSM. Numerical results have been presented to show the effect of model parameters.

2.   Description of the model

We consider a multiple working vacation $GI/M^{(a,c,b)}/1$ queueing system with change over times and Bernoulli schedule vacation interruption. It is assumed that the inter-arrival times of customers are independent and identically distributed random variables with cumulative distribution function $A(x)$, probability density function $a(x)$, $x \geqslant 0$, Laplace-Stiltjes transform $A^*(\theta)$, $Re(\theta) \geqslant 0$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$, where $h^{(1)}(x_0)$ denotes the first derivative of $h(x)$ at $x = x_0$. The service begins only if there are at least c units in the queue. The customers are served in batches with a minimum size a and a maximum size b $(a \leqslant c \leqslant b)$. The various states that a server observes are given below: At a service completion epoch during a regular busy period if the queue size (j) is

- $a \leqslant j \leqslant b$: the server continues to serve.
- $0 \leqslant j \leqslant a - 2$: the server goes for a working vacation.
- $j = a - 1$: server will wait for some time in the system called change over time which is exponentially distributed with rate α_1. It starts service on finding an arrival during this change over time, otherwise it will go for a working vacation.

On returning from a working vacation, if the queue size j is

- $0 \leqslant j \leqslant c - 2$: another working vacation resumes and this process continues until at least $c - 1$ customers are left in the queue.
- $j = c - 1$: the server is in change over time, which is exponentially distributed with rate α_2. Service starts on finding an arrival during this period, otherwise another vacation follows.
- $j \geqslant c$: $min\{j, b\}$ customers are served according to first-come, first-served rule.

Service times during regular busy period, during vacation and vacation times are exponentially distributed with rate μ, η and ϕ, respectively. In the working vacation, a customer is serviced at a lower rate and at the instants of a service completion, the vacation is interrupted and the server is resumed to a regular busy period with probability $\bar{q} = 1 - q$ (if there are at least c customers in the queue), or continues the vacation with probability q. Further, the inter-arrival times, service times, change over times and working vacation times are mutually independent of each other. The traffic intensity is given by $\rho = \lambda/b\mu$. The state of the system at time t is described by the following random variables:

- $X(t) =$ number of customers present in the queue,
- $U(t) =$ remaining inter-arrival time for the next arrival,
- $Y(t) = \begin{cases} 0, & \text{if the server is in working vacation,} \\ 1, & \text{if the server is busy,} \\ 2, & \text{if the server is in change over times.} \end{cases}$

At steady state, let us define

$$\omega_n(x)dx = \lim_{t \to \infty} Pr\Big\{X(t) = n,\ x < U(t) \leq x + dx, Y(t) = 0\Big\},\ n \geq 0,$$

$$\pi_n(x)dx = \lim_{t \to \infty} Pr\Big\{X(t) = n,\ x < U(t) \leq x + dx, Y(t) = 1\Big\},\ n \geq 0,$$

$$\nu_n(x)dx = \lim_{t \to \infty} Pr\Big\{X(t) = n,\ x < U(t) \leq x + dx, Y(t) = 2\Big\},\ n = a - 1\ \text{ or }\ c - 1.$$

Let $\omega_n^*(\theta)$, $\pi_n^*(\theta)$ and $\nu_n^*(\theta)$ be the Laplace-Stiltjes transforms of $\omega_n(x)$, $\pi_n(x)$ and $\nu_n(x)$, respectively so that $\omega_n \equiv \omega_n^*(0)$, $\pi_n \equiv \pi_n^*(0)$, and $\nu_n \equiv \nu_n^*(0)$ are the steady state probabilities that n customers are in the queue and the server is in working

vacation, regular busy period and in change over times, respectively, at an arbitrary epoch.

3. Analysis of the model

In this section, we shall discuss the evaluation of steady state queue length distributions at pre-arrival and arbitrary epochs using the supplementary variable technique and the recursive method. Relating the states of the system at two consecutive time epochs t and $t + dt$ and using the probabilistic arguments, we set up the following differential-difference equations at steady state:

$$-\omega_0^{(1)}(x) = \mu\pi_0(x) + q\eta\sum_{n=c}^{b}\omega_n(x),$$

$$-\omega_n^{(1)}(x) = a(x)\omega_{n-1}(0) + \psi(1 \leqslant n \leqslant a-2)\mu\pi_n(x) + \psi(n = a-1)\alpha_1\nu_n(x)$$
$$+q\eta\omega_{n+b}(x), \quad 1 \leqslant n \leqslant a-1,$$

$$-\omega_n^{(1)}(x) = a(x)\omega_{n-1}(0) + q\eta\omega_{n+b}(x), \quad a \leqslant n \leqslant c-2,$$

$$-\omega_n^{(1)}(x) = a(x)\omega_{n-1}(0) + \psi(n = c-1)\alpha_2\nu_n(x) + q\eta\omega_{n+b}(x) - \phi\omega_n(x)$$
$$-\psi(n \geqslant c)\eta\omega_n(x), \quad n \geqslant c-1,$$

$$-\pi_0^{(1)}(x) = a(x)(\nu_{a-1}(0) + \nu_{c-1}(0)) + \mu(\sum_{n=a}^{b}\pi_n(x) - \pi_0(x)) + (\phi + \bar{q}\eta)\sum_{n=c}^{b}\omega_n(x),$$

$$-\pi_n^{(1)}(x) = a(x)\pi_{n-1}(0) + \mu\pi_{n+b}(x) + (\phi + \bar{q}\eta)\omega_{n+b}(x) - \mu\pi_n(x), n \geqslant 1,$$

$$-\nu_{a-1}^{(1)}(x) = \mu\pi_{a-1}(x) - \alpha_1\nu_{a-1}(x),$$

$$-\nu_{c-1}^{(1)}(x) = \phi\omega_{c-1}(x) - \alpha_2\nu_{c-1}(x),$$

where $\psi(\chi)$ is equal to 1 when the expression χ is satisfied, otherwise its value is 0. Further, $\omega_n(0), \ n \geqslant 0$; $\pi_n(0), \ n \geqslant 0$ and $\nu_n(0), \ n = a-1$ or $n = c-1$ are the respective probabilities with the remaining inter-arrival time equal to zero, i.e., an arrival is about to occur. Multiplying the above equations by $e^{-\theta x}$, integrating

with respect to x from 0 to ∞ yields

$$-\theta\omega_0^*(\theta) = \mu\pi_0^*(\theta) + q\eta\sum_{n=c}^{b}\omega_n^*(\theta) - \omega_0(0), \tag{1}$$

$$-\theta\omega_n^*(\theta) = A^*(\theta)\omega_{n-1}(0) + \psi(n = a - 1)\alpha_1\nu_n^*(\theta) + q\eta\omega_{n+b}^*(\theta)$$
$$+\psi(1 \leqslant n \leqslant a - 2)\mu\pi_n^*(\theta) - \omega_n(0), \quad 1 \leqslant n \leqslant a - 1, \tag{2}$$

$$-\theta\omega_n^*(\theta) = A^*(\theta)\omega_{n-1}(0) + q\eta\omega_{n+b}^*(\theta) - \omega_n(0), \quad a \leqslant n \leqslant c - 2, \tag{3}$$

$$(\phi - \theta)\omega_n^*(\theta) = A^*(\theta)\omega_{n-1}(0) + \psi(n = c - 1)\alpha_2\nu_n^*(\theta) + q\eta\omega_{n+b}^*(\theta)$$
$$-\psi(n \geqslant c)\eta\omega_n^*(\theta) - \omega_n(0), \quad n \geqslant c - 1, \tag{4}$$

$$(\mu - \theta)\pi_0^*(\theta) = A^*(\theta)(\nu_{a-1}(0) + \nu_{c-1}(0)) + \mu\sum_{n=a}^{b}\pi_n^*(\theta)$$

$$+(\phi + \bar{q}\eta)\sum_{n=c}^{b}\omega_n^*(\theta) - \pi_0(0), \tag{5}$$

$$(\mu - \theta)\pi_n^*(\theta) = A^*(\theta)\pi_{n-1}(0) + \mu\pi_{n+b}^*(\theta) + (\phi + \bar{q}\eta)\omega_{n+b}^*(\theta)$$
$$-\pi_n(0), \quad n \geqslant 1, \tag{6}$$

$$(\alpha_1 - \theta)\nu_{a-1}^*(\theta) = \mu\pi_{a-1}^*(\theta) - \nu_{a-1}(0), \tag{7}$$

$$(\alpha_2 - \theta)\nu_{c-1}^*(\theta) = \phi\omega_{c-1}^*(\theta) - \nu_{c-1}(0). \tag{8}$$

Adding equations (1) - (8), taking limit as $\theta \to 0$ and using the normalizing condition, $\sum_{n=0}^{\infty}\omega_n + \sum_{n=0}^{\infty}\pi_n + \nu_{a-1} + \nu_{c-1} = 1$, we obtain

$$\sum_{n=0}^{\infty}\omega_n(0) + \sum_{n=0}^{\infty}\pi_n(0) + \nu_{a-1}(0) + \nu_{c-1}(0) = \lambda. \tag{9}$$

It may be noted here that the left-hand side of (9) represents the probability that an arrival is about to occur, which is equal to the arrival rate of customers. This is used in the sequel to obtain a relation between arrival is about to occur and pre-arrival epoch probabilities.

3.1 *Steady state distribution at pre-arrival epochs*

Let ω_n^- (π_n^-) be the probability that n ($n \geqslant 0$) customers are present in the queue at pre-arrival epoch and the server is on working vacation (regular busy period) and ν_n^- denotes the probability that n ($n = a - 1$ or $c - 1$) customers in the queue at pre-arrival epoch and the server is in change over times. Using Bayes' theorem and equation (9), we express the pre-arrival epoch probabilities as below:

$$\omega_n^- = \omega_n(0)/\lambda, \; n \geq 0; \; \pi_n^- = \pi_n(0)/\lambda, \; n \geq 0;$$
$$\nu_n^- = \nu_n(0)/\lambda, \; n = a - 1, \; c - 1. \tag{10}$$

To obtain ω_n^-, π_n^- and ν_n^- we need to evaluate $\omega_n(0)$, $\pi_n(0)$ and $\nu_n(0)$ which is discussed below.

We define the displacement operator E as $E^x\omega_n = \omega_{n+x}$, and rewrite equation

(4) for $n \geqslant c$ as

$$(\eta + \phi - \theta - q\eta E^b)\omega_n^*(\theta) = (A^*(\theta) - E)\omega_{n-1}(0). \tag{11}$$

Setting $\theta = \eta + \phi - q\eta E^b$ in (11), we get

$$\omega_n(0) = Cr^n, \quad n \geqslant c - 1, \tag{12}$$

where C is an arbitrary constant and r is a real root inside the unit circle of the equation $A^*(\eta + \phi - q\eta z^b) - z = 0$.

Substituting equation (12) in (11), we obtain

$$\omega_n^*(\theta) = \frac{(A^*(\theta) - r)Cr^{n-1}}{\tau_1 - \theta}, \quad n \geqslant c, \tag{13}$$

where $\tau_1 = \eta(1 - qr^b) + \phi$.

Similarly from equation (6) we obtain

$$\pi_n(0) = k\xi^n - C\tau_3 r^{n+b}, \quad n \geqslant 0, \tag{14}$$

$$\pi_n^*(\theta) = \frac{k\xi^{n-1}(A^*(\theta) - \xi)}{\mu - \theta - \mu\xi^b} - \frac{C\tau_3(A^*(\theta) - r)r^{n+b-1}}{\tau_1 - \theta}, \quad n \geqslant 1, \tag{15}$$

where k is a constant, ξ is the unique real root of the equation $A^*(\mu - \mu z^b) - z = 0$ inside the unit circle, $\tau_2 = \tau_1 - \mu(1 - r^b)$ and $\tau_3 = (\phi + q\eta)/\tau_2$.

Now inserting $\theta = \alpha_1$ in equation (7), we get

$$\nu_{a-1}(0) = kL_1 - CL_2, \tag{16}$$

where

$$L_1 = \frac{\mu\xi^{a-2}(A^*(\alpha_1) - \xi)}{\mu - \alpha_1 - \mu\xi^b} \quad \text{and} \quad L_2 = \frac{\mu\tau_3(A^*(\alpha_1) - r)r^{a+b-2}}{\tau_1 - \alpha_1}.$$

Putting $\theta = \mu$ in (5) and using (13), (14), (15) and (16), $\nu_{c-1}(0)$ is obtained as

$$\nu_{c-1}(0) = kL_3 + CL_4, \tag{17}$$

where

$$L_3 = \frac{1}{A^*(\mu)} \left(1 - A^*(\mu)L_1 + \frac{(A^*(\mu) - \xi)(\xi^{a-1} - \xi^b)}{\xi^b(1 - \xi)} \right),$$

$$L_4 = \frac{1}{A^*(\mu)} \left(A^*(\mu)L_2 - \tau_3 r^b + \frac{\mu\tau_3(A^*(\mu) - r)(r^{a+b-1} - r^{2b})}{(\tau_1 - \mu)(1 - r)} \right.$$

$$\left. - \frac{\tau_3\tau_2(A^*(\mu) - r)(r^{c-1} - r^b)}{(\tau_1 - \mu)(1 - r)} \right).$$

From (7), (15) and (16) we have

$$\nu^*_{a-1}(\theta) = \frac{k}{\alpha_1 - \theta}\left[\frac{\mu(A^*(\theta) - \xi)\xi^{a-2}}{\mu - \theta - \mu\xi^b} - L_1\right]$$
$$+ \frac{C}{\alpha_1 - \theta}\left[L_2 - \frac{\mu\tau_3(A^*(\theta) - r)r^{a+b-2}}{\tau_1 - \theta}\right]. \tag{18}$$

Setting $\theta = 0$ in equation (1), we get

$$\mu\pi_0 = \omega_0(0) - q\eta C(r^{c-1} - r^b)/\tau_1. \tag{19}$$

From equation (5) setting $\theta = 0$ and using equations (13), (14)-(17), we get

$$\pi_0 = (kL_5 + CL_6)/\mu, \tag{20}$$

where

$$L_5 = L_1 + L_3 - 1 + (\xi^{a-1} - \xi^b)/(1 - \xi^b)$$
$$L_6 = L_4 - L_2 + \tau_3\big(\tau_1 r^b + \tau_2(r^{c-1} - r^b) - \mu(r^{a+b-1} - r^{2b})\big)/\tau_1.$$

From equations (19) and (20) it follows that

$$\omega_0(0) = kL_5 + CL_7, \tag{21}$$

where

$$L_7 = L_6 + q\eta(r^{c-1} - r^b)/\tau_1.$$

Setting $\theta = 0$ in equation (2) for $1 \leqslant n \leqslant a - 2$, we get

$$\omega_n(0) = k\left(L_5 + \frac{1 - \xi^n}{1 - \xi^b}\right) + C\left(L_7 + \frac{(1 - r^n)r^b}{\tau_1}(q\eta - \mu\tau_3)\right). \tag{22}$$

Inserting $\theta = 0$ in equation (2) for $n = a - 1$, $\omega_{a-1}(0)$ can be obtained as

$$\omega_{a-1}(0) = kL_8 + CL_9, \tag{23}$$

where

$$L_8 = L_5 - L_1 + \frac{1 - \xi^{a-1}}{1 - \xi^b} \quad\text{and}\quad L_9 = L_7 + L_2 + (q\eta - \mu\tau_3)(r^b - r^{a+b-1})/\tau_1.$$

Setting $\theta = 0$ in equation (3) and using equations (13) and (23), we get

$$\omega_n(0) = kL_8 + C\left(L_9 + \frac{q\eta(1 - r^{n-a+1})r^{a+b-1}}{\tau_1}\right), \quad a \leqslant n \leqslant c - 2. \tag{24}$$

Using equation (9) we have

$$kL_{10} + CL_{11} = \lambda, \tag{25}$$

where

$$L_{10} = L_1 + L_3 + (a-1)L_5 + (c-a)L_8 + \frac{1}{1-\xi} + \frac{1}{1-\xi^b} \sum_{n=1}^{a-2}(1-\xi^n),$$

$$L_{11} = L_4 - L_2 + (a-1)L_7 + (c-a)L_9 + \frac{r^b}{\tau_1}(q\eta - \mu\tau_3) \sum_{n=1}^{a-2}(1-r^n)$$

$$+\frac{q\eta r^{a+b-1}}{\tau_1} \sum_{n=a}^{c-2}(1-r^{n-a+1}) + \frac{r^{c-1}}{1-r} - \frac{\tau_3 r^b}{1-r}.$$

Setting $\theta = \phi + \alpha_2$ in (4) for $n = c-1$ and (8), and after simplification, we get

$$kL_{12} + CL_{13} = 0, \tag{26}$$

where

$$L_{12} = \alpha_2 L_3/\phi + A^*(\phi + \alpha_2)L_8,$$
$$L_{13} = A^*(\phi + \alpha_2)\left(L_9 + \frac{q\eta(1-r^{c-a-1})r^{a+b-1}}{\tau_1}\right) + \frac{q\eta(A^*(\phi+\alpha_2) - r)r^{c+b-2}}{\tau_1 - \phi - \alpha_2}$$
$$-r^{c-1} + \alpha_2 L_4/\phi.$$

Solving (25) and (26), we obtain the values of k and C as

$$k = \frac{\lambda L_{13}}{L_{10}L_{13} - L_{11}L_{12}} \quad \text{and} \quad C = \frac{\lambda L_{12}}{L_{11}L_{12} - L_{10}L_{13}}.$$

We are now in a position to obtain the pre-arrival epoch probabilities ω_n^-, π_n^-, ν_j^- from the probabilities $\omega_n(0)$, $\pi_n(0)$, $\nu_j(0)$, $n \geqslant 0$, $j = a-1, c-1$.

THEOREM 3.1 *The pre-arrival epoch queue length distributions ω_n^- that an arrival sees n customers in the queue and the server is in working vacation, π_n^- that the server is busy and ν_j^- ($j = a-1, c-1$) that the server is in change over times are given by*

$$\omega_n^- = \left[k(L_5 + \frac{1-\xi^n}{1-\xi^b}) + C(L_7 + \frac{(1-r^n)r^b(q\eta - \mu\tau_3)}{\tau_1})\right]/\lambda, \ 0 \leqslant n \leqslant a-2,$$

$$\omega_n^- = \left[kL_8 + C(L_9 + \frac{q\eta(1-r^{n-a+1})r^{a+b-1}}{\tau_1})\right]/\lambda, \ a-1 \leqslant n \leqslant c-2,$$

$$\omega_n^- = Cr^n/\lambda, \ n \geqslant c-1,$$

$$\pi_n^- = \left[k\xi^n - C\tau_3 r^{n+b}\right]/\lambda, \ n \geqslant 0,$$

$$\nu_{a-1}^- = \left[kL_1 - CL_2\right]/\lambda,$$

$$\nu_{c-1}^- = \left[kL_3 + CL_4\right]/\lambda.$$

Proof Using (10) in (12), (14), (16), (17) and (21) - (24), we obtain the result of the theorem. ∎

3.2 *Steady state distribution at arbitrary epochs*

To obtain the queue length distribution at arbitrary epochs, we develop the relations between distributions of number of customers in the queue at pre-arrival and arbitrary epochs. This is discussed in the following theorem.

THEOREM 3.2 *The arbitrary epoch probabilities are given by*

$$\omega_n = Cr^{n-1}(1-r)/\tau_1, \quad n \geqslant c, \tag{27}$$

$$\pi_n = \frac{k\xi^{n-1}(1-\xi)}{\mu(1-\xi^b)} - \frac{C\tau_3(1-r)r^{n+b-1}}{\tau_1}, \quad n \geqslant 1, \tag{28}$$

$$\nu_{a-1} = kL_{14} + CL_{15}, \tag{29}$$

where

$$L_{14} = \frac{1}{\alpha_1}\Big(\frac{(1-\xi)\xi^{a-2}}{(1-\xi^b)} - L_1\Big) \quad and \quad L_{15} = \frac{1}{\alpha_1}\Big(L_2 - \frac{\mu\tau_3(1-r)r^{a+b-2}}{\tau_1}\Big).$$

Proof Setting $\theta = 0$ in (13), (15) and (18), we obtain the result of the theorem. ∎

One may note here that from Theorem 3.2 we can not get $\{\omega_n\}_0^{c-1}$, π_0 and ν_{c-1}. We have already obtained π_0 in equation (20) and the remaining can be obtained using the following theorem.

THEOREM 3.3 *The arbitrary epoch probabilities $\{\omega_n\}_0^{c-1}$ and ν_{c-1} are given by*

$$\omega_0 = -kL_{16} - C\Big[L_{17} + \frac{q\eta(r^{c-1} - r^b)\tau_4}{1-r}\Big], \tag{30}$$

$$\omega_n = k\Big[\frac{L_5}{\lambda} + \frac{1 - \xi^{n-1}}{\lambda(1-\xi^b)} - \xi^{n-1}\frac{\lambda(1-\xi) - \mu(1-\xi^b)}{\lambda\mu(1-\xi^b)^2} - L_{14}\psi(n = a-1)\Big]$$

$$+ C\Big[\frac{L_7}{\lambda} - L_{15}\psi(n = a-1) + \frac{r^b(q\eta - \mu\tau_3)(1 - r^{n-1})}{\lambda\tau_1}$$

$$+ r^{b+n-1}\tau_4[\mu\tau_3 - q\eta]\Big], \quad 1 \leqslant n \leqslant a-1, \tag{31}$$

$$\omega_n = \Big(1 - \frac{\phi}{\alpha_2 + \phi}\psi(n = c-1)\Big)\Big(k\Big[\frac{L_8}{\lambda} + \frac{L_3}{\alpha_2}\psi(n = c-1)\Big]$$

$$+ C\Big[L(n) + \frac{L_4}{\alpha_2}\psi(n = c-1)\Big]\Big), \quad a \leqslant n \leqslant c-1, \tag{32}$$

$$\nu_{c-1} = \frac{\phi}{\alpha_2 + \phi}\Big(k\Big[\frac{L_8}{\lambda} - \frac{L_3}{\phi}\Big] + C\Big[L(c-1) - \frac{L_4}{\phi}\Big]\Big), \tag{33}$$

where

$$\tau_4 = \frac{\lambda(1-r) - \tau_1}{\lambda\tau_1^2},$$

$$L_{16} = \frac{L_5}{\mu} - \frac{L_1}{\lambda} - \frac{L_3}{\lambda} + \frac{\mu(\xi^{a-1} - \xi^b)[\lambda(1-\xi) - \mu(1-\xi^b)]}{\lambda\mu^2(1-\xi)(1-\xi^b)^2},$$

$$L_{17} = \frac{L_6}{\mu} + \frac{L_2 - L_4}{\lambda} + \frac{\tau_3\tau_4}{1-r}\big(\tau_2(r^{c-1} - r^b) - \mu(r^{a+b-1} - r^{2b})\big),$$

$$L(n) = \frac{1}{\lambda}\Big(L_9 + \frac{q\eta r^{a+b-1}(1 - r^{n-a}(1 + \lambda\tau_1\tau_4))}{\tau_1}\Big).$$

Proof Differentiating equations (13), (15), (7) and (5), and setting $\theta = 0$ yields

$$\omega_n^{*(1)}(0) = Cr^{n-1}\Big(\frac{\lambda(1-r) - \tau_1}{\lambda\tau_1^2}\Big), \quad n \geqslant c,$$

$$\pi_n^{*(1)}(0) = k\xi^{n-1}\Big(\frac{\lambda(1-\xi) - \mu(1-\xi^b)}{\lambda\mu^2(1-\xi^b)^2}\Big) - C\tau_3 r^{n+b-1}\tau_4, \quad n \geqslant 1,$$

$$\nu_{a-1}^{*(1)}(0) = \frac{k}{\alpha_1}\Big(\frac{\xi^{a-2}[\lambda(1-\xi) - \mu(1-\xi^b)]}{\lambda\mu(1-\xi^b)^2} + L_{14}\Big) - \frac{C}{\alpha_1}\Big(\mu\tau_3 r^{a+b-2}\tau_4 - L_{15}\Big),$$

$$\mu\pi_0^{*(1)}(0) = kL_{16} + CL_{17}.$$

Differentiating (1) - (3), setting $\theta = 0$, using the above derivatives and the probabilities $\omega_n(0)$, $\pi_n(0)$, $\nu_j(0)$, $n \geqslant 0$, $j = a-1, c-1$, we get (30) - (32), respectively.

Adding and differentiating (4) for $n = c-1$ and (8) and setting $\theta = 0$, we get

$$\omega_{c-1} + \nu_{c-1} = -A^{*(1)}(0)\omega_{c-2}(0) - q\eta\omega_{c+b-1}^{*(1)}(0). \tag{34}$$

Setting $\theta = 0$ in equation (8) yields

$$\alpha_2\nu_{c-1} = \phi\omega_{c-1} - \nu_{c-1}(0). \tag{35}$$

By combining equations (34) and (35) one can obtain ω_{c-1} and ν_{c-1}. ∎

This completes the evaluation of the arbitrary epoch probabilities.

4. Special cases

The following special cases are deduced from our model by taking specific values of the parameters a, c, b, q, α_1, α_2, ϕ and η.

Case 1: $a = c = b = 1$, $q = 1$ and $\alpha_1, \alpha_2 \to \infty$, that is, the batch size is one, no change over time and no vacation interruption. The model reduces to $GI/M/1/\infty$ queue with multiple working vacations and our results match numerically with [1].

Case 2: If $q = 1$, $a = c$ and $\alpha_1, \alpha_2 \to \infty$ then the model reduces to $GI/M^{(a,b)}/1/\infty/MWV$ queue.

Case 3: $q = 1$, $a = c = b = 1$, $\phi \to \infty$, $\eta = 0$ and $\alpha_1, \alpha_2 \to \infty$ that is, the batch size is one, the average duration of working vacation is zero and no change over time. In this case, the system reduces to $GI/M/1/\infty$ queue and our results match with the results available in the literature.

5. Performance measures and cost model

Once the queue length probabilities are known, we can evaluate the various performance measures. The average queue length when the server is on working vacation (L_{qv}), the average queue length when the server is busy (L_{qb}), the average queue length when the server is in change over times (L_{qc}) and the average number of customers in the queue at an arbitrary epoch (L_q) are given by

$$L_{qv} = \sum_{n=0}^{\infty} n\omega_n, \, L_{qb} = \sum_{n=0}^{\infty} n\pi_n, \, L_{qc} = (a-1)\nu_{a-1} + (c-1)\nu_{c-1}, \, L_q = L_{qv} + L_{qb} + L_{qc}.$$

The average waiting time in the queue (W_q) of a customer using Little's rule is given by $W_q = L_q/\lambda$. The probability that the server is on working vacation (P_{wv}), the probability that the server is busy (P_b) and the probability that the server is in change over times (P_c) are respectively, given by

$$P_{wv} = \sum_{n=0}^{\infty} \omega_n^-, \quad P_b = \sum_{n=0}^{\infty} \pi_n^- \quad \text{and} \quad \mathrm{P_c} = \nu_{a-1}^- + \nu_{c-1}^-.$$

Cost model

We develop the total expected cost function per unit time with an objective to determine the optimum values of μ, ϕ, a and c so that the expected cost function is minimized. Let us define

$C_1=$ the unit time cost of every customer in the queue,

$C_2=$ the service cost per unit time when the server is in normal busy period,

$C_3=$ the service cost per unit time when the server is in working vacation,

$C_4=$ fixed cost per unit time when the server is in working vacation,

$C_5=$ fixed cost per unit time when the server is in change over time (after service completion epoch),

$C_6=$ fixed cost per unit time when the server is in change over time (after vacation completion epoch).

Let F be the total expected cost per unit time. Using the definitions of each cost element and its corresponding system characteristics, we have

$$F = C_1 L_q + C_2 \mu P_b + (C_3 \eta + C_4 \phi) P_{wv} + (C_5 \alpha_1 + C_6 \alpha_2) P_c.$$

We have considered the following optimization problems:

- Minimize $F(\phi)$ subject to the constraint $0.1 < \phi \leqslant 2$.
- Minimize $F(\mu)$ subject to the constraint $0.5 < \mu \leqslant 1.5$.
- Minimize $F(a, c)$ subject to the constraint $1 \leqslant a, c \leqslant 17$ and $a < c$.

The numerical searching approach is implemented using quadratic fit search method and genetic algorithm on the computer software Mathematica with μ, ϕ, a and c as the decision variables. We have used these two optimization techniques so as to ensure the reliability of the results and the numerical results obtained from these two techniques are very close as shown in Tables 2 and 3.

Table 1.   Optimal values (a^*, c^*) and F^* for different pairs of μ and ϕ.

	$b = 20$, $\eta = 0.2$, Erlang-3					
	(μ, ϕ)	(0.25, 2.5)	(0.3, 2.5)	(0.45, 2.5)	(0.3, 3.0)	(0.3, 3.5)
MWV	(a^*, c^*)	(13,17)	(9,11)	(3,5)	(8,10)	(6,8)
	F^*	370.49	251.50	149.57	254.92	260.40
VI	(a^*, c^*)	(13,17)	(8,10)	(3,5)	(8,10)	(7,9)
	F^*	373.16	252.94	149.71	255.65	260.54

Table 2.   The optimal values ϕ^* and F^* for various values of η.

	$q = 1$, MWV				$q = 0$, VI			
	QFSM		GA		QFSM		GA	
η	ϕ^*	F^*	ϕ^*	F^*	ϕ^*	F^*	ϕ^*	F^*
0.2	1.1646	118.60	1.1646	118.60	1.1930	119.01	1.1929	119.01
0.4	0.9326	117.53	0.9316	117.53	0.9989	118.58	0.9989	118.58
0.6	0.6862	116.10	0.6886	116.10	0.8048	118.15	0.8048	118.15
0.8	0.4146	114.00	0.4172	114.00	0.6110	117.72	0.6134	117.72
1.0	0.0810	110.15	0.1076	110.18	0.4174	117.29	0.4175	117.29

Table 3.   The optimal values μ^* and F^* for various values of λ.

	$q = 1$, MWV				$q = 0$, VI			
	QFSM		GA		QFSM		GA	
λ	μ^*	F^*	μ^*	F^*	μ^*	F^*	μ^*	F^*
1.2	0.5099	74.13	0.5101	74.13	0.5206	72.97	0.5206	72.97
1.4	0.5790	78.00	0.5791	78.00	0.5906	76.96	0.5912	76.96
1.6	0.6449	82.05	0.6440	82.05	0.6573	81.01	0.6574	81.01
1.8	0.7071	86.29	0.7072	86.29	0.7210	85.11	0.7211	85.11
2.0	0.7657	90.72	0.7635	90.72	0.7822	89.27	0.7840	89.27

6.   Numerical results

To demonstrate the applicability of the theoretical investigation made in the previous sections, we present some numerical results in the form of tables and graphs. We have considered the following cost parameters: $C_1 = 15$, $C_2 = 50$, $C_3 = 25$, $C_4 = 35$, $C_5 = 5$ and $C_6 = 8$. We have taken $a = 3$, $c = 6$, $b = 10$, $\lambda = 3.5$, $\rho = 0.2$, $\phi = 0.2$, $\eta = 0.5$, $\alpha_1 = 1.5$, $\alpha_2 = 2.5$ for all the tables and figures, unless they are considered as variables or their values are mentioned in the respective figures and tables. The optimal values (a^*, c^*) and F^* for different pairs of μ and ϕ which are obtained by employing genetic algorithm are shown in Table 1. One can see that (i) as μ increases the average cost decreases, the optimal values a^* and c^* also decrease, (ii) as ϕ increases the average cost increases, the optimal values a^* and c^* decrease.

Using QFSM and GA, the optimal values of ϕ (μ) and the minimum expected cost F^* are shown in Table 2 (3) for various values of η (λ) and for hyperexponential (exponential) inter-arrival distribution with $\mu = 0.8$ for Table 2. From Table 2 we observe that both the optimal mean vacation rate and the minimum expected cost decrease as η increases. From Table 3 one can see that as the mean arrival rate increases both the optimal mean service rate and the minimum expected cost increase. Here, one may note that the increase in the optimal service rate with λ is as expected in view of the stability of the system.

Figure 1 illustrates the influence of the minimum threshold c on the expected
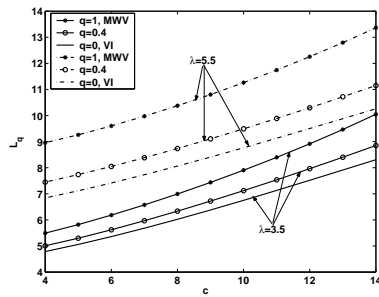
Figure 1. Effect of c on L_q.
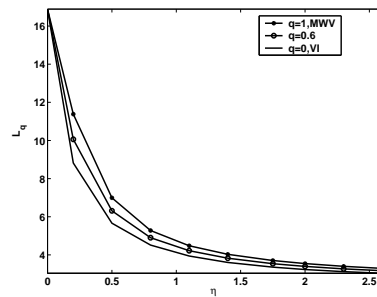
Figure 2. Impact of η on L_q.
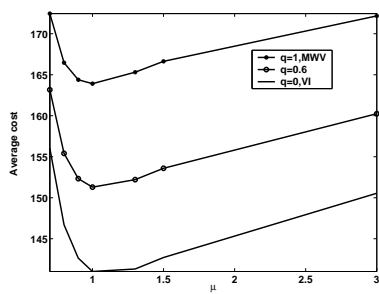
Figure 3. Average cost vs μ.
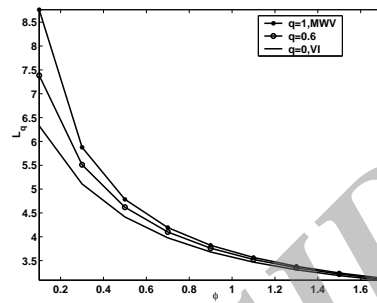
Figure 4. Impact of ϕ on L_q.

queue length for different values of λ and vacation interruption probability, q, with exponential inter-arrival distribution and $b = 15$. Here, L_q increases with c, λ and q. Since c is the minimum threshold to start service, increasing c will result in greater accumulation of customers in the queue thereby increasing L_q. From the figure one can also infer that L_q is low for the model with vacation interruption for fixed λ and c.

Figure 2 provides the expected queue length with a change of service rate in working vacation η at different q for exponential inter-arrival time distribution. As expected, L_q decreases with the increase of η and the larger the probability q is, the larger L_q becomes. That is the model with vacation interruption ($q = 0$) performs better than the model without vacation interruption ($q = 1$). Moreover, when η is zero then clearly q has no effect on the average queue length.

Figure 3 shows the average cost as a function of the mean service rate μ for different values of q with exponential inter-arrival time distribution. The figure demonstrates that there is an optimal mean service rate and the average cost increases with q.

The effect of ϕ on L_q for different values of q is presented in Figure 4 for Erlang-3 inter-arrival time distribution. We see that L_q decreases with the increase of ϕ. This is due to the fact that mean vacation time decreases and the server is available with shorter breaks.

Figures 5 and 6 depict the impact of α_1 and α_2 on W_q for multiple working vacations without and with vacation interruption models, respectively. The inter-arrival time considered is deterministic. In both the cases, W_q increases with α_1 and α_2. As α_1 (α_2) gets larger the mean duration of change over times becomes smaller so that the server will go for another vacation with out waiting for an arrival for some reasonable duration of time. This contributes for W_q to increase.

From the above numerical discussion we note the following:

In terms of L_q, the model with vacation interruption performs better. Therefore to offer a better service under the working vacation policy, one can consider vacation
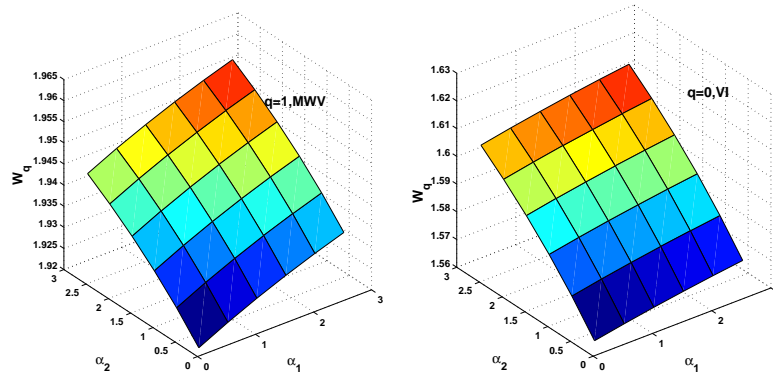
Figure 5. (α_1, α_2) vs W_q $(q = 1)$.   Figure 6. (α_1, α_2) vs W_q $(q = 0)$.

interruption policy which utilizes the server and decreases the queue size effectively.

7.   Conclusion

In this paper, we have analyzed a renewal input batch service multiple working vacations queue with change over times and Bernoulli schedule vacation interruption that has potential applications in production, manufacturing, traffic signals and telecommunication systems, etc. We have developed a recursive method, using the supplementary variable technique and treating the remaining inter-arrival time as the supplementary variable, to find the steady state queue length distributions at pre-arrival and arbitrary epochs. The recursive method is powerful and easy to implement. Some special cases of the model have been discussed. Various performance measures such as the average queue length and the average waiting time in the queue have been obtained along with a suitable cost function. The quadratic fit search method and genetic algorithm are applied to search for the optimal values of the system parameters. The method of analysis used in this paper can be applied to $GI^X/M^{(a,c,b)}/1$ and $MAP/M^{(a,c,b)}/1$ queues with multiple working vacations that are left for future investigation.

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