

## Tension Trigonometric Splines Interpolation Method for Solving a Linear Boundary Value Problem

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**Abstract.** By using the trigonometric uniform splines of order 3 with a real tension factor, a numerical method is developed for solving a linear second order boundary value problems (2BVP) with Dirichlet, Neumann and Cauchy types boundary conditions. The moment at the knots is approximated by central finite-difference method. The order of convergence of the method and the theory is illustrated by solving test examples. Experimental results demonstrate that our method is more effective for the problems where the exact solution is trigonometric or hyperbolic.

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### 1. Introduction

In this paper, where the second derivative is approximated by the three-point central difference scheme, and where we need to build an approximate equation for each type of boundary condition, we study a method based on the trigonometric B-splines of order 3 (lower order) with a tension factor  $\rho$  for constructing numerical solutions to second-order boundary value problems (2BVPs) of the form:

$$y^{(2)}(\theta) + f(\theta)y'(\theta) + g(\theta)y(\theta) = p(\theta), \quad (1)$$

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subject to the three boundary conditions:

$$\text{Dirichlet : } y(a) = a_0, \quad y(b) = b_0, \quad (2)$$

$$\text{Neumann : } y'(a) = a_1, \quad y'(b) = b_1, \quad (3)$$

$$\text{Cauchy : } y(a) = a_0, \quad y'(b) = b_1, \quad (4)$$

where  $f(\theta)$ ,  $g(\theta)$  and  $p(\theta)$  are given continuous functions defined in the bounded interval  $[a, b]$ ,  $a_i$  and  $b_i$ ,  $i = 0, 1$  are real constants.

In many engineering applications including the study of beam deflections, heat flow, and various dynamic systems, we need solving the second order two-point boundary value problems. Much attention have been given to solve the second-order boundary value problems (2VBP) with Dirichlet, Neumann and Cauchy types boundary conditions, which have application in various branches of applied sciences. These problems are generally arise in the mathematical modeling of viscoelastic flows [10]. A spline has been widely applied for the numerical solutions of some ordinary and partial differential equations in the numerical analysis [5, 7, 16]. Many authors have used numerical and approximate methods to solve second and third BVPs. Recently in [19] M.M. Rahman et al. solved linear differential equation numerically by Galerkin method with Hermite polynomial as trial functions. Lima and Carpentier in [18] obtained a Numerical solution of a singular boundary value problem in non-Newtonien fluid mechanics. Feng and Li in [13] solved a second order Neumann boundary value problem with singular nonlinearity for exact three positive solutions. In a series of paper by Dabounou et al. [8, 9] BVPs of order fourth and sixth were solved using third, and fourth order hyperbolic splines. The numerical solution of the boundary value problem by splines interpolation and quasi-interpolation has been considered by many authors; see, [14–17] for example, and the references given in these papers. Despite the importance of the trigonometric and hyperbolic B-splines, little attention has been paid to developing efficient numerical methods based on these B-splines for solving the boundary value problem, except those of Refs. [8, 9, 12] for special cases. This motivates us to use trigonometric B-splines(tension) of order 3 (lower order) to solve these problems. We apply tension trigonometric B-splines of order 3 to develop a new numerical method for obtaining efficient approximations to the solution of boundary value problem. The new method is of order first for arbitrary a real tension factor  $\rho > 0$ .

The paper has been arranged in the following way. In Section 2, we give a explicit representation of trigonometric B-splines(tension) of order 3. The interpolation trigonometric B-splines is developed in Section 3. Solutions of (2VBP) with Dirichlet, Neumann and Cauchy types boundary conditions are presented in Section 4. To illustrate our algorithm, numerical examples with various values of a real tension factor  $\rho$  are presented in Section 5.

## 2. Explicit Representation of Trigonometric B-Splines (Tension) of Order 3

To develop the numerical solutions based on trigonometric B-splines (tension) of order 3 for the solution of second order linear Dirichlet, Neumann and Cauchy boundary value problem equations, let  $k$  one intergre such that  $k \geq 1$ . Let  $n_k =$

$3.2^k + 2$  and  $h_k = \frac{b-a}{n_k-2}$ . We consider a general uniform mesh

$$\begin{cases} \theta_{-2}^k = \theta_{-1}^k = \theta_0^k = a, \\ \theta_i^k = a + ih_k, i = 1 \dots n_k - 3, \\ \theta_{n_k-2}^k = \theta_{n_k-1}^k = \theta_{n_k}^k = b, \end{cases} \tag{5}$$

We introduce the trigonometric (tension) splines space of order 3 is defined as follows

$$\mathcal{V}_k = \{s \in \mathcal{C}^1(I) : s|_{[\theta_i^k, \theta_{i+1}^k]} \in \Gamma_3\} \text{ where } \Gamma_3 = \text{span}\{1, \cos(\rho\theta), \sin(\rho\theta)\}.$$

where  $\rho$  is a positive real tension factor ( $\rho > 0$ ).

The dimension of  $\mathcal{V}_k$  is  $n_k$  and we denote by  $\phi_{i,k}$ ,  $i = 0, 1, \dots, n_k - 5$ , the third-order trigonometric (tension) B-splines:

$$\phi_{i,k}(\theta) = C_k \begin{cases} -1 + \cos(\rho(\theta - \theta_i^k)), & \theta \in [\theta_i^k, \theta_{i+1}^k[; \\ 2(\cos(\rho h_k) - \cos(\rho \frac{h_k}{2})) \cos(\rho(\theta_{i+1}^k + \frac{h_k}{2} - \theta)), & \theta \in [\theta_{i+1}^k, \theta_{i+2}^k[; \\ -1 + \cos(\rho(\theta - \theta_{i+3}^k)), & \theta \in [\theta_{i+2}^k, \theta_{i+3}^k[; \\ 0, & \text{otherwise.} \end{cases}$$

The respective left and right hand side boundary trigonometric B-splines are

$$\phi_{-2,k}(\theta) = C_k \begin{cases} 2(-1 + \cos(\rho(h_k - \theta + a))), & \theta \in [\theta_0^k, \theta_1^k[; \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_{-1,k}(\theta) = C_k \begin{cases} 1 + 2 \cos(\rho h_k) - 2 \cos(\rho(a + h_k - \theta)) - \cos(\rho(-a + \theta)), & \theta \in [\theta_0^k, \theta_1^k[; \\ -1 + \cos(\rho(a + 2h_k - \theta)), & \theta \in [\theta_1^k, \theta_2^k[; \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_{n_k-4,k}(\theta) = C_k \begin{cases} -1 + \cos(\rho(b - 2h_k - \theta)), & \theta \in [\theta_{n_k-4}^k, \theta_{n_k-3}^k[; \\ 1 + 2 \cos(\rho h_k) - \cos(\rho(b - \theta)) - 2 \cos(\rho(b - h_k - \theta)), & \theta \in [\theta_{n_k-3}^k, \theta_{n_k-2}^k[; \\ 0, & \text{otherwise.} \end{cases}$$

$$\phi_{n_k-3,k}(\theta) = C_k \begin{cases} 2(-1 + \cos(\rho(b - h_k - \theta))), & \theta \in [\theta_{n_k-3}^k, \theta_{n_k-2}^k[; \\ 0, & \text{otherwise.} \end{cases}$$

where  $C_k = \frac{1}{2(-1 + \cos(\rho h_k))}$ .

The trigonometric B-splines(tension) of order 3 possess all the desirable properties of classical polynomial B-splines, see [20]. In this paper, we limit ourselves to list some of them

- $\phi_{i,k}(\theta)$  is supported on the interval  $[\theta_i^k, \theta_{i+1}^k]$ ;
- Positivity :  $\phi_{i,k}(\theta) \geq 0, \forall \theta \in [\theta_i^k, \theta_{i+1}^k]$ ;
- Partition of unity:  $\sum_{i=-2}^{n_k-3} \phi_{i,k}(\theta) = 1$ .

Table 1. The values of  $\phi_{i,k}(\theta)$  and  $\phi'_{i,k}(\theta)$  at the knots

	$\theta_i^k$	$\theta_{i+1}^k$	$\theta_{i+2}^k$	$\theta_{i+3}^k$	else
$\phi_{i,k}(\theta)$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\phi'_{i,k}(\theta)$	0	$\frac{\rho}{2 \tan(\rho \frac{h_k}{2})}$	$\frac{-\rho}{2 \tan(\rho \frac{h_k}{2})}$	0	0

### 3. Trigonometric Interpolation Method

The key point of the discretization of our scheme is how to get an approximate formula of  $y^{(2)}(\theta_j^k)$  and  $y'(\theta_j^k)$  by using Taylor series expansion.

According to Schoenberg-Whitney theorem (see [11]), for a given function  $y(\theta)$  sufficiently smooth there exists a unique trigonometric spline

$$s(\theta) = \sum_{i=-2}^{n_k-3} \mu_i \phi_{i,k}(\theta) \in \mathcal{V}_k$$

satisfying the interpolation conditions:

$$s(\theta_j^k) = y(\theta_j^k), \quad j = 0, 1, \dots, n_k - 3; \quad (6)$$

$$s'(a) = y'(a), \quad s'(b) = y'(b). \quad (7)$$

For  $j = 0, 1, \dots, n_k - 3$ , let  $m_j = s'(\theta_j^k)$  and for  $j = 0, 1, \dots, n_k - 3$ , let  $M_j$  the approximate moment at the knot  $\theta_j^k$ , setting :

$$M_j = \frac{s(\theta_j^k + h_k) - 2s(\theta_j^k) + s(\theta_j^k - h_k)}{h_k^2}.$$

By using the Taylor series expansion we have:

$$m_j = s'(\theta_j^k) = y'(\theta_j^k) - \frac{1}{180} h_k^4 y^{(5)}(\theta_j^k) + O(h_k^6); \quad (8)$$

$$M_j = y''(\theta_j^k) + \frac{1}{12} h_k^2 y^{(4)}(\theta_j^k) + \frac{1}{360} h_k^4 y^{(6)}(\theta_j^k) + O(h_k^6); \quad (9)$$

By Table 1 and this equations, we get:

Table 2. The approximation values of  $y(\theta_j^k)$ ,  $y'(\theta_j^k)$  and  $y''(\theta_j^k)$ .

	$y(\theta_j^k)$	$y'(\theta_j^k)$	$y''(\theta_j^k)$
Approximate value	$s(\theta_j^k)$	$m_j$	$M_j$
Representation in $\mu_j$	$\frac{\mu_{j-1} + \mu_{j-2}}{2}$	$\frac{\rho(\mu_{j-1} - \mu_{j-2})}{2 \tan(\rho \frac{h_k}{2})}$	$\frac{\mu_{j-3} - \mu_{j-2} - \mu_{j-1} + \mu_j}{2h_k^2}$
Error order	$O(h_k^4)$	$O(h_k^4)$	$O(h_k^2)$

### 4. Formulation of 2 BVP in Matrix form

In this section, we study a method based on trigonometric (tension) B-splines of order 3 for constructing numerical solutions to linear Dirichlet, Neumann and Cauchy boundary value problem 2 BVP of the form (1).

4.1 **Trigonometric (Tension) Solution with Dirichlet Boundary Condition**

We define a trigonometric (tension) spline interpolant  $s(\theta) = \sum_{i=-2}^{n_k-3} \mu_i \phi_{i,k}(\theta)$  satisfying

boundary conditions (2.2) and  $\tilde{s}(\theta) = \sum_{i=-2}^{n_k-3} \tilde{\mu}_i \phi_{i,k}(\theta)$  be the approximate spline of  $s(\theta)$ . To do this we discretize (1) at  $\theta_j^k$  for  $j = 1, 2, \dots, n_k - 3$ , we get the spline approximation as :

$$y^{(2)}(\theta_j^k) + f(\theta_j^k)y'(\theta_j^k) + g(\theta_j^k)y(\theta_j^k) = p(\theta_j^k), \quad j = 1, 2, \dots, n_k - 3. \tag{10}$$

Substituting  $y^{(2)}(\theta_j^k)$ ,  $y'(\theta_j^k)$  and  $y(\theta_j^k)$  in equation (10) and using the following notations:

$$f_j = f(\theta_j^k), \quad g_j = g(\theta_j^k) \text{ and } p_j = p(\theta_j^k),$$

we arrive at the following linear system :

$$\frac{\mu_{j-3} - \mu_{j-2} - \mu_{j-1} + \mu_j}{2h_k^2} + f_j \frac{\rho(\mu_{j-1} - \mu_{j-2})}{2 \tan(\rho \frac{h_k}{2})} + g_j \frac{\mu_{j-1} + \mu_{j-2}}{2} = p_j + O(h_k^2) \tag{11}$$

Consequently,

$$(\mu_{j-3} - \mu_{j-2} - \mu_{j-1} + \mu_j) + \alpha_{j,k}(\mu_{j-1} - \mu_{j-2}) + \beta_{j,k}(\mu_{j-1} + \mu_{j-2}) = \gamma_{j,k} + \xi_k. \tag{12}$$

Where  $\alpha_{j,k} = f_j \frac{\rho h_k^2}{\tan(\rho \frac{h_k}{2})}$ ,  $\beta_{j,k} = g_j h_k^2$ ,  $\gamma_{j,k} = 2h_k^2 p_j$ , and  $\xi_k = O(2h_k^4)$ .

By dropping  $\xi_k$  from (12), we yield a linear system with  $n_k - 3$  linear equations in  $n_k$  unknowns  $\mu_j$ ,  $j = -2, -1, \dots, n_k - 3$ . So three more equations are needed.

On the other hand, by using the Dirichlet boundary conditions (2), we get

$$\begin{cases} y(a) = a_0; \\ y(b) = b_0. \end{cases} \text{ Thus, } \begin{cases} \mu_{-2} = a_0; \\ \mu_{n_k-3} = b_0. \end{cases} \tag{13}$$

To obtain unique solution we propose the following formula

$$y'_{j-1} + 4y'_j + y'_{j+1} = \frac{3}{h_k}(y_{j+1} - y_{j-1}) + O(h_k^4) \tag{14}$$

which can be easily demonstrated using a Taylor series expansion.

Combining the scheme (14) with Table 2, we can construct an approximate formulae for  $y^{(2)}(a)$ , as follows

$$y^{(2)}(a) = -4M_1 - M_2 + \frac{3}{h_k}(m_2 - m_0) + O(h_k^4). \tag{15}$$

For smaller  $h_k$ , we have  $\tan(\frac{h_k}{2}) \sim \frac{h_k}{2}$  and by turning (15) the coefficients are determined as follows

$$(2\mu_{-2} - 3\mu_{-1} - \mu_0 + 3\mu_1 - \mu_2) + 2h_k f_0(\mu_{-1} - \mu_{-2}) = 2h_k^2(p_0 - g_0 a_0) + O(h_k^4) \tag{16}$$

Take (12) and (16), we get  $n_k - 2$  linear equations with  $\mu_i$ ,  $i = -1, 0, \dots, n_k - 5, n_k - 4$ , as unknowns since  $\mu_{-2}$ , and  $\mu_{n_k-3}$  have been yielded from (13).

Let  $C = [\mu_{-1}, \mu_0, \dots, \mu_{n_k-5}, \mu_{n_k-4}]^T$ ,  $\tilde{C} = [\tilde{\mu}_{-1}, \tilde{\mu}_0, \dots, \tilde{\mu}_{n_k-5}, \tilde{\mu}_{n_k-4}]^T$ ,  $D = [d_1, d_2, \dots, d_{n_k-2}]^T$ ,  $E = [e_1, e_2, \dots, e_{n_k-2}]^T$ . We can write our method in matrix form as :

$$(A_1 + \lambda_k A_2 F + h_k^2 B G)C = D + E; \tag{17}$$

$$(A_1 + \lambda_k A_2 F + h_k^2 B G)\tilde{C} = D, \tag{18}$$

with

$$\lambda_k = \frac{\rho h_k^2}{\tan(\rho \frac{h_k}{2})}$$









Table 4. Maximum absolute error and Order of convergence for Problem (30).

$k$	Error for $\rho = 10^{-3}$	Error for $\rho = 1$	Order of convergence for $\rho = 1$
3	2.475e-002	2.475e-002	1.1087
4	1.246e-002	1.246e-002	1.0536
5	6.256e-002	6.256e-003	1.0276
6	3.133e-003	3.133e-003	1.0136
7	1.595e-003	1.568e-003	1.0070
8	7.303e-004	7.843e-004	1.0036
9	6.082e-004	3.922e-004	

Example 5.3 We consider the following Dirichlet boundary-value problem

$$\begin{cases} d^2y/dx^2 - dy/dx = -\exp(x - 1) - 1, x \in [0, 1]; \\ y(0) = y(1) = 0. \end{cases} \tag{31}$$

The exact solution is  $y(x) = x(1 - \exp(x - 1))$ .

Result has been shown for different values of  $k$  and with various values of tension factor  $\rho$  in Table 5.

Table 5. Maximum absolute error for Problem (31).

$k$	Error for $\rho = 10^{-4}$	Error for $\rho = 1$
3	1.018e-002	1.018e-002
4	5.185e-003	5.190e-003
5	2.599e-003	2.620e-003
6	1.477e-003	1.316e-003
7	7.828e-004	6.597e-004
8	3.302e-004	3.302e-004
9	1.605e-004	1.652e-004

Example 5.4 Consider the following Neumann boundary-value problem discussed in [19].

$$\begin{cases} y^{(2)}(x) + y(x) = \cos(x), x \in [0, 5]; \\ y'(0) = 0, y'(5) = 0. \end{cases} \tag{32}$$

for which the exact solution is

$$y(x) = \frac{(-6 \cos^2(5) + 2 + 10 \cot(5) \sin(10) + 2 \cos(10)) \cos(x)}{4} + \frac{2 \cos^3(x) + 2x \sin(x) + \sin(x) \sin(2x)}{4}.$$

The observed maximum errors in absolute values computed at various points of the interval  $[0, 5]$ , for problem (32), and the convergence order are summarized in Table 6.

Example 5.5 We consider the following Neumann boundary-value problem discussed in [19].

$$\begin{cases} y^{(2)}(x) + y(x) = x^2 \exp(-x), x \in [0, 10]; \\ y'(0) = 0, y'(10) = 0. \end{cases} \tag{33}$$

The exact solution is  $y(x) = -\frac{\cos(10)+99 \exp(-10)}{2 \sin(10)} \cos(x) - \frac{\sin(x)}{2} + \frac{\exp(-x)(1-x)^2}{2}$ .

Result has been shown for different values of  $k$  and with various values of  $\rho$  in Table 8.

Table 6. Maximum absolute error and Order of convergence for Problem (32).

$k$	Error for $\rho = 10^{-4}$	Error for $\rho = 1$	Order of convergence for $\rho = 1$
3	2.531e-001	2.531e-001	0.9725
4	1.340e-001	1.340e-001	0.9768
5	6.944e-002	6.944e-002	0.9862
6	3.541e-002	3.541e-002	0.9924
7	1.805e-002	1.789e-002	0.9960
8	9.072e-003	8.993e-003	0.9978
9	8.553e-003	4.509e-003	

Table 7. Error in Galerkin method developed in [19] for Problem (32).

$x$	Maximum absolute error for $n = 12$
0.1	6.7277e-001
0.2	2.1316e+000
0.3	1.3334e+001
0.4	4.8501e+000
0.5	2.5722e+000

Table 8. Maximum absolute error for Problem (33).

$k$	Error for $\rho = 10^{-3}$	Error for $\rho = 1$
3	2.062e-001	2.062e-001
4	1.478e-001	1.478e-001
5	9.614e-002	9.614e-002
6	5.700e-002	5.700e-002
7	3.149e-002	3.149e-002
8	1.663e-002	1.663e-002
9	8.557e-003	8.558e-003

Table 9. Error in Galerkin method developed in [19] for Problem (33).

$x$	Maximum absolute error for $n = 12$
0.1	1.5213e+000
0.2	8.9251e-002
0.3	2.8328e-001
0.4	3.8058e-001
0.5	3.0792e-001

*Example 5.6* We consider the following Cauchy boundary-value problem

$$\begin{cases} y^{(2)}(x) + xy(x) = -(x^3 + 3x) \exp(x), & x \in [0, 1]; \\ y(0) = 0, & y'(1) = -\exp(1). \end{cases} \quad (34)$$

The exact solution is  $y(x) = (-x^2 + x) \exp(x)$ .

In Table 10, we give the corresponding errors for different values of  $k$  and  $\rho$  and the computed convergence orders.

*Example 5.7* We consider the following Cauchy boundary-value problem

$$\begin{cases} y^{(2)}(x) - 2y(x) = 2(1 + \tan^2(x)), & x \in [0, 1]; \\ y(0) = 0, & y'(1) = \tan(1). \end{cases} \quad (35)$$

Table 10. Maximum absolute error and Order of convergence for Problem (34).

$k$	Error for $\rho = 10^{-2}$	Error for $\rho = 1$	Order of convergence for $\rho = 1$
3	2.714e-002	2.715e-002	1.0311
4	1.434e-002	1.434e-002	0.9966
5	7.473e-003	7.473e-003	0.9940
6	3.828e-003	3.828e-003	0.9960
7	1.939e-003	1.939e-003	0.9973
8	9.757e-004	9.763e-004	0.9988
9	4.909e-004	4.898e-004	

The exact solution is  $y(x) = (x - 1) \tan(x)$ .

The results for different values of  $k$  and with various values of  $\rho$  are summarized in Table 11.

Table 11. Maximum absolute error for Problem (35).

$k$	Error for $\rho = 10^{-3}$	Error for $\rho = 1$
3	6.328e-002	6.328e-002
4	3.481e-002	3.481e-002
5	2.157e-002	2.157e-002
6	1.515e-002	1.515e-002
7	1.198e-002	1.999e-002
8	1.043e-002	1.041e-002
9	9.566e-003	9.635e-003

The norms of the approximated solutions were compared with those of finite difference (FDM) and cubic B-spline interpolation methods (CBIM). The results from these methods were generated by solving Examples (29) and (31) by using the methods explained in [4] and [6]. The results are shown in Table 12 and Table 13.

Table 12. Max-Norm and  $L^2$ -Norm for Problem (29).

Method	Max-Norm	$L^2$ -Norm
FDM(Burden-Faires [4])	5.1880e-005	1.1764e-004
CBIM(Caglar et al. [6])	5.2011e-005	1.1794e-004
Our method(for $\rho = 1$ and $k = 10$ )	3.9190e-005	1.0670e-004

Table 13. Max-Norm and  $L^2$ -Norm for Problem (31).

Method	Max-Norm	$L^2$ -Norm
FDM(Burden-Faires [4])	1.6265e-004	3.6971e-004
CBIM(Caglar et al. [6])	2.8996e-004	6.6089e-004
Our method(for $\rho = 1$ and $k = 9$ )	1.6524e-004	5.1775e-004
Our method(for $\rho = 1$ and $k = 10$ )	8.2645e-005	2.5885e-004

## 6. Conclusion

Despite a lower order of used splines, our method produced better with Neumann boundary conditions (see examples 4 and 5), in comparison with the largest value of  $n$  degree of

Hermite polynomials in Galerkin method developed in [19], also the Galerkin method is not applied to any example with Dirichlet boundary condition, for more details see [19]. The purpose of this paper is to present a new simple numerical method, suitable and accurate to solve (2VBP) boundary value problem with Dirichlet, Neumann and Cauchy conditions by using trigonometric splines of order 3 with a tension factor  $\rho$ . Experimental results confirm the first order of convergence. We notice also that the influence of the value of tension factor  $\rho$  is remarkable when the value of  $k$  exceeds the value 8. The results in this paper can be easily extended to other classes of systems of boundary value problems of higher order and a multi-dimensional boundary problems.

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