

$(\delta, \gamma, 2)$ -Bessel Lipschitz Functions in the Space $L_{2,\alpha}(\mathbb{R}^+)$

M. El Hamma^{a,*} and R. Daher^b

^aDepartment of Mathematics, Faculty of sciences Ain Chock, University of Hassan II,
Casablanca, Morocco;

^bDepartment of Mathematics, Faculty of sciences Ain Chock, University of Hassan II,
Casablanca, Morocco.

Abstract. Using a generalized translation operator, we obtain a generalization of Theorem 5 in [4] for the Bessel transform for functions satisfying the $(\delta, \gamma, 2)$ -Bessel Lipschitz condition in $L_{2,\alpha}(\mathbb{R}^+)$.

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Index to information contained in this paper

- 1 Introduction and Preliminaries
- 2 Main Results
- 3 Conclusion

1. Introduction and Preliminaries

Integral transforms and their inverses are widely used solve various in calculus, mechanics, mathematical, physics, and computational mathematics (see, e.g., [7, 9, 10]).

In [4], we proved theorem related the Bessel transform and (k, γ) -Bessel Lipschitz functions. In this paper, we prove a generalization of this theorem for this transform in the space $L_{2,\alpha}(\mathbb{R}^+)$. For this purpose, we use a generalized translation operator.

Assume that $L_{2,\alpha} = L_{2,\alpha}(\mathbb{R}^+)$, $\alpha > -\frac{1}{2}$, is the Hilbert space of measurable functions $f(x)$ on \mathbb{R}^+ with the finite norm

$$\|f\|_{2,\alpha} = \left(\int_0^\infty |f(x)|^2 x^{2\alpha+1} dx \right)^{1/2}.$$

*Corresponding author. Email: m.elhamma@yahoo.fr.

Let

$$B = \frac{d^2}{dt^2} + \frac{(2\alpha + 1)}{t} \frac{d}{dt},$$

be the Bessel differential operator.

For $\alpha > -\frac{1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(x) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n + \alpha + 1)},$$

where $\Gamma(x)$ is the gamma-function (see [6]).

LEMMA 1.1 [1] *The following inequalities are valid for Bessel function j_α*

- (1) $|j_\alpha(x)| \leq 1$
- (2) $1 - j_\alpha(x) = O(x^2)$; $0 \leq x \leq 1$.

LEMMA 1.2 *The following inequality is true*

$$|1 - j_\alpha(x)| \geq c,$$

with $x \geq 1$, where $c > 0$ is a certain constant.

Proof Analog of Lemma 2.9 in [3] ■

The Bessel transform of a function $f \in L_{2,\alpha}$ is defined (see [5, 6, 8]) by the formula

$$\hat{f}(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt; \quad \lambda \in \mathbb{R}^+.$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^\alpha \Gamma(\alpha + 1))^{-2} \int_0^\infty \hat{f}(\lambda) j_\alpha(\lambda t) \lambda^{2\alpha+1} d\lambda.$$

From [5], we have the Parseval's identity

$$\|\hat{f}\|_{2,\alpha} = (2^\alpha \Gamma(\alpha + 1)) \|f\|_{2,\alpha}.$$

In $L_{2,\alpha}$, consider the generalized translation operator T_h defined by

$$T_h f(t) = c_\alpha \int_0^\pi f(\sqrt{t^2 + h^2 - 2th \cos \varphi}) \sin^{2\alpha} \varphi d\varphi,$$

where

$$c_\alpha = \left(\int_0^\pi \sin^{2\alpha} \varphi d\varphi \right)^{-1} = \frac{\Gamma(\alpha + 1)}{\Gamma(1/2)\Gamma(\alpha + \frac{1}{2})}$$

From [2], we have

$$\widehat{(\mathbb{T}_h f)}(\lambda) = j_\alpha(\lambda h) \widehat{f}(\lambda) \quad (1)$$

We note the important property of the Bessel transform: If $f \in L_{2,\alpha}$ then

$$\widehat{\mathbb{B}f}(\lambda) = (-\lambda^2) \widehat{f}(\lambda). \quad (2)$$

The finite differences of the first and higher orders are defined as follows

$$\Delta_h f(x) = \mathbb{T}_h f(x) - f(x) = (\mathbb{T}_h - \mathbb{E}_{2,\alpha}) f(x),$$

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (\mathbb{T}_h - \mathbb{E}_{2,\alpha})^k f(x) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \mathbb{T}_h^i f(x), \quad (3)$$

where $\mathbb{T}_h^0 f(x) = f(x)$, $\mathbb{T}_h^i f(x) = \mathbb{T}_h(\mathbb{T}_h^{i-1} f(x))$; $i = 1, 2, \dots, k$; $k = 1, 2, \dots$ and $\mathbb{E}_{2,\alpha}$ is a unit operator in $L_{2,\alpha}$.

Let $W_{2,\alpha}^k$ be the Sobolev space constructed by the Bessel operator \mathbb{B} , i.e.,

$$W_{2,\alpha}^k = \{f \in L_{2,\alpha}, \mathbb{B}^j f \in L_{2,\alpha}; j = 1, 2, \dots, k\}.$$

In [4], we have the following result

THEOREM 1.3 *Let $f \in L_{2,\alpha}$. Then the following are equivalent*

- (1) $f \in Lip(k, \gamma, 2)$, $0 < k < 1$, $\gamma \geq 0$,
- (2) $\int_r^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O(r^{-2k} (\log r)^{2\gamma})$ as $r \rightarrow +\infty$,

where

$$Lip(k, \gamma, 2) = \{f \in L_{2,\alpha}, \|\mathbb{T}_h f(t) - f(t)\|_{2,\alpha} = O\left(\frac{h^k}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \rightarrow 0\}.$$

The main aim of this paper is to establish a generalization of Theorem 1.3

2. Main Results

In this section we present the main result of this paper. We first need to define the $(\delta, \gamma, 2)$ -Bessel Lipschitz class.

DEFINITION 2.1 Let $0 < \delta < 1$, $\gamma \geq 0$ and $r = 0, 1, \dots, k$. A function $f \in W_{2,\alpha}^k$ is said to be in the $(\delta, \gamma, 2)$ -Bessel Lipschitz class, denoted by $Lip(\delta, \gamma, 2)$; if

$$\|\Delta_h^k B^r f(x)\|_{2,\alpha} = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \rightarrow 0.$$

LEMMA 2.2 Let $f \in W_{2,\alpha}^k$. Then

$$\|\Delta_h^k B^r f(x)\|_{2,\alpha}^2 = \frac{1}{(2^\alpha \Gamma(\alpha + 1))^2} \int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda,$$

where $r = 0, 1, \dots, k$

Proof From formula (2), we have

$$\widehat{B^r f}(\lambda) = (-1)^r \lambda^{2r} \widehat{f}(\lambda) \quad (4)$$

By formulas (1) and (4), we conclude that

$$\widehat{T_h^i B^r f}(\lambda) = (-1)^r t^{2r} j_\alpha^i(th) \widehat{f}(\lambda), 1 \leq i \leq k. \quad (5)$$

From formulas (3) and (5) follows that the Bessel transform of $\Delta_h^k B^r f(x)$ is $(-1)^r \lambda^{2r} (j_\alpha(\lambda h) - 1)^k \widehat{f}(\lambda)$.

By Parseval's identity we have the result. ■

THEOREM 2.3 Let $f \in W_{2,\alpha}^k$. Then the followings are equivalent

- (1) $f \in Lip(\delta, \gamma, 2)$,
- (2) $\int_s^\infty \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right)$ as $s \rightarrow +\infty$.

Proof 1) \implies 2) Assume that $f \in Lip(\delta, \gamma, 2)$. Then we have from Lemma 2.2

$$\|\Delta_h^k B^r f(x)\|_{2,\alpha}^2 = \frac{1}{(2^\alpha \Gamma(\alpha + 1))^2} \int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda,$$

If $\lambda \in [\frac{1}{h}, \frac{2}{h}]$ then $\lambda h \geq 1$ and Lemma 1.2 implies that

$$1 \leq \frac{1}{c^{4k}} |1 - j_\alpha(\lambda h)|^{2k}.$$

Then

$$\begin{aligned} \int_{1/h}^{2/h} \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda &\leq \frac{1}{c^{4k}} \int_{1/h}^{2/h} \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &\leq \frac{1}{c^{4k}} \int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &= O\left(\frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}}\right). \end{aligned}$$

We have

$$\int_s^{2s} \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \leq C \frac{s^{-2\delta}}{(\log s)^{2\gamma}},$$

where C is a positive constant.

So that

$$\begin{aligned} \int_s^\infty \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda &= \left[\int_s^{2s} + \int_{2s}^{4s} + \int_{4s}^{8s} + \dots \right] |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &\leq C \frac{s^{-2\delta}}{(\log s)^{2\gamma}} + C \frac{(2s)^{-2\delta}}{(\log 2s)^{2\gamma}} + C \frac{(4s)^{-2\delta}}{(\log 4s)^{2\gamma}} + \dots \\ &\leq C \frac{s^{-2\delta}}{(\log s)^{2\gamma}} \left(1 + 2^{-2\delta} + (2^{-2\delta})^2 + (2^{-2\delta})^3 + \dots \right) \\ &\leq CC_\delta \frac{s^{-2\delta}}{(\log s)^{2\gamma}}, \end{aligned}$$

where $C_\delta = (1 - 2^{-2\delta})^{-1}$ since $2^{-2\delta} < 1$.

This proves that

$$\int_s^\infty \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right) \text{ as } s \rightarrow +\infty$$

2) \implies 1) Suppose now that

$$\int_s^\infty \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right) \text{ as } s \rightarrow +\infty$$

We write

$$\int_0^\infty \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = I_1 + I_2$$

where

$$I_1 = \int_0^{1/h} \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda,$$

and

$$I_2 = \int_{1/h}^{\infty} \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

Estimate the summands I_1 and I_2 from above. It follows from the formula $|j_\alpha(\lambda h)| \leq 1$ that

$$\begin{aligned} I_2 &= \int_{1/h}^{\infty} \lambda^{4r} |1 - j_\alpha(\lambda h)|^{2k} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &\leq 4^k \int_{1/h}^{\infty} \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda \\ &= O\left(\frac{h^{2\delta}}{(\log \frac{1}{h})^{2\gamma}}\right) \end{aligned}$$

To estimate I_1 , we use the inequality (2) of Lemma 1.1
Set

$$\psi(x) = \int_x^{\infty} \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda.$$

Using integration by parts, we obtain

$$\begin{aligned} I_1 &\leq -C_1 h^{4k} \int_0^{1/h} t^{4k} \psi'(t) dt \\ &\leq C_1 \psi\left(\frac{1}{h}\right) + 4C_1 k h^{4k} \int_0^{1/h} t^{4k-1} \psi(t) dt \\ &\leq C_2 h^{4k} \int_0^{1/h} t^{4k-1} t^{-2\delta} (\log t)^{-2\gamma} dt \\ &\leq C_3 h^{-2\delta} (\log \frac{1}{h})^{-2\gamma}, \end{aligned}$$

where C_1, C_2 and C_3 are positive constants and this ends the proof. ■

COROLLARY 2.4 Let $f \in W_{2,\alpha}^k$, and let

$$f \in Lip(\delta, \gamma, 2)$$

Then

$$\int_s^\infty |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{s^{-2\delta-4r}}{(\log s)^{2\gamma}}\right) \text{ as } s \longrightarrow +\infty$$

3. Conclusion

In this work we have succeeded to generalise the theorem 5 in [4] for the Bessel transform in the Sobolev space $W_{2,\alpha}^k$ constructed by the Bessel operator B . We proved that $f \in Lip(\delta, \gamma, 2)$ if and only if $\int_s^{+\infty} \lambda^{4r} |\widehat{f}(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = O\left(\frac{s^{-2\delta}}{(\log s)^{2\gamma}}\right)$ as $s \longrightarrow +\infty$.

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