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Plane Wave Propagation Through a Planer Slab

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Abstract. An approximation technique is considered for computing transmission and reflection coefficients for plane waves propagating through stratified slabs.The propagation of elastic pulse through a planar slab is derived from first principles using straightforward time-dependent method. The paper ends with calculations of enhancement factor for the elastic plane wave and it is shown that it depends on the velocity ratio of the wave in two different media but not the incident wave form.The result, valid for quite arbitrary incident pulses and quite arbitrary slab inhomogeneities, agrees with that obtained by time-independent methods, but uses more elementary methods.

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1. Introduction

Wave propagation in inhomogeneous medium is a challenge for both theoretical research and engineering practice. With the rapid development of science and technology, wave motionstudy of the heterogeneous medium (atmosphere, ocean, earth-crust, functionally graded materials and cycle grid structure, etc.) seems much more important[26].Mathematically, the problem is treated by solving Helmholtz equation with variable coefficient [2], which is explored by a few scholars to try to find a generalized method applied in all cases. Meanwhile, all of parameters changed in uniaxial coordinate. In astrophysics, Gans [5] discussed phenomenon of light wave under normal incidence and

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oblique incidence conditions in continuously variable medium. It is found that there is no reflection but total reflection in geometrical optics in inhomogeneous medium and the total reflection condition is given. Epstien [4] investigated reflection wave in an inhomogeneous absorbing medium by solving wave equation with variable coefficient based on hypergeometric function. The procedure represented that the reflection is always very insignificant, except the case when conductivity is small and where we have conditions very near to total reflection, which is the same as mechanism of transmission of acoustic or electromagnetic wave in earth atmosphere. When refraction index varied with the form as parabola, the asymptotic expansions of Weber's function [25] was developed bythe method of the steepest descent. The solution of the radio wave propagation in inhomogeneous electromagnetic field was expressed in the form of the residue series. In terms of uniform of seawater, Potter and Murphy [17] employed variables separation and elliptic coordinates conversion to investigate wave equation in a medium with a particular velocity variation. The result corresponded in part to actual underwater measurements and it yielded a shadow zone as well as propagation of acoustic wave in atmosphere without acoustic wave propagation. In elastic solid medium, Caviglia and Morro [3] studied an elastic wave propagation in case that a uniaxially-inhomogeneous layer with certain thickness,sandwiched between two homogeneous half infinite spaces. Then existence and uniqueness for the solution were proved. The similar physical model has been established by Mieczyslaw C. [15]. The couple systems of ordinary differential equation for amplitudes of forward and backward waves were derived to obtain the analytical solution and explicit expressions for reflection and transition coefficient.Robins [18] discussed the Helmholtz equation for the case of horizontal stratification, both sound speed and density varying continuously with depth.

The analytical solutions to forms of sound-speed and density were outlined in terms of well-known special functions such as Bessel and Airy functions, which were capable of giving good agreement with real density and speed profiles in marine sediments. Watanabe and Payton [23] derived impulsive and time-harmonic Green's functions for SH waves in an inhomogeneous elastic solid. A critical frequency that distinguishes the wave nature of the response was found in the case of a linear velocityvariation.Rovithiset al. [19] investigated a vertical seismic wave response of inhomogeneous soil deposits over a homogeneous layer on a rigid base. The problem is treated analytically leading to a closed form analytical solution for the base-to-surface transform function. Peng and Liu [16] introduced WKBJ approximate theory to investigate dispersion relations of Love surface wave, when a vertical heterogeneous half-space with medium parameters that varied continuously was covered with a certain thickness of homogeneous and isotropic elastic medium.

Researchers had discussed the theory of plane waves such as; Sinha [20] studied the transmission of elastic waves through a homogenous layer sandwiched in homogenous media. Tooly et al., [22] discussed reflection and transmission of plane compressional waves. Gupta [8] solved the problem of Reflection of elastic waves from a linear transition layer. Agemi [1] studied the problem on the global existence of nonlinear elastic waves. Gedroit et al., [6] solved the problem of finite-amplitude elastic wave amplitude in solids. Gol'dberg [7] had taken interaction of plane longitudinal and transverse elastic waves, Johnson et al., [10] discussed the nonlinear generation of elastic waves in crystalline rock. Hughes [9] had taken the case of second-order elastic deformations in solids. Jones and Kobett [11] studied the interaction of plane elastic waves in an isotropic solid. John [12] discussed the interaction of elastic waves in an isotropic medium. Kakar and Kakar [13] studied propagation of Love waves in a non-homogeneous elastic media. Kakar and Gupta [14] also discussed propagation of Love waves in a non-homogeneous orthotropic layer under 'P' overlying semi-infinite non-homogeneous medium.

Many scientists had solved the problems of reflections and transmissions of elastic waves from interface by using typical methods [21,24], but in the present paper, we drive the solution of Navier's equations by using time-dependent methods (which describes the propagation of an elastic pulse through a planar slab of finite width). These methods are much easier than the earlier methods used. We consider first the case of homogeneous slab then inhomogeneous slab. The velocity is constant for homogeneous case but it is continuously varying for non-homogeneous case. The time-dependent methods are applied to solve the transmitted and reflected pulses.

2. Basic Concept

Consider an infinite absolutely rigid plane plate (screen/surface, which is well wielded contact with the surrounding elastic medium. Let x-y-plane coincide with the plate (where central part of the plane is shown). The z-axis is taken normal to the plate in the upward direction. As horizontal section of the interface is shown and the media are taken in the x-y-plane $(-\infty<x<\infty,\ -\infty<y<\infty)$. If we disturb the plate sufficiently rapidly in such a manner that it remains parallel to itself (plane parallel moment; horizontal plane), then at any instant of time the displacement of any point of the interface will be same. The displacement vector u_i is taken to be independent of x and y. the medium in front of the interface will of course be compressed, while behind it, on the negative z-axis will be stretched. The state will be transmitted in the medium in directions parallel to z-axis. The problem is formulated by assuming the following assumptions.

- Media are taken to be continuous at the interface due to perfect welded contact, with surrounding elastic medium, during the transmission of motion through the interface. The media do not slip relative to each other, so that at the interfaceresultant horizontal motions above and below are equal in pairs.
- The condition of the interaterrestrial contacts for the vertical motions are analogous, there can be neither exploitation nor formation if intermediately cavities at the interface during motion, then $\overline{w}_1 - \overline{w}_2 = 0$, where $\overline{w}_1$ and $\overline{w}_2$ are the resultant vertical motions in the lower and upper media respectively.

3. Governing Equations and Used Method

The equations of motion of three-dimensional elasticity

$$\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i, \tag{1}$$

The stress-strain relations (Hooke's law)

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \tag{2}$$

The strain-displacement relations (Cauchy's relations)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

Where, λ, μ are Lamb's Constant and ρ is the density of the medium, u_i are the displacement components, δ_{ij} are elastic constants, σ_{ij} are the stress tensor components,

ε_{ij} are the deformation tensor components, ε_{kk} is the trace of deformation tensor, f_i are the volume force components.

Substituting Equation (3) in Equation (1), we get

$$\sigma_{ij} = \lambda u_{i,i}\delta_{ij} + \mu(u_{i,j} + u_{j,i}), \tag{4}$$

Substituting Equation (4) and Equation (1) and simplifying, the Navier's equation of motion in terms of displacements can be obtained in the form:

$$(\lambda+\mu)u_{j,ji} + u_{i,jj} + \rho f_i = \rho \ddot{u}_i, \tag{5}$$

In vector form:

$$(\lambda+\mu)\nabla\nabla\cdot u + \mu\nabla^2 u + \rho f = \rho \ddot{u}_i, \tag{6}$$

In terms of rectangular Cartesian coordinates (6) can be written as

$$\begin{aligned}
(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + \mu\nabla^2 u + \rho f_x = \rho\frac{\partial^2 u}{\partial t^2}, \\
(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + \mu\nabla^2 v + \rho f_y = \rho\frac{\partial^2 v}{\partial t^2}, \\
(\lambda+\mu)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}\right) + \mu\nabla^2 w + \rho f_z = \rho\frac{\partial^2 w}{\partial t^2}.
\end{aligned} \tag{7}$$

where, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a Laplacian operator.

In the absence of body forces the equation of motion in vector form reduces to

$$(\lambda+\mu)\nabla\nabla\cdot u + \mu\nabla^2 u = \rho \ddot{u}_i, \tag{8}$$

The solutions of Equation (8) are given by

$$\begin{aligned}
& w = \alpha_z\left[f_1\left(t - \frac{z}{a}\right) + f_2\left(t + \frac{z}{a}\right)\right] \\
& or \\
& w = \alpha_z\left[f_1\left(z - at\right) + f_2\left(z + at\right)\right]
\end{aligned} \tag{9}$$

$$(u,v) = (\alpha_x, \alpha_y)\left[f_1\left(t - \frac{z}{b}\right) + f_2\left(t + \frac{z}{b}\right)\right] \tag{10}$$

or

$$(u,v)=(\alpha_x,\alpha_y)\left[f_1(z-bt)+f_2(z+bt)\right]$$

The first term $f_1(z-at), f_1(z-bt), f_1\left(t-\frac{z}{a}\right), f_1\left(t-\frac{z}{b}\right)$ in the above expressions represents the transmission of waves in the positive z-direction i.e. outgoing wave or advance wave and the second term $f_2(z+at)$, $f_2(z+bt)$, $f_2\left(t+\frac{z}{a}\right)$, $f_2\left(t+\frac{z}{b}\right)$ represents the transmission in the negative z-direction i.e. incoming wave or retarding wave. Here u,v,w are the components of u_i and they vary with time but they differ only in the cosine of angles made by u_i with the axis of co-ordinates $\alpha_x,\alpha_y,\alpha_z$. For sake of convenience, the coefficient of α's $(\alpha_x,\alpha_y,\alpha_z)$ are taken to be unity as they do not affect the general behavior of the field variables. Since the terms of the above solution functions are arbitrary therefore they have bounded derivatives up to second order.
In case of the present problem, the displacements are assumed as:

1. Incident Wave ; $(z=\bar{a})$ in the medium M_1 $(-\infty<z\le a,-\infty<x,y<\infty)$:

   $W_I=W_I(z-c_0t)$

   Where, c_0 is the velocity of propagation in medium M_1
2. Reflected Wave; $(z=\bar{a})$ in the medium M_1

   $W_R=W_R(z+c_0t)$
3. Transmitted Wave into the slab S $(a\le z\le b,-\infty<x,y<\infty)$:

   $W_+=W_+(z-c_1t)$

   Where, c_0 is the velocity of propagation in M_2
4. Wave reflected from the upper boundary $(z=\bar{b})$ of slab into the slab:

   $W_-=W_-(z+c_1t)$
5. Wave transmitted into the medium M_2 from slab:

   $W_T=W_T(z-c_0t)$ i.e. medium M_1 is similar to M_2

4. Case of Single-Layer Slab

We shall assume that the slab lies perpendicular throughout to the z-axis in $\Re^3$, with faces at $z=a>0$ and $z=b>a$, and is isotropic in the horizontal *x* and *y* directions for slab $'S'$ $(a\le z\le b,-\infty<x,y<\infty)$. The incident wave has finite energy and propagates in the positive *z*-direction, normal to the slab and incident from below.Under the above assumptions the problem essentially becomes one directional. The propagation velocity is c_0 outside the slab and c_1 inside the slab, where c_0 and c_1 are constants with $0<c_1<c_0$ (see Fig-1). The general form of the solution is taken as:

$$W(z,t)=\begin{cases} W_I(z-c_0t)+W_R(z+c_0t) & if\ (-\infty<z\le a)\quad for\,(M_1)\\ W_+(z-c_1t)+W_-(z+c_1t) & if\ (a\le z\le b)\quad for\,(S)\\ W_T=W_T(z-c_0t) & if\ (b\le z<\infty)\quad for\,(M_2)\end{cases} \tag{11}$$

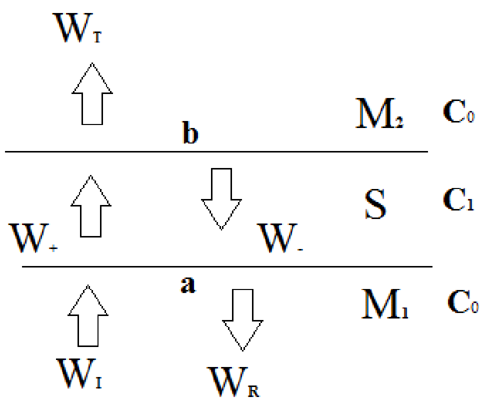

Figure 1. Single layer slab

4.1 Solution of the problem

The field variables W_R, W_+, W_- and W_T for the given value of W_I can be found from the displacement and stress-boundary conditions at the interfaces. But in this case, we have taken the coefficient of α's $(\alpha_x,\alpha_y,\alpha_z)$ equal to unity. Therefore, we apply the displacement boundary conditions coupled with travel-time of wave and using the lag in time for the waves travelling in the same direction with different velocities of propagation. At the interfaces $z=a$ and $z=b$ we assume that is $W(z,t)$ continuous at all times *t*. Therefore, at $z=a$ this leads to

$$\frac{1}{c_0}W_I(z-c_0t)+\frac{1}{c_0}W_R(z+c_0t)=\frac{1}{c_0}W_+(z-c_1t)+\frac{1}{c_0}W_-(z+c_1t) \tag{12}$$

$$\frac{1}{c_0}W_I(z-c_0t)-\frac{1}{c_0}W_R(z+c_0t)=\frac{1}{c_1}W_+(z-c_1t)-\frac{1}{c_1}W_-(z+c_1t) \tag{13}$$

Adding Equation (12) and Equation (13), we get

$$\frac{2}{c_0}W_I(z-c_0t)=\frac{c_0+c_1}{c_0c_1}W_+(z-c_1t)-\frac{c_0-c_1}{c_0c_1}W_-(z+c_1t) \tag{14}$$

or

$$W_+(z-c_1t)=\frac{2c_1}{c_0+c_1}W_I(z-c_0t)+\frac{c_0-c_1}{c_0+c_1}W_-(z+c_1t) \tag{15}$$

Subtracting Equation (12) and Equation (13), we get

$$\frac{2}{c_0}W_R\left(z+c_0t\right)=\frac{c_1-c_0}{c_0c_1}W_+\left(z-c_1t\right)+\frac{c_1+c_0}{c_0c_1}W_-\left(z+c_1t\right) \tag{16}$$

or

$$W_R\left(z+c_0t\right)=\frac{c_1-c_0}{2c_1}W_+\left(z-c_1t\right)+\frac{c_1+c_0}{2c_1}W_-\left(z+c_1t\right) \tag{17}$$

Combining Equation (15) and Equation (17), we get

$$W_R\left(z+c_0t\right)=\frac{c_0-c_1}{c_0+c_1}W_I\left(z-c_0t\right)+\frac{2c_0}{c_0+c_1}W_-\left(z+c_1t\right) \tag{18}$$

Now Equation (15) and Equation (18) must hold at all times t. therefore put $u=a-c_1t,$ then $t=\left(a-u\right)/c_1$ and Equation (15) becomes

$$W_+(u)=\frac{2c_1}{c_0+c_1}W_I\left(a-\frac{c_0}{c_1}(a-ut)\right)+\frac{c_0-c_1}{c_0+c_1}W_-(2a-u) \tag{19}$$

Since this holds for all *u,* we can put $u=z-c_1t$ and get

$$W_+(z-c_1t)=\frac{2c_1}{c_0+c_1}W_I\left(a+\frac{c_0}{c_1}(z-a-c_1t)\right)+\frac{c_0-c_1}{c_0+c_1}W_-(a-z+c_1t). \tag{20}$$

Similarly if we put $v=a+c_0t,$ then $t=\left(v-a\right)/c_0$ and Equation (18) becomes

$$W_R(v)=\frac{c_0-c_1}{c_0+c_1}W_I(2a-v)+\frac{2c_0}{c_0+c_1}W_-\left(a-\frac{c_1}{c_0}(a-v)\right) \tag{21}$$

When $v=z+c_0t$, we get

$$W_R(z+c_0t)=\frac{c_0-c_1}{c_0+c_1}W_I(2a-z-c_0t)+\frac{2c_0}{c_0+c_1}W_-\left(a+\frac{c_1}{c_0}(z-a-c_0t)\right) \tag{22}$$

Equation (20) and Equation (22) give W_R and W_+ in terms of W_I and W_- It must be noted that all the above relations are hold for values of z and t.

Similarly, at the other interface $z=b,$ we get

$$W_-(z+c_1t)=\frac{c_0-c_1}{c_0+c_1}W_+(2b-z-c_1t), \tag{23}$$

$$W_T(z-c_0t)=\frac{2c_0}{c_0+c_1}W_+\left(b+\frac{c_1}{c_0}(z-b-c_0t)\right) \tag{24}$$

giving W_- and W_T in terms of W_+ for all z and t.
Now if we combine Equation (20) and Equation (23) we get

$$W_+(z-c_1t)=W_0(z-c_1t)+\left(\frac{c_0-c_1}{c_0+c_1}\right)^2 W_+(2b-2a+z-c_1t), \tag{25}$$

where,

$$W_0(z-c_1t)=\frac{2c_1}{c_0+c_1}W_I\left(a+\frac{c_0}{c_1}(z-a-c_1t)\right). \tag{26}$$

Equation (25) can be solved for W_+ by iteration, we get

$$=\sum_{n=0}^{\infty}\left(\frac{c_0-c_1}{c_0+c_1}\right)^{2n} W_0(2n(b-a)+z-c_1t) \tag{27}$$

$$=\frac{2c_1}{c_0+c_1}\sum_{n=0}^{\infty}\left(\frac{c_0-c_1}{c_0+c_1}\right)^{2n} W_I\left(a+2n\frac{c_0}{c_1}(b-a)+\frac{c_0}{c_1}(z-a-c_1t)\right). \tag{28}$$

Also, Equation (23) can be solved for W_- by iteration, we get

$$W_-(z+c_1t)=\left(\frac{c_0-c_1}{c_0+c_1}\right)W_+(2b-z-c_1t)$$
$$=\sum_{n=0}^{\infty}\left(\frac{c_0-c_1}{c_0+c_1}\right)^{2n+1} W_0(2n(b-a)+2b-z-c_1t). \tag{29}$$

Using, Equation (27) and Equation (28) to find W_T from Equation (19) and W_R from Equation (17).

Discussion

1. If the incident wave is W_I bounded, then W_0 in the Equation (26) is also bounded, and hence series in Equation (27) and in Equation (28) are convergent.
2. If the incident wave W_I is a periodic having time period $(2b-2a)/c_0$ then W_0 will also be periodic with time period $(2b-2a)/c_1$. Hence, Equation (27) reduces to

$$W_+(z-c_1t)=\sum_{n=0}^{\infty}\left(\frac{c_0-c_1}{c_0+c_1}\right)^{2n}W_0(z-c_1t)$$

$$\Rightarrow W_+(z-c_1t)=\frac{(c_0+c_1)^2}{4c_0c_1}W_0(z-c_1t)$$

or

$$W_+(z-c_1t)=\frac{c_0+c_1}{2c_0}W_I\left(a+\frac{c_0}{c_1}(z-a-c_1t)\right) \tag{30}$$

The factor $\frac{(c_0+c_1)^2}{4c_0c_1}$ in Equation (29) is called an amplitude enhancement factor. The enhancement factor depends on the ratio of c_1/c_0 but not the incident wave form. Hence, enhancement factor can be written as

$$\xi=\frac{(c_0+c_1)^2}{4c_0c_1}=\frac{(1+\eta)^2}{4\eta}$$

Where, $\eta=\frac{c_1}{c_0}$

As, $\eta=\frac{c_1}{c_0}$ increases, the amplitude enhancement factor decreases and vice-versa.

Using Equation (29) and Equation (24), the transmitted wave is

$$W_T(z-c_0t)=\frac{2c_0}{c_0+c_1}\frac{(c_0+c_1)^2}{4c_0c_1}W_0(z-c_1t)=W_I\left(a+\frac{c_0}{c_1}(b-a)+z-c_0t)\right) \tag{31}$$

Using Equation (29) and Equation (22), the reflected wave is

$$W_R(z+c_0t)=\frac{c_0-c_1}{c_0+c_1}\left(W_I(2a-z-c_0t)-\frac{4c_0c_1}{(c_0+c_1)^2}\frac{(c_0+c_1)^2}{4c_0c_1}W_I(2a-z-c_0t)\right)=0 \tag{32}$$

We observe that the transmitted wave has the same amplitude as that of the incident wave but it lags in time due to the width of the slab. The amplitude of the reflected wave is zero. This means that the slab is transparent to any pulse train with resonant time period.

5. Case of Multiple-Layer Slab

We now consider a multiple-layer slab having n layers with interfaces a_i ,

$0 < a_0 < a_1 < \cdots\cdots\cdots\cdots < a_n$ and propagation velocity c_j in the j^{th} layer.

The general form of the solution is taken as:

$$W(z,t) = \begin{cases} W_I(z - c_0 t) + W_R(z + c_0 t) & if\,(-\infty < z \le a_0) \\ W_+^j(z - c_j t) + W_-^j(z + c_j t) & if\,(a_{j-1} \le z \le a_j) \quad for\,(M_2) \\ W_T = W_T(z - c_{n+1} t) & if\,(a_n \le z < \infty) \end{cases} \tag{33}$$

5.1. Solution of the problem

The solution has been found in the same way as it is done in the previous case for single layer; here we just piece together the solutions of previous section. Now from Equation (20) we have

$$W_+^{(k)}(z - c_k t) = \frac{2c_k}{c_k + c_{k-1}} W_+^{(k-1)}\left(a_{k-1} + \frac{c_{k-1}}{c_k}(z - a_{k-1} - c_k t)\right) + \frac{c_k - c_{k-1}}{c_k + c_{k-1}} W_-^{(k)}(a_{k-1} - (z - a_{k-1} + c_k t)). \tag{34}$$

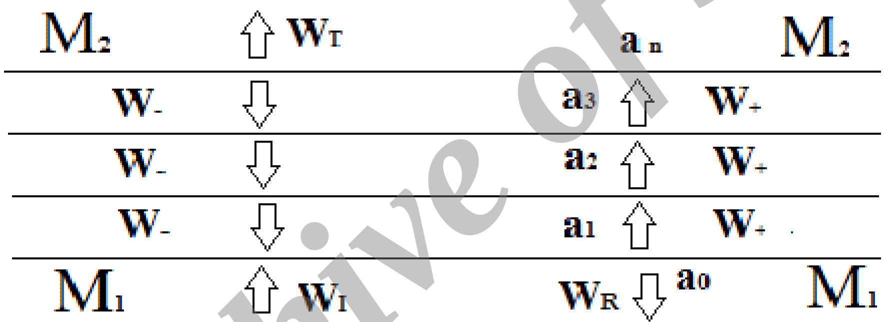

Figure2.Multiple layers slab

A similar expression gives $W_+^{(k-1)}$ in terms of $W_+^{(k-2)}$ and $W_-^{(k-1)}$ if we combine the expressions obtained for $W_+^{(j)}$ $(1 \le j \le m)$ and set $W_+^{(0)} = W_I$, we get

$$W_+^{(m)}(z - c_m t) = \prod_{i=0}^{m-1}\left(\frac{2c_{i+1}}{c_{i+1} + c_i}\right) W_I\left(a_0 + \sum_{i=0}^{m-2} \frac{c_0}{c_{i+1}} \Delta a_i + \frac{c_0}{c_m}(z - a_{m-1} + c_m t)\right)$$
$$-\sum_{j=1}^{m}\left(\prod_{i=j}^{m-1} \frac{2c_{i+1}}{c_{i+1} + c_i}\right)\left(\frac{\Delta c_{j-i}}{c_j - c_{j-i}}\right) \times W_-^{(j)}\left(a_{j-1} - \sum_{i=j}^{m-1} \frac{c_j}{c_i} \Delta a_i - \frac{c_j}{c_m}(z - a_{m-1} + c_m t)\right). \tag{35}$$

Here we have $\Delta a_j = a_{j+1} - a_j$ and $\Delta c_j = c_{j+1} - c_j$ and for simplicity we put $\Sigma_{l=m}^{m-i} = 0$, In the same way we have

$$W_-^{(k)}(z+c_k t)=\frac{2c_k}{c_k+c_{k-1}}W_-^{(k-1)}\left(a_k+\frac{c_{k+1}}{c_k}(z-a_k+c_k t)\right)$$
$$-\frac{c_{k+1}-c_k}{c_{k+1}+c_k}W_+^{(k)}(a_k-(z-a_k+c_k t)). \tag{36}$$

Similar expression gives $W_-^{(j)}$ in terms of $W_-^{(j+1)}$ and $W_-^{(k-1)}$ for $W_+^{(j)}$ $(k\le j\le n)$ and combining these expressions and setting $W_-^{(n+1)}=0$, we get

$$W_-^{(k)}(z-c_k t)=\sum_{j=k}^{n}\left(\prod_{j=k}^{j-1}\frac{2c_i}{c_{i+1}+c_i}\right)\left(\frac{\Delta c_j}{c_{j+1}+c_j}\right)$$
$$\times W_-^{(j)}\left(a_j+\sum_{i=k}^{k-1}\frac{c_j}{c_i}\Delta a_i-\frac{c_j}{c_k}(z-a_k+c_k t)\right). \tag{37}$$

Put Equation (36) and Equation (29), we get

$$W_+^{(m)}(z-c_m t)=C_{m-1}W_I\left(a_j+\sum_{i=0}^{m-2}\frac{c_0}{c_{i+1}}\Delta a_i+\frac{c_0}{c_m}(z-a_{m-1}+c_m t)\right)$$
$$-\sum_{k=1}^{m}\sum_{j=k}^{n}(C_{m-1}/C_{k-1})(D_{j-1}/D_{k-1})\left(\frac{\Delta c_{k-i}}{c_k+c_{k-i}}\right)\left(\frac{\Delta c_j}{c_{j+1}+c_j}\right)$$
$$\times W_+^{(j)}\left(a_j+\sum_{i=k+1}^{j-1}\frac{c_j}{c_i}\Delta a_j+\sum_{i=k}^{m-1}\frac{c_j}{c_i}\Delta a_j+\frac{c_j}{c_m}(z-a_{m-1}+c_m t)\right). \tag{38}$$

Where,

$$C_{j-1}=\prod_{i=0}^{j-1}\frac{2c_{i+1}}{c_{i+1}+c_i}\ \text{and}\ D_{j-1}=\prod_{i=0}^{j-1}\frac{2c_{i+1}}{c_{i+1}+c_i} \tag{39}$$

Equation (38) can be solved by iteration, as we did for Equation (25), the solution is very complicated therefore for sake of convenience, we developed, as for Equation (27), the series solution is

$$W_+^{(m)}(z-c_m t)=\sum_{p=0}^{\infty}W_{2p}^{(m)}(z-c_m t) \tag{40}$$

Where,

$$W_0^{(m)}(z-c_m t)=C_{m-1}W_I\left(a_0+\sum_{i=0}^{m-2}\frac{c_0}{c_{i+1}}\Delta a_i+\frac{c_0}{c_m}(z-a_{m-1}+c_m t)\right) \tag{41}$$

there is no reflection at the interface, and

$$W_{2p}^{(m)}(z-c_m t) = -\sum_{k=1}^{m}\sum_{j=k}^{n}(C_{m-1}/C_{k-1})(D_{j-1}/D_{k-1})\left(\frac{\Delta c_{k-i}}{c_k + c_{k-i}}\right)\left(\frac{\Delta c_j}{c_{j+1}+c_j}\right)$$

$$\times W_{2p}^{(m)}\left(a_j + \sum_{i=k+1}^{j-1}\frac{c_j}{c_i}\Delta a_j + \sum_{i=k}^{m-1}\frac{c_j}{c_i}\Delta a_j + \frac{c_j}{c_m}(z-a_{m-1}+c_m t)\right).D_{j-1} = \prod_{i=0}^{j-1}\frac{2c_{i+1}}{c_{i+1}+c_i} \quad (42)$$

involves 2p reflections at interfaces within the slab (see fig. 2)

W_R and W_T are calculated as done in Equation (22) and Equation (24)

$$W_R(z+c_0 t) = \frac{\Delta c_0}{c_0+c_1}W_I(2a-z-c_0 t) + \frac{2c_0}{c_0+c_1}W_-\left(a + \frac{c_1}{c_0}(z-a-c_0 t)\right). \quad (43)$$

$$W_T(z-c_{n+1}t) = \frac{2c_{n+1}}{c_{n+1}+c_n}W_+\left(b + \frac{c_{n+1}}{c_{n+1}+c_n}(z-b-c_{n+1}t)\right). \quad (44)$$

Discussion

If the incident wave W_I is a periodic having time period $(2\Delta a_j)/c_0$ then j^th layer is resonant, and will appear transparent to the waveforms W_+^{j-1} and W_-^{j+1}. The delay in each pulse time is

$$a_0 + \sum(2\Delta a_j)/c_j \quad (45)$$

6. Case of Continuous Slab

Finally, we take the case of continuous slab in which the wave velocity varies continuously and differentially across the slab.

$$c = c(z)\begin{vmatrix} c(a) & if\ (-\infty < z \le a) \\ c(z) & if\ (a \le z \le b) \\ c(b) & if\ (b \le z < \infty) \end{vmatrix} \quad (46)$$

6.1. Solution of the Problem

This case be treated as the limiting case of multiple slab of preceding section and it can be solved by replacing a_i by z, and let $n \to \infty, \Delta a_i dz, \Delta c_i/\Delta a_i \to \frac{dc(z)}{dz}$ but $a = a_0$ and $b = a_n$ remain constant. Therefore for limiting case,

$$\sum_{i=0}^{m-1}\frac{c_0}{c_i}\Delta a_i \to \int_a^z \frac{c(a)}{c(z)}dz \quad (47)$$

$$\frac{\Delta c_i / \Delta a_i}{c_{i+1} + c_i} \rightarrow \frac{c'(z)}{2c(z)} \tag{48}$$

Also,

$$\frac{2c_{i+1}}{c_{i+1} + c_i} = \left(1 - \frac{\Delta c_j}{2c_{i+1}}\right)^{-1} \tag{49}$$

Hence,

$$(C_{m-1})^{-1} = \left(\prod_{i=0}^{m-1} \frac{2c_{i+1}}{c_{i+1} + c_i}\right)^{-1} = \prod_{i=0}^{m-1} \left(1 - \frac{1}{2c_{i+1}} \frac{\Delta c_i}{\Delta a_i} \Delta a_i\right)$$
$$= 1 - \sum_{i=0}^{m-1} \frac{1}{2c_{i+1}} \Delta a_i + \sum_{i=0}^{m-1} \sum_{j=0}^{i-1} \left(\frac{1}{2c_{i+1}} \frac{\Delta c_i}{\Delta a_i} \Delta a_i\right)\left(\frac{1}{2c_{i+1}} \frac{\Delta c_j}{\Delta a_j} \Delta a_j\right) \tag{50}$$

For $n \rightarrow \infty$ Equation (50) reduces to

$$(C_{m-1})^{-1} \rightarrow 1 - \int_a^z \frac{c'(u)}{2c(u)} du + \int_a^z \int_a^u \frac{c'(u)}{2c(u)} \frac{c'(v)}{2c(v)} dv du - \cdots$$
$$= \exp\left(-\frac{1}{2} \int_a^z \frac{c'(u)}{2c(u)} du\right) = \exp\left(-\frac{1}{2}(\log c(z) - \log c(a))\right) = \left(\frac{c(a)}{c(z)}\right)^{1/2} \tag{51}$$

Similarly we can find that

$$\frac{2c_{i+1}}{c_{i+1} + c_i} = \left(1 + \frac{\Delta c_j}{2c_{i+1}}\right)^{-1} \tag{52}$$

From Equation (51), it follows that

$$(D_{m-1})^{-1} \rightarrow \left(\frac{c(a)}{c(z)}\right)^{1/2} \tag{53}$$

Therefore, Equation (34) becomes by using Equation (52) and Equation (53)

$$W_+(z - c(z)t) = \left(\frac{c(a)}{c(z)}\right)^{1/2} W_I\left(a + \int_a^z \frac{c(u)}{c(u)} du - c(a)t\right)$$
$$-\int_a^z \left(\frac{c(z)}{c(y)}\right)^{1/2} \frac{c'(y)}{2c(y)} du W_-\left(y - \int_y^z \frac{c(y)}{c(u)} du + c(y)t\right) dy \tag{54}$$

And Equation (36) becomes by using Equation (52) and Equation (53)

$$W_-(y - c(y)t) = +\int_y^b \left(\frac{c(y)}{c(x)}\right)^{1/2} \frac{c'(x)}{2c(x)} du W_+\left(x + \int_y^x \frac{c(x)}{c(u)} du - c(x)t\right) dx \tag{55}$$

Comparing Equation (54) and Equation (55)

$$W_+(z-c(z)t)=\left(\frac{c(a)}{c(z)}\right)^{1/2}W_I\left(a+\int_a^z\frac{c(u)}{c(u)}du-c(a)t\right)$$
$$-\int_a^z\int_y^b\left(\frac{c(z)}{c(x)}\right)^{1/2}\frac{c^{/}(y)}{2c(y)}\frac{c^{/}(x)}{2c(x)}\times W_+\left(x+\int_y^x\frac{c(x)}{c(u)}du+\int_y^z\frac{c(x)}{c(v)}du-c(x)t\right)dxdy \tag{56}$$

Equation (56) can be solved for W_+ by iteration. Hence, we have

$$W_+(z-c(z)t)=\sum_{p=0}^{\infty}W_{2p}(z-c(z)t)\left(\frac{c(a)}{c(z)}\right)^{1/2} \tag{57}$$

Where,

$$W_0(z-c(z)t)=\left(\frac{c(a)}{c(z)}\right)^{1/2}W_I\left(a+\int_a^z\frac{c(u)}{c(u)}du-c(a)t\right) \tag{58}$$

involves no reflections, and

$$\sum_{p=0}^{\infty}W_{2p}(z-c(z)t)=-\int_a^z\int_y^b\left(\frac{c(z)}{c(x)}\right)^{1/2}\frac{c^{/}(y)}{2c(y)}\frac{c^{/}(x)}{2c(x)}$$
$$\times W_{2p-2}\left(x+\int_y^x\frac{c(x)}{c(u)}du+\int_y^z\frac{c(x)}{c(v)}du-c(x)t\right)dxdy \tag{59}$$

involves 2p reflections.

Equation (43) and Equation (44) gives

$$W_R(z+c(a)t)=W_-(z+c(a)t) \qquad if \quad z\le a \tag{60}$$

$$W_T(z-c(a)t)=W_+(z-c(a)t) \qquad if \quad z\ge b \tag{61}$$

7. Conclusion

We observe that the transmitted wave has the same amplitude as that of the incident wave but it lags in time due to the width of the slab. The amplitude of the reflected wave is zero. This means that the slab is transparent to any pulse train with resonant time period. The time dependent method is much easier than other methods.
The incident wave W_I is a periodic having time period $(2\Delta a_j)/c_0$ for multiple slab. The j^{th} layer is resonant in the multiple slab, and will appear transparent to the waveforms W_+^{j-1} and W_-^{j+1} . The delay in each pulse time is $a_0+\sum(2\Delta a_j)/c_j$.

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