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Differential Transformation Method For Solving Fuzzy Fractional Heat Equations

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Abstract. In this paper, the differential transformation method (DTM) is applied to solve fuzzy fractional heat equations. The elementary properties of this method are given. The approximate and exact solutions of these equations are calculated in the form of series with easily computable terms. The proposed method is also illustrated by some examples. The results reveal that DTM is a highly effective scheme for obtaining approximate analytical solutions of fuzzy fractional heat equations.

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1. Introduction

Zadeh published his pioneering study in fuzzy theory in [19], hundreds of examples have been supplied where the nature of uncertainty in the behavior of a given system processes is fuzzy rather than stochastic nature. The theoretical framework of fuzzy initial value problems (FIVPs) has been an active research field over the last few years. The concept of fuzzy derivative was first introduced by Chang and Zadeh in [6]. Dubois and Prade [7], defined and used the extension principle. Other techniques have been discussed by Puri and Ralescu [16] and Goetshel and Voxman [11].

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The most important contribution on these numerical methods is the Euler method provided by Ma, M. Friedman, A. Kandel in [14]. S. Abbasbandy, T. Allahviranloo in [1] developed four-stage order Runge-Kutta methods for a Cauchy problem with a fuzzy initial value. T. Allahviranloo in [4] introduced DTM for solving fuzzy differential equations (FDEs) and further methods can be found in [8–10].

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering [10, 18]. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively. Fuzzy fractional differential equations have been studied by scientists and engineers such as T. Allahviranloo et al. in [5] and R.W. Ibrahim and H.A. Jalab in [12]. Our idea is to find the approximate solution of fuzzy heat-like equations with differential transform method. The differential transform method was first introduced by Zhou [20] who solved linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in form of polynomial expressions such as Taylor series expansion. But procedure is easier than the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expensive for higher orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solution of ordinary or partial differential equations.

The rest of the paper is organized as follows. In Section 2, we call some definition on fuzzy numbers and fuzzy Caputo's derivative. In Section 3, differential transformation method for fuzzy heat-like equations and fuzzy heat like equations are illustrated. Examples are shown in Section 4, and finally, conclusion is given in Section 5.

2. Preliminaries

The basic definitions and notations of fuzzy numbers [11] and fuzzy Caputo's derivative are given in this section.

2.1 Definitions and notations

Definition 2.1 The fuzzy set A in X is a set of ordered pairs $A = \{(x, \mu_A(x)) | x \in X\}$, where μ_A is called the membership function of x in A and the range of μ_A is a subset of the nonnegative real number.

Definition 2.2 An arbitrary fuzzy number u in parametric form is an ordered pair $(\underline{u}(r); \overline{u}(r))$ of functions $\underline{u}(r)$, $\overline{u}(r)$; $0 \leqslant r \leqslant 1$ which satisfy the following conditions [2]:
1. $u(r)$ is a bounded left-continuous non-decreasing function over $[0, 1]$.
2. $u(r)$ is a bounded left-continuous non-increasing function over $[0, 1]$.
3. $\underline{u}(r) \leqslant \overline{u}(r), \leqslant r \leqslant 1$. The set of all such fuzzy numbers is represented by E^1.

2.2 Fuzzy Caputo's derivative

We denote $C^{\mathbb{F}}[a, b]$ as a space of all fuzzy-valued functions which are continuous on $[a, b]$, and the space of all Lebesque integrable fuzzy-valued functions on the bounded interval $[a, b] \subset \mathbb{R}$ by $L^{\mathbb{F}}[a, b]$, we denote the space of fuzzy-value functions $f(x)$ which have continuous H-derivative up to order $n-1$ on $[a, b]$ such that

$f^{(n-1)}(x) \in AC^{\mathbb{F}}([a,b])$ by $AC^{(n)\mathbb{F}}([a,b])$, where $AC^{\mathbb{F}}([a,b])$ denote the set of all fuzzy-valued functions which are absolutely continuous.

Definition 2.3 Let $f(x) \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$, the fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:

$$\big(I^{\alpha}_{a+}f\big)(x;r) = \big[\big(I^{\alpha}_{a+}\underline{f}\big)(x;r), \big(I^{\alpha}_{a+}\overline{f}\big)(x;r)\big],$$

where $0 \leqslant r \leqslant 1$ and

$$\big(I^{\alpha}_{a+}\underline{f}\big)(x;r) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\underline{f}(t)\mathrm{d}t}{(x-t)^{1-\alpha}}, \quad 0 \leqslant r \leqslant 1,$$
$$\big(I^{\alpha}_{a+}\overline{f}\big)(x;r) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\overline{f}(t)\mathrm{d}t}{(x-t)^{1-\alpha}}, \quad 0 \leqslant r \leqslant 1.$$

Let $f(x) \in C^{\mathbb{F}}((0,a]) \cap L^{\mathbb{F}}(0,a)$, be a given function such that $f(t;r) = [\underline{f}(t;r), \overline{f}(t;r)]$ for all $t \in (0,a]$ and $0 \leqslant r \leqslant 1$. We define the fuzzy fractional Riemann-Liouville derivative of order $0 < \alpha < 1$ of f in the parametric form,

$$^{RL}D^{\alpha}f(t;r) =: \frac{1}{\Gamma(1-\alpha)}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha}\underline{f}(s;r)\mathrm{d}s, \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha}\overline{f}(s;r)\mathrm{d}s\Big],$$

provided that equation defines a fuzzy number $^{RL}D^{\alpha}f(t) \in \mathbb{E}$. In fact,

$$^{RL}D^{\alpha}f(t;r) := \big[^{RL}D^{\alpha}\underline{f}(t;r), {}^{RL}D^{\alpha}\overline{f}(t;r)\big].$$

Obviously, $^{RL}D^{\alpha}f(t) = \frac{\mathrm{d}}{\mathrm{d}t}I^{1-\alpha}f(t)$ for $t \in (0,a]$. For more information, see [3].

3. Differential transformation method and fuzzy fractional heat equation

3.1 Generalized two-dimensional differential transformation method

In this section we shall derive the generalized two-dimensional DTM that we have developed for the numerical solution of linear partial differential equations with space and time-fractiol derivatives.

Consider a function of two variables $u(x,y)$, and suppose that it can be represented as a product of two single-variable functions, that is, $u(x,y) = f(x)g(y)$. Based on the properties of generalized two-dimensional differential transform, the function $u(x,y)$ can be represented as:

$$u(x,y) = \sum_{k=0}^{\infty} F_{\alpha}(k)(x-x_0)^{k\alpha}\sum_{h=0}^{\infty} G_{\beta}(h)(y-y_0)^{h\beta}$$

$$= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta} \tag{1}$$

where $0 < \alpha, \beta \leqslant 1, U_{\alpha,\beta} = F_\alpha(k)G_\beta(h)$ is called the spectrum of $u(x, y)$. The generalized two-dimensional differential transform of the function $u(x, y)$ is given by

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D^{\alpha}_{x_0})^k (D^{\beta}_{y_0})^h u(x,y)]_{(x_0,y_0)}, \qquad (2)$$

where $(D^{\alpha}_{x_0})^k = D^{\alpha}_{x_0} D^{\alpha}_{x_0} \cdots D^{\alpha}_{x_0}, k$ -times. In case of $\alpha = 1$, and $\beta = 1$ the generalized two-dimensional differential transform (1) reduces to the classical two-dimensional differential transform [15]. Next, we give some useful theorems about writing the generalized differential transform in equivalent forms under certain conditions from [13].

Theorem 3.1 *Suppose that $U_{\alpha,\beta}(k, h), V_{\alpha,\beta}(k, h)$, and $W_{\alpha,\beta}(k, h)$ are the differential transformations of the functions $u(x, y), v(x, y)$, and $w(x, y)$, respectively, then*
1 if $u(x, y) = v(x, y) \pm w(x, y)$, then $U_{\alpha,\beta}(k, h) = V_{\alpha,\beta}(k, h) \pm W_{\alpha,\beta}(k, h)$,
2 if $u(x, y) = \lambda v(x, y),\ \lambda \in \mathbb{R}$ then $U_{\alpha,\beta}(k, h) = \lambda V_{\alpha,\beta}(k, h)$
3. if $u(x, y) = v(x, y)w(x, y)$, then

$$U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(k-r,s)$$

4. if $u(x, y) = (x - x_0)^{m\alpha}(y - y_0)^{n\beta}$, then

$$U_{\alpha,\beta}(k,h) = \delta(k-m,h-n) = \begin{cases} 1, k=m, h=n \\ 0, \quad otherwise \end{cases}$$

Theorem 3.2 *If $u(x, y) = D^{\alpha}_{x_0} v(x, y), 0 < \alpha \leqslant 1$, then the generalized differential transform (2) can be written as*

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+1,h).$$

Theorem 3.3 *If $u(x, y) = D^{\gamma}_{x_0} v(x, y), m - 1 < \gamma \leqslant m$, and $v(x, y) = f(x)g(y)$, then the generalized differential transform (2) can be written as*

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+\frac{\gamma}{\alpha},h).$$

3.2 Fuzzy fractional heat equation

Consider the fuzzy fractional heat equation with the indicated initial conditions [17]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + k, \quad 0 < x < 1, \quad t > 0, \quad 0 < \alpha < 1,$$

$$u(x,0) = f(x), \quad 0 < x < 1,$$

where $f(x) \in C^{\mathbb{F}}((0, 1])$.

4. Examples

Example 1. Consider the following fuzzy fractional heat equation with indicated initial condition [17],

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \tag{3}$$

$$u(x,0) = \widetilde{f}(x) = \widetilde{k}\sin(\pi x), \quad 0 < x < 1. \tag{4}$$

We can write the initial condition as follows:

$$u(x,0) = \widetilde{f}(x) = \widetilde{k}\Big(\sum_{i=0}^{\infty} \frac{\sin(i\pi x/2)(\pi x)^i}{i!}\Big), \quad 0 < x < 1. \tag{5}$$

Taking the differential transform of (3), we have

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U(k,h+1) = (k+1)(k+2)U(k+2,h), \tag{6}$$

From the initial condition, we find

$$U(i,0) = \widetilde{k}\frac{\pi^i}{i!}\sin(\frac{\pi i}{2}) \quad i = 0,1,2,\cdots \tag{7}$$

By applying Eqs. (7) into Eqs. (6), we can obtain some values of $U(k,h)$ as follows:

$$U(i,0) = 0, \quad i = even,$$

$$U(1,0) = \widetilde{k}\pi, \qquad U(3,0) = \widetilde{k}\frac{\pi^3}{3!}, \qquad U(5,0) = \widetilde{k}\frac{\pi^5}{5!}, \cdots$$

$$U(1,1) = \widetilde{k}\frac{\pi^3}{\Gamma(\alpha+1)}, \quad U(3,1) = \widetilde{k}\frac{\pi^5}{3!\Gamma(\alpha+1)}, \cdots U(i,1) = 0, \quad i = even$$

$$U(1,2) = \widetilde{k}\frac{\pi^5}{\Gamma(2\alpha+1)}, \quad U(3,2) = \widetilde{k}\frac{\pi^7}{3!\Gamma(2\alpha+1)}, \cdots U(i,2) = 0, \quad i = even$$

$$\vdots$$

Consequently substituting all $U(k,h)$ into Eq. (1), we obtain the series form solutions of Eq. (3) and (4) as

$$u(x,t) = \widetilde{k}(\pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \cdots)(1 - \frac{\pi^2 t^\alpha}{\Gamma(\alpha+1)} + \frac{\pi^4 t^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} t^{n\alpha}}{\Gamma(n\alpha+1)} \widetilde{k}\sin(\pi x)$$

The approximate solutions DTM of order 6 and 8 are compared and plotted for $\alpha = 0.85, 0.90, 0.95, 1,\ x = 0.5,\ t = 0.5$ and $\widetilde{k}(r) = (0.75 + 0.25r, 1.25 - 0.25r)$ in

Figure 1.

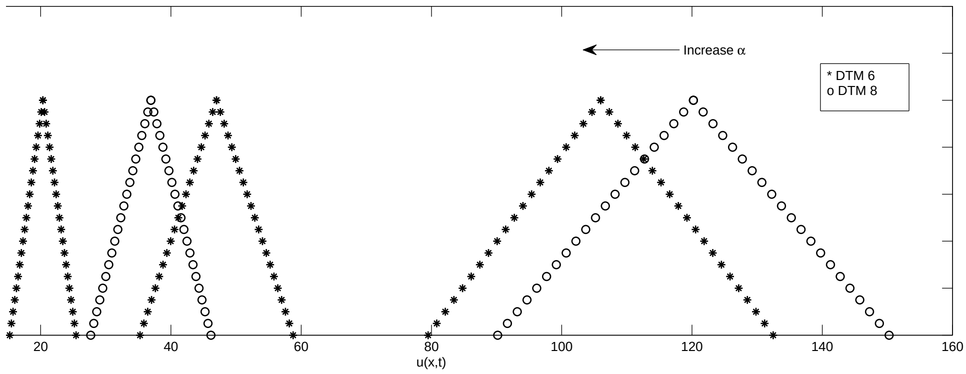

Figure 1. The approximate solutions for different values of α of Exp. 1.

Example 2. Consider the following fuzzy fractional heat equation with the indicated initial condition [17],

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \quad 0 < \alpha < 1, \tag{8}$$

$$u(x,0) = f(x) = \widetilde{k}x^2, \quad 0 < x < 1. \tag{9}$$

The parametric form of (8) is

$$\frac{\partial^\alpha \overline{u}}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 \overline{u}}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \tag{10}$$

$$\frac{\partial^\alpha \underline{u}}{\partial t^\alpha} = \frac{x^2}{2}\frac{\partial^2 \underline{u}}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \tag{11}$$

Taking the differential transform of (10) and (11), we have

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}\overline{U}(k,h+1)$$
$$= \frac{1}{2}\sum_{s=0}^{h}\sum_{r=0}^{k}\delta(r-2,h-s)(k+1-r)(k+2-r)\overline{U}(k+2-r,s), \tag{12}$$

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}\underline{U}(k,h+1)$$
$$= \frac{1}{2}\sum_{s=0}^{h}\sum_{r=0}^{k}\delta(r-2,h-s)(k+1-r)(k+2-r)\underline{U}(k+2-r,s), \tag{13}$$

From the initial conditions, we find

$$U(k,0)=\begin{cases}\widetilde{k}, & k=2,\\ 0, & k=0,1,3,4,\cdots.\end{cases} \tag{14}$$

By applying Eq. (14) into Eqs. (12) and (13), we can obtain some values of $U(k,h)$ as follows:

$$U(k,1)=\begin{cases}\frac{\widetilde{k}}{\Gamma(\alpha+1)}, & k=2,\\ 0, & k=0,1,3,4,\cdots.\end{cases} \tag{15}$$

$$U(k,2)=\begin{cases}\frac{\widetilde{k}}{\Gamma(2\alpha+1)}, & k=2,\\ 0, & k=0,1,3,4,\cdots.\end{cases} \tag{16}$$

$$\vdots$$

Consequently substituting all $U(k,h)$ into Eq. (1), we obtain the series form solutions of Eq. (8) and (9) as

$$\begin{aligned} u(x,t) &= \widetilde{k}x^2\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\right)\\ &= \widetilde{k}x^2\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \end{aligned}$$

Therefore, the exact solution is given by

$$u(x,t)=\widetilde{k}x^2\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

The approximate solutions DTM of order 6 and 8 are compared and plotted for $\alpha=0.85,0.90,0.95,1,\ x=1.0,\ t=2.0$ and $\widetilde{k}(r)=(0.75+0.25r,1.25-0.25r)$ in Figure 2.

Example 3. Consider the inhomogeneous fuzzy fractional heat equation

$$\frac{\partial^{\alpha}U}{\partial t^{\alpha}}=\frac{\partial^2 U}{\partial x^2}+\tilde{k},\quad 0<x<1,\quad t>0,\quad 0<\alpha<1, \tag{17}$$

with initial condition,

$$U(x,0)=\tilde{k}x^2,\quad 0<x<1, \tag{18}$$

and $\tilde{k}=(0.75+0.25r,1.25-0.25r)$.
The parametric form of (17) is

$$\frac{\partial^{\alpha}\overline{U}}{\partial t^{\alpha}}=\frac{\partial^2\overline{U}}{\partial x^2}+(1.25-0.25r),\quad 0<x<1,\quad t>0, \tag{19}$$

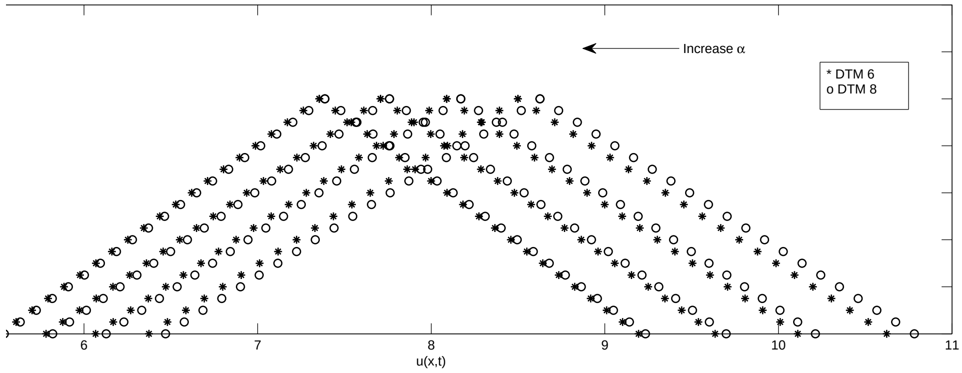

Figure 2. The approximate solutions for different values of α of Exp. 2.

$$\frac{\partial^\alpha \underline{U}}{\partial t^\alpha} = \frac{\partial^2 \underline{U}}{\partial x^2} + (0.75 + 0.25r), \quad 0 < x < 1, \quad t > 0, \tag{20}$$

One can readily find the differential transform of (19) and (20), as follows,

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} \overline{U}(k, h+1) = \\ (k+1)(k+2)\overline{U}(k+2, h) + (1.25 - 0.25r)\delta(k, h), \tag{21}$$

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} \underline{U}(k, h+1) = \\ (k+1)(k+2)\underline{U}(k+2, h) + (0.75 + 0.25r)\delta(k, h). \tag{22}$$

From the initial conditions, we find

$$\overline{U}(k,0) = \begin{cases} 1.25 - 0.25r, & k = 2, \\ 0, & k = 0, 1, 3, 4, \cdots. \end{cases} \tag{23}$$

$$\underline{U}(k,0) = \begin{cases} 0.75 + 0.25r, & k = 2, \\ 0, & k = 0, 1, 3, 4, \cdots. \end{cases} \tag{24}$$

By applying Eqs. (23) and (24) into Eqs. (21) and (22), we can obtain some value of $U(k, h)$ as follows:

$$\overline{U}(k,1) = \begin{cases} \frac{3(1.25-0.25r)}{\Gamma(\alpha+1)}, & k = 0, \\ 0, & k = 1, 2, 3, \cdots. \end{cases} \tag{25}$$

$$\underline{U}(k,1) = \begin{cases} \frac{3(0.75+0.25r)}{\Gamma(\alpha+1)}, & k = 0, \\ 0, & k = 1, 2, 3, \cdots. \end{cases} \tag{26}$$

$$\overline{U}(k,2) = 0, \quad k = 0, 1, 2, \cdots. \tag{27}$$

$$\underline{U}(k,2) = 0, \quad k = 0,1,2,\cdots. \tag{28}$$

$$\vdots$$

Consequently substituting all $U(k,h)$ into Eq. (1), we obtain the series form solutions of Eq. (17) and (18) as

$$\overline{u}(x,t) = (1.25 - 0.25r)(\frac{3t^\alpha}{\Gamma(\alpha+1)} + x^2), \quad \underline{u}(x,t) = (0.75 + 0.25r)(\frac{3t^\alpha}{\Gamma(\alpha+1)} + x^2)$$

which is the exact solution of (17) with initial condition (18).

5. Conclusion

In this work, we used the differential transformation method for approximate solution of fuzzy fractional heat equations and illustrated by some numerical examples. The results showed that the DTM is remarkably effective and very simple.

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