International Journal of
Mathematical Modelling & Computations
Vol. 05, No. 02, Spring 2015, 173- 184

Analysis of Finite Buffer Renewal Input Queue with Balking and Markovian Service Process

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Abstract. This paper presents the analysis of a finite buffer renewal input queue wherein the customers can decide either to join the queue with a probability or to balk. The service process is Markovian service process (MSP) governed by an underlying m-state Markov chain. Employing the supplementary variable and embedded Markov chain techniques, the steady-state system length distributions at pre-arrival and arbitrary epochs are obtained. Based on the system length distributions, some performance measures of the model and waiting-time analysis are presented. Finally, numerical results are displayed to show the effect of model parameters on the key performance measures.

Received:1 August 2014, Revised:9 December 2014, Accepted:6 Janurary 2015.

Keywords: Queue, Finite buffer, Markovian service process, Balking, Waiting-time

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1. Introduction

Impatience is the most prominent characteristic as individuals always feel anxious and impatient during waiting for service in real life. The customers' impatient acts should be involved in the study of queueing systems to model real situations exactly. The importance of such systems appear in many real-life problems such as the situations involving impatient telephone switchboard customers, hospital emergency rooms handling critical patients, inventory systems that store perishable goods, etc. Balking is one such impatient phenomenon where customers decide either to join the queue or not to join the queue with a probability. Modeling balking is worthwhile because one obtains new managerial insights and the lost revenues

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due to balking in various industries can be enormous. While making decision for the number of servers needed in the service system to meet time-varying demand, the balking probabilities can be used to estimate the amount of lost business in more practical considerations for the managers as given in [12].

Performance analysis of queueing systems with balking has attracted many researchers owing to their wide applications in real life congestion problems. [10] first presented the $M/M/1$ queue with balking. [7] extended this work to $GI/M/1/N$ queue with balking using embedded Markov chain. [2] obtained the transient solution of a single server state dependent system with balking. An $M/M/1$ queue with random balking has been studied by [13]. They obtained the stationary characteristics of the system and maximum likelihood estimate of the balking parameter. [11] have analyzed a single server queue with two types of services and restricted admissibility using supplementary variable technique.

During the last few decades queueing systems under various type of arrival and service processes have been investigated due to their applicability in various networking situations, production, manufacturing systems, etc. Traditional queueing analysis using Poisson process is not powerful enough to capture the correlated nature of arrival (service) processes. The correlated arrival and/or service processes in queueing systems have been shown empirically and theoretically to have a significant impact on the queueing behavior. The Markovian arrival process (MAP) has been introduced due to the limitations of Poisson process in modelling correlated arrivals. Like MAP, MSP is a versatile service process and can capture the correlation among the service times. Several other service processes like Poisson process, Markov modulated Poisson process ($MMPP$), Phase (PH) type renewal process, etc., can be considered as special cases of MSP. For details on MSP, readers are referred to [5] and [1].

In recent years there has been a great interest in analyzing various queueing models with MSP. [3] discussed the asymptotic behavior of queues with MAP and/or MSP using perturbation theory. [9] have analyzed a $GI/MSP/1$ queue with finite and infinite buffers using a combination of embedded Markov chain and supplementary variable techniques for the finite buffer system, and the matrix-geometric method and the renewal-theory for the infinite buffer system. [15] have analyzed a finite buffer $GI/MSP/1$ queue with accessible and non-accessible batch service using embedded Markov chain and supplementary variable technique. [4] studied a $GI/BMSP/1$ queue with state dependent arrivals using a combination of matrix geometric method, Markov renewal theory and semi Markov process for obtaining the queue length distributions at various epochs. [6] presented closed-form analysis for evaluating the pre-arrival epoch probabilities of infinite buffer $GI/MSP/1$ queue based on the roots of the characteristic equation. Using the classical Markov renewal theory, they obtained the steady-state system length distribution at an arbitrary epoch. Recently, [8] analyzed a discrete-time finite buffer $GI/MSP/1$ queue with N threshold policy. They obtained the system length distributions at pre-arrival and arbitrary epochs using supplementary variable and embedded Markov chain techniques.

The present literature shows that the impatient behavior of customers has not been considered in finite buffer $GI/MSP/1$ queues so far, to the best of our knowledge. Motivated by this, we aim to incorporate balking in a finite buffer queue where the input follows a renewal process and the departures form an MSP. The model is analyzed using embedded Markov chain and supplementary variable techniques. The former technique has been adopted for obtaining the steady-state probabilities at pre-arrival epoch while the latter technique is used for obtaining the arbitrary epoch probabilities. Some performance measures and the analysis of waiting-time distribution in the system have been discussed. Numerical results have been presented in the form of tables and graphs to show the effect of model parameters on the performance indices.

The rest of the paper is organized as follows: Section 2 presents model description and the notations used to describe the model parameters. The analytical analysis of the model is carried out in Section 3. In Section 4, some performance measures of the model and waiting-time analysis are discussed. The behavior of the performance measures against the variation of model parameters is studied in Section 5 through some numerical results. Finally, Section 6 concludes the paper.

2. Model Description

Let us consider a $GI/MSP/1/N$ queue with balking. We assume that the inter-arrival times of successive arrivals are independent and identically distributed random variables with cumulative distribution function $A(u)$, probability density function $a(u),\ u \geq 0$, Laplace Stieltjes transform (LST) $A^*(\theta)$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$, where $h^{(1)}(0)$ denotes the first derivative of $h(\theta)$ evaluated at $\theta = 0$. If a customer on arrival finds n customers in the system then it decides either to join the queue with probability b_n or to balk with probability $\bar{b}_n = 1 - b_n$. Further, we assume that $b_0 = 1,\ 0 \leq b_{n+1} \leq b_n \leq 1,\ 1 \leq n \leq N-1$ and $b_N = 0$.

Customer are served by a single server according to First-Come First-Serve (FCFS) service discipline. The service process is MSP and is governed by an underlying m-state Markov chain having transition rate $L_{ij}, 1 \leq i, j \leq m,\ i \neq j$, with a transition from state i to j without service completion and having transition rate $M_{ij}, 1 \leq i, j \leq m$, with a transition from state i to j with a service completion. The matrix $\mathbf{L} = [L_{ij}]$ has nonnegative off-diagonal and negative diagonal elements. The matrix $\mathbf{M} = [M_{ij}]$ has nonnegative elements, and both have at least one positive entry. Let $X(t)$ denote the number of customers served in $(0, t]$ with state space $\{n : n \geq 0\}$ and let $J(t)$ be the state of the underlying Markov chain at time t with state space $\{i : 1 \leq i \leq m\}$. Then $\{X(t), J(t)\}$ is a two-dimensional Markov process with state space $\{(n, i) : n \geq 0, 1 \leq i \leq m\}$. The infinitesimal generator of the above Markov process is given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{L} & \mathbf{M} & 0 & 0 & \dots \\ 0 & \mathbf{L} & \mathbf{M} & 0 & \dots \\ 0 & 0 & \mathbf{L} & \mathbf{M} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have $(\mathbf{L} + \mathbf{M})\mathbf{e} = \mathbf{0}$, where $\mathbf{e}$ is a $m \times 1$ vector with all its components equal to 1. Since $(\mathbf{L} + \mathbf{M})$ is the infinitesimal generator of the underlying Markov chain $\{J(t)\}$, there exists a stationary probability vector $\overline{\boldsymbol{\pi}}$ such that $\overline{\boldsymbol{\pi}}(\mathbf{L} + \mathbf{M}) = \mathbf{0},\ \overline{\boldsymbol{\pi}}\mathbf{e} = \mathbf{1}$. The fundamental service rate of the stationary MSP is given by $\mu^* = \overline{\boldsymbol{\pi}}\mathbf{Me}$. The case when the server remains idle for a certain time interval, then a customer enters and the service process starts with the initial phase distribution $f_j,\ j = 1, 2 \dots, m,\ \sum_{j=1}^{m} f_j = 1$, independently of the path followed in the previous service period. Thus, an MSP is characterized by the matrices $\mathbf{L}, \mathbf{M}$ and the phase distribution vector $\mathbf{f} = (f_1, f_2, \dots, f_m)$.

The customers are served individually according to MSP with mean service time $1/\mu^*$. The state of the system at time t is described by the following random variables:

- $N(t)$= number of customers present in the system including the one in service,

- $\zeta(t)$= phase of the service process,
- $U(t)$=remaining inter-arrival time for the next arrival.

We now define the joint probability densities of the system length $N(t)$, phase of the server $\zeta(t)$ and the remaining inter-arrival time $U(t)$, respectively, by

$$\pi_{n,i}(x,t)dx = Pr\{N(t) = n, \zeta(t) = i, x \le U(t) \le x + dx\},\ 0 \le n \le N,$$
$$1 \le i \le m,\ x \ge 0.$$

As $t \to \infty$, the above probabilities are denoted by $\pi_{n,i}(x)$. Further, let $\boldsymbol{\pi}_n(x)$ $(0 \le n \le N)$ denote the row vectors of order $1 \times m$ whose i^{th} component is $\pi_{n,i}(x)$.

3. Analysis of the Model

In this section, we shall carry out the analytical analysis of the model and obtain the steady-state system length distributions at various epochs.

3.1 *Steady-State Probabilities at Pre-Arrival Epoch*

Consider the system just before an arrival of a customer which are taken as embedded points. Let $t_0,\ t_1,\ \ldots$ be the time epochs at which successive arrivals occur and $t_n^-,\ t_n^+$ be the time epochs just before and after the arrival instant t_n, respectively. The inter-arrival times $T_{n+1} = t_{n+1} - t_n,\ n \ge 0$, are independently and identically distributed random variables with common distribution function $A(x)$. Let there be given a non-increasing sequence $\{b_n\},\ 0 \le n \le N$ of non-negative real numbers with $b_0 = 1$ and $b_N = 0$. The sequence $\{b_n\}$ is called balking sequence. Thus, $b_n = Pr\{t_n^+ = t_n^- + 1/t_n^- = n\}$, denotes the probability that if a customer on arrival finds n customers in the system joins the queue. The state of the system at t_i^- is defined as $\{N_s(t_i^-),\ \zeta(t_i^-)\}$, where $N_s(t_i^-)$ is the number of customers in the system, $\zeta(t_i^-)$ indicates the phase of the service process. In the limiting case, let

$$\pi_{n,j}^- = \lim_{t\to\infty} Pr\{N_s(t_i^-) = n,\ \zeta(t_i^-) = j\},\ 0 \le n \le N,\ 1 \le j \le m,$$

where $\pi_{n,j}^-$ denotes the pre-arrival epoch probabilities that that there are n customers in the system when the service process is in phase j. Let $\boldsymbol{\pi}_n^-(0 \le n \le N)$ be the row vector of order $1 \times m$ whose i^{th} component is $\pi_{n,i}^-$.

Let $\mathbf{S}_n(n \ge 0)$ denote an $m \times m$ matrix whose $(i,j)^{th}$ element represents the conditional probability that n customers have been served during an inter-arrival time and the underlying Markov chain of the service process is in phase j just before the arrival, given that the underlying Markov chain was in phase i at the previous pre-arrival epoch.

Observing the state of the system at two consecutive embedded points, we have an embedded Markov chain whose state space is equivalent to $\Omega = \{(i,j),\ 0 \le i \le N,\ 1 \le j \le m\}$. The elements $[\mathbf{P}_{ij}]_{m\times m}$ of the one step transition probability matrix (TPM) $\boldsymbol{\mathcal{P}}_{(N+1)m\times(N+1)m}$ of the above mentioned Markov chain is given by

$$\mathbf{P}_{ij} = \begin{cases} \mathbf{S}_0 & : \ i=0, \ j=1, \\ b_i\mathbf{S}_0 & : \ 1\le i\le N-1, \ 1\le j\le N, \ i+1=j, \\ \boldsymbol{\Delta}_{ij} & : \ 1\le i\le N-1, \ 1\le j\le N, \ (i+1)>j, \\ \mathbf{S}_{N-j} & : \ i=N, \ 1\le j\le N, \\ \boldsymbol{\Gamma}_i & : \ 0\le i\le N, j=0, \\ 0 & : \ \text{otherwise}, \end{cases}$$

where $\boldsymbol{\Delta}_{ij} = b_i\mathbf{S}_{i+1-j} + \bar{b}_i\mathbf{S}_{i-j}$ and $\boldsymbol{\Gamma}_i = \left(\mathbf{I}_m - \sum_{k=0}^{i-1}\mathbf{S}_k - b_i\mathbf{S}_i\right)\mathbf{ef}$, $\mathbf{ef}$ is a stochastic matrix and has the invariant vector $\mathbf{f}$ and $\mathbf{I}_m$ is the identity matrix of order m. It may be remarked here that if any of the $f_j = 0$ $(1\le j\le m)$ then the j^{th} column of the matrices $\boldsymbol{\Gamma}_i$ will be equal to zero and hence $(0,j)^{th}$ column of $\boldsymbol{\mathcal{P}}$ will also be equal to zero. In order to obtain the stochastic matrix $(0,j)^{th}$ row and column should be deleted from the matrix $\boldsymbol{\mathcal{P}}$ and the corresponding component of the pre-arrival epoch probability $\pi^-_{0,j}$ will be equal to zero.

The matrices $\mathbf{S}_n$ involved in the TPM, in general, for arbitrary inter-arrival time distribution requires numerical integration and can be carried out along the lines given in [14]. However, when the inter-arrival time distribution is of PH type, these matrices can be evaluated without any numerical integration as follows:

Let $A(x)$ have a PH distribution with irreducible representation $(\boldsymbol{\alpha}, \ \mathbf{T})$, where $\boldsymbol{\alpha}$ and $\mathbf{T}$ are of dimensions β. Then, the matrices $\mathbf{S}_n$ are given by

$$\mathbf{S}_n = \mathbf{U}_n\left(\mathbf{I}_m\otimes\mathbf{T}^0\right), \ 0\le n\le N-1,$$

where for $0\le n\le N-1$,

$$\mathbf{U}_0 = -\left(\mathbf{I}_m\otimes\boldsymbol{\alpha}\right)\left(\mathbf{L}\otimes\mathbf{I}_\beta + \mathbf{I_m}\otimes\mathbf{T}\right)^{-1},$$
$$\mathbf{U}_n = -\mathbf{U}_{n-1}\left(\mathbf{M}\otimes\mathbf{I}_\beta\right)\left(\mathbf{L}\otimes\mathbf{I}_\beta + \mathbf{I_m}\otimes\mathbf{T}\right)^{-1}, \ 1\le n\le N-1.$$

$\mathbf{T}^0$ is given by $\mathbf{T}^0 = -\mathbf{Te}$ and $\otimes$ denotes the Kronecker product of two matrices. It may be noted here that the various inter-arrival time distributions arising in practical applications can be approximated by PH distributions.

The pre-arrival epoch probabilities $\boldsymbol{\pi}^-_n (0\le n\le N)$ can be evaluated by solving the system of equations $(\boldsymbol{\pi}^-_0, \boldsymbol{\pi}^-_1, \ldots, \boldsymbol{\pi}^-_N) = (\boldsymbol{\pi}^-_0, \boldsymbol{\pi}^-_1, \ldots, \boldsymbol{\pi}^-_N)\boldsymbol{\mathcal{P}}$. We have used GTH algorithm for solving the system of equations.

3.2 *Steady-State Probabilities at Arbitrary Epoch*

To obtain steady-state distribution at an arbitrary epoch we will develop the relations between distribution of number of customers in the system at pre-arrival and arbitrary epochs. For this we use supplementary variable technique and relate the states of the system at two consecutive time epochs t and $t+dt$. Using probabilistic arguments, matrices, vector notations and taking limit as $t\to\infty$, we have the following differential-difference equations at steady-state:

$$-\boldsymbol{\pi}_0^{(1)}(x) = \boldsymbol{\pi}_1(x)\mathbf{M},$$
$$-\boldsymbol{\pi}_n^{(1)}(x) = \boldsymbol{\pi}_n(x)\mathbf{L} + \boldsymbol{\pi}_{n+1}(x)\mathbf{M} + \boldsymbol{\pi}_{n-1}(0)b_{n-1}a(x) + \boldsymbol{\pi}_n(0)(1-b_n)a(x),$$
$$1 \le n \le N-1,$$
$$-\boldsymbol{\pi}_N^{(1)}(x) = \boldsymbol{\pi}_N(x)\mathbf{L} + \boldsymbol{\pi}_{N-1}(0)b_{N-1}a(x) + \boldsymbol{\pi}_N(0)a(x),$$

where $\boldsymbol{\pi}_n(0)$ are the respective probabilities with remaining inter-arrival time equal to zero. Let us define the LST of $\boldsymbol{\pi}_n(x)$ as $\boldsymbol{\pi}_n^*(\theta) = \int_0^\infty e^{-\theta x}\boldsymbol{\pi}_n(x)dx$, so that $\boldsymbol{\pi}_n = \boldsymbol{\pi}_n^*(0)$, where $\boldsymbol{\pi}_n$ is the $1\times m$ vector whose i^{th} component $\pi_{n,i}$ denotes the probability that n customers are in the system and the service process is in phase i at an arbitrary time. Multiplying the above system of equations by $e^{-\theta x}$ and integrating with respect to x over 0 to ∞ yields

$$-\theta\boldsymbol{\pi}_0^*(\theta) = \boldsymbol{\pi}_1^*(\theta)\mathbf{M} - \boldsymbol{\pi}_0(0), \tag{1}$$
$$-\theta\boldsymbol{\pi}_n^*(\theta) = \boldsymbol{\pi}_n^*(\theta)\mathbf{L} + \boldsymbol{\pi}_{n+1}^*(\theta)\mathbf{M} + \boldsymbol{\pi}_{n-1}(0)b_{n-1}A^*(\theta) + \boldsymbol{\pi}_n(0)(1-b_n)A^*(\theta)$$
$$-\boldsymbol{\pi}_n(0),\ 1 \le n \le N-1, \tag{2}$$
$$-\theta\boldsymbol{\pi}_N^*(\theta) = \boldsymbol{\pi}_N^*(\theta)\mathbf{L} + \boldsymbol{\pi}_{N-1}(0)b_{N-1}A^*(\theta) + \boldsymbol{\pi}_N(0)A^*(\theta) - \boldsymbol{\pi}_N(0). \tag{3}$$

Post multiplying (1) to (3) by the vector $\mathbf{e}$, adding them and using $(\mathbf{L}+\mathbf{M})\mathbf{e} = \mathbf{0}$, we obtain

$$\sum_{n=0}^{N}\boldsymbol{\pi}_n^*(\theta)\mathbf{e} = \frac{1-A^*(\theta)}{\theta}\sum_{n=0}^{N}\boldsymbol{\pi}_n(0)\mathbf{e}.$$

Taking limit as $\theta \to 0$ and using the normalization condition $\sum_{n=0}^{N}\boldsymbol{\pi}_n\mathbf{e} = 1$ yields

$$\sum_{n=0}^{N}\boldsymbol{\pi}_n(0)\mathbf{e} = \lambda. \tag{4}$$

3.2.1 Relations between system length distribution at arbitrary and pre-arrival epochs

The pre-arrival epoch probabilities $\boldsymbol{\pi}_n^-$ and the rate probabilities $\boldsymbol{\pi}_n(0)$ are related by:

$$\boldsymbol{\pi}_n^- = \frac{1}{\lambda}\boldsymbol{\pi}_n(0),\ 0 \le n \le N, \tag{5}$$

where λ is given in (4). Setting $\theta = 0$ in (2), (3) and using (5), we obtain the arbitrary epoch probabilities as

$$\boldsymbol{\pi}_N = \lambda b_{N-1}\boldsymbol{\pi}_{N-1}^-(-\mathbf{L})^{-1},$$
$$\boldsymbol{\pi}_n = \left(\boldsymbol{\pi}_{n+1}\mathbf{M} + \lambda(b_{n-1}\boldsymbol{\pi}_{n-1}^- - b_n\boldsymbol{\pi}_n^-)\right)(-\mathbf{L})^{-1},\ n = N-1,\ N-2,\ldots,1.$$

As the only unknown $\boldsymbol{\pi}_0$ cannot be obtained explicitly from the steady-state equations (1) to (3), we evaluate it using normalization condition as $\boldsymbol{\pi}_0 = \overline{\boldsymbol{\pi}} - \sum_{n=1}^{N}\boldsymbol{\pi}_n$. This completes the evaluation of steady-state probabilities at various epochs.

Remark 1: Taking $b_n = 1,\ 0 \le n \le N-1$, our model reduces to $GI/MSP/1/N$ queue and the results match numerically with [9], see Table 1.
Remark 2: Taking MSP representation for Poisson service times, our model reduces to $GI/M/1/N$ queue with balking and our results match numerically with [7], see Table 2.

4. Performance Measures

Performance measures are important features of any queueing system as they reflect the efficiency of the queueing system under consideration. Once the steady-state probabilities at different epochs are known, various performance measures of the system can be obtained. The blocking probability (P_{loss}), the expected number of customers in the system (L_s) and the expected waiting-time in the system (W_{sl}) using Little's rule are, respectively, given by $P_{loss} = \boldsymbol{\pi}_N^- \mathbf{e}$; $L_s = \sum_{n=1}^{N} n\boldsymbol{\pi}_n \mathbf{e}$; $W_{sl} = L_s/\lambda'$, where $\lambda' = \lambda(J.N.)$ and $J.N. = \sum_{n=0}^{N-1} b_n \boldsymbol{\pi}_n^- \mathbf{e}$. The average balking rate $(B.R.)$ is given by $B.R. = \sum_{n=1}^{N} \lambda(1-b_n)\boldsymbol{\pi}_n \mathbf{e}$.

4.1 *Waiting-Time Analysis*

In this subsection, we obtain the expected waiting-time in the system of an arriving customer (who joins the queue), under the FCFS discipline. Let $W_s^*(\theta)$ and W_s be the LST and the expected waiting-time in the system of a customer who joins the queue and waits for service, respectively. Let $\boldsymbol{\delta}_k(\theta)$ be the LST of the probability that k customers will be served within a time x and the service process upon completion of service passes to phase j, provided k customers were in the system and the service process was in phase i at the beginning of service. Since the probability that the service of a customer who joins the queue is completed in the interval $[x, x+dx]$ is given by the matrix $e^{\mathbf{L}x}\mathbf{M}dx$ and the total service time of k customers is the sum of their service times, we have

$$\boldsymbol{\delta}_1(\theta) = \int_0^\infty e^{-\theta x} e^{\mathbf{L}x}\mathbf{M}dx = (\theta \mathbf{I}_m - \mathbf{L})^{-1}\mathbf{M}, \quad \boldsymbol{\delta}_k(\theta) = \boldsymbol{\delta}_1^k(\theta),\ k \ge 2.$$

Therefore, $W_s^*(\theta)$ is given by

$$W_s^*(\theta) = \frac{1}{J.N.} \sum_{n=0}^{N-1} b_n \boldsymbol{\pi}_n^- \boldsymbol{\delta}_1^{n+1}(\theta)\mathbf{e}.$$

The expected waiting-time in the system is given by

$$W_s = \frac{1}{J.N.} \sum_{n=0}^{N-1} \sum_{k=0}^{n} b_n \boldsymbol{\pi}_n^- (-\mathbf{L}^{-1}\mathbf{M})^k (-\mathbf{L})^{-1}\mathbf{e}. \tag{6}$$

It may be noted that the expected waiting-time obtained from (6) matches exactly with W_{sl} obtained from Little's rule.

Table 1. Steady-state probabilities of $PH/MSP/1/80$ queue with and without balking

	with balking		without balking	
n	$\boldsymbol{\pi}_n^-\mathbf{e}$	$\boldsymbol{\pi}_n\mathbf{e}$	$\boldsymbol{\pi}_n^-\mathbf{e}$	$\boldsymbol{\pi}_n\mathbf{e}$
0	0.503032	0.503761	0.460374	0.461098
1	0.261868	0.261440	0.181168	0.180868
2	0.139937	0.139756	0.109279	0.109142
3	0.062684	0.062605	0.074485	0.074399
4	0.022952	0.022924	0.052031	0.051971
5	0.007078	0.007069	0.036510	0.036468
10	0.000003	0.000003	0.006241	0.006234
15	0.000000	0.000000	0.001067	0.001066
20	0.000000	0.000000	0.000182	0.000182
⋮	⋮	⋮	⋮	⋮
80	0.000000	0.000000	0.000000	0.000000
Sum	1.000000	1.000000	1.000000	1.000000
L_s	0.871176		1.732004	
W_s	2.881797		4.024256	
$B.R.$	0.127905		0.000000	

Table 2. Steady-state probabilities of $M/M/1/5$ queue with balking

n	0	1	2	3	4	5	Sum
$\boldsymbol{\pi}_n^-\mathbf{e}$	0.423277	0.364018	0.156528	0.044871	0.009647	0.001659	1.000000
$\boldsymbol{\pi}_n\mathbf{e}$	0.423277	0.364018	0.156528	0.044871	0.009647	0.001659	1.000000

$L_s = 0.858573,\ W_s = 2.0,\ B.R. = 0.141638.$

5. Numerical Results

To demonstrate the applicability of the analytical results obtained in the previous sections, extensive graphical and numerical work has been done. The balking function is taken as $b_n = 1/(n+1), 1 \le n \le N-1$ with $b_0 = 1$. Further, it is assumed that for $n \ge N,\ b_n = 0$.

The steady-state probabilities of $PH/MSP/1/80$ queue with and without balking is presented in Table 1. The MSP representation is taken as $\mathbf{L} = \begin{bmatrix} -3.39 & 0.0 \\ 0.0 & -0.21 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} 3.19 & 0.2 \\ 0.2 & 0.01 \end{bmatrix}$ and $\mathbf{f} = (0.6, 0.4)$ with $\mu^* = 1.8$. The PH representation is taken as $\boldsymbol{\alpha} = (0.3, 0.7)$, $\mathbf{T} = \begin{bmatrix} -0.6 & 0.074 \\ 0.0575 & -0.45 \end{bmatrix}$ with $\lambda = 0.430391$. The fourth and fifth columns of the table exactly match with [9]. Table 2 displays the system length distributions of $M/M/1/5$ queue with balking. The MSP representation of exponential distribution is taken as $\mathbf{L} = [-0.5],\ \mathbf{M} = [0.5]$ and $\mathbf{f} = (1.0)$ with $\mu^* = 0.5$ and $\lambda = 0.43$. One may note that the pre-arrival and arbitrary epoch probabilities are the same due to memoryless property of exponential distribution.

The effect of N on P_{loss} and W_s is depicted in Figures 1 and 2, respectively, in $PH/MSP/1$ queue. For Figure 1, the following three types of service time distributions have been considered:

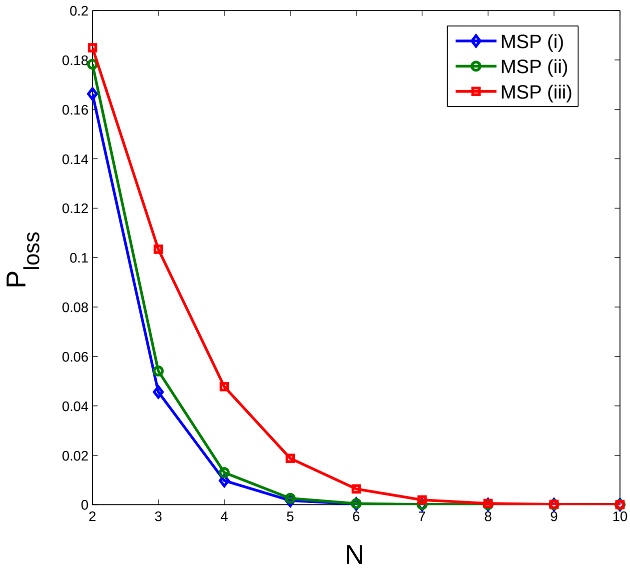

Figure 1. N versus P_{loss}

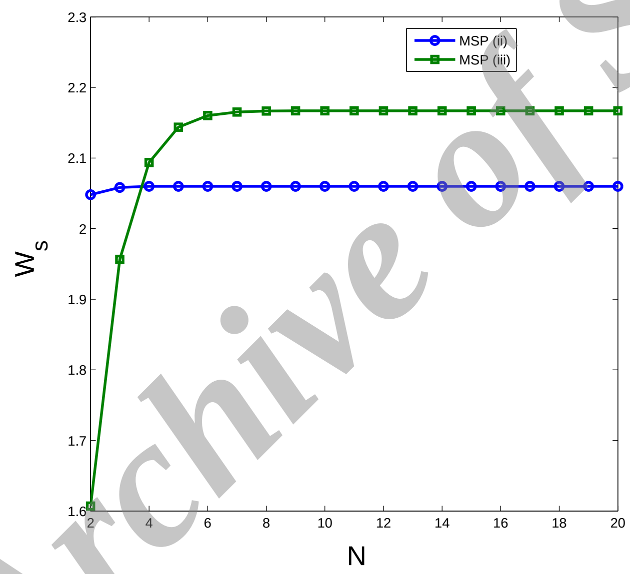

Figure 2. Effect of N on W_s

(i) Poisson with MSP representation $\mathbf{L} = [-0.5]$, $\mathbf{M} = [0.5]$ and $\mathbf{f} = (1.0)$,

(ii) MSP with $\mathbf{L} = \begin{bmatrix} -1.5125 & 0.750 \\ 0.875 & -1.025 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} 0.7625 & 0.0 \\ 0.125 & 0.025 \end{bmatrix}$ and $\mathbf{f} = (1.0, 0.0)$,

(iii) MSP with $\mathbf{L} = \begin{bmatrix} -6.9375 & 0.9375 \\ 0.0625 & -0.1958 \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} 6.0 & 0.0 \\ 0.0 & 0.1333 \end{bmatrix}$ and $\mathbf{f} = (1.0, 0.0)$.

MSPs (ii) and (iii) have lag 2 correlation coefficients equal to 0.0000050 and 0.143181, respectively. The above three service time distributions have equal mean service rate $\mu^* = 0.5$. The PH representation is taken as $\boldsymbol{\alpha} = (0.3, 0.7)$, $\mathbf{T} = \begin{bmatrix} -0.6 & 0.07 \\ 0.06 & -0.45 \end{bmatrix}$ with $\lambda = 0.43$. For Figure 2, we have considered MSP (ii) and

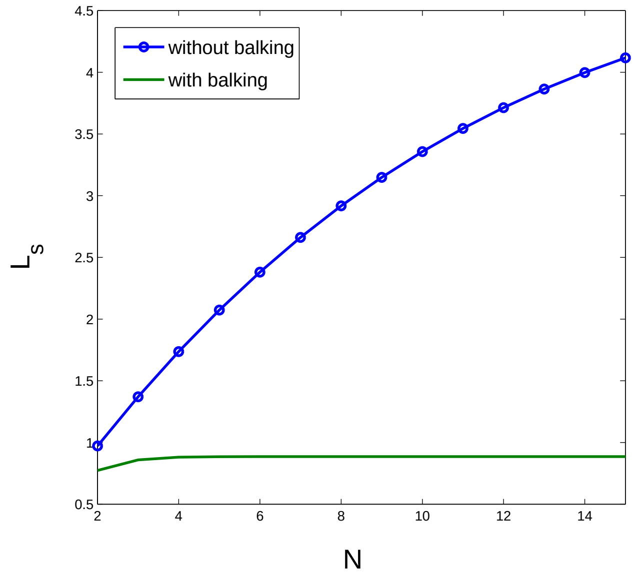

Figure 3. Impact of N on L_s

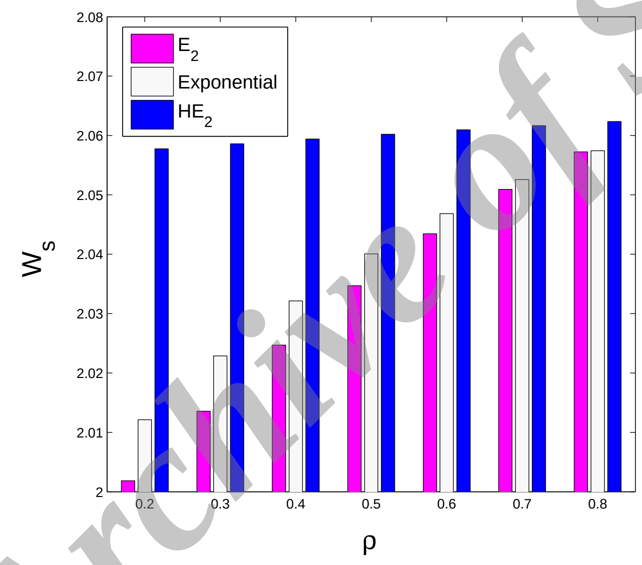

Figure 4. ρ versus W_s

(iii) representations and the PH representation is taken as in Figure 1. It may be observed from the figure that W_s is lower for highly correlated MSP (iii) for $N = 2, 3, 4$ and for $N > 4$, W_s is lower for MSP (ii).

The dependence of expected system length on buffer capacity is presented in Figure 3 for $E_3/MSP/1$ queue with and without balking. The MSP representation is taken as in case (ii) and the Erlang-3 inter-arrival time representation is taken as $\boldsymbol{\alpha}= (1.0, 0.0, 0.0)$, $\mathbf{T} = \begin{bmatrix} -1.29 & 1.29 & 0.0 \\ 0.0 & -1.29 & 1.29 \\ 0.0 & 0.0 & -1.29 \end{bmatrix}$ with $\lambda = 0.43$. The expected system lengths increase with N in both the models. Moreover, for fixed N, the system lengths are lower in the case of queues with balking compared to queues without balking as it should be.

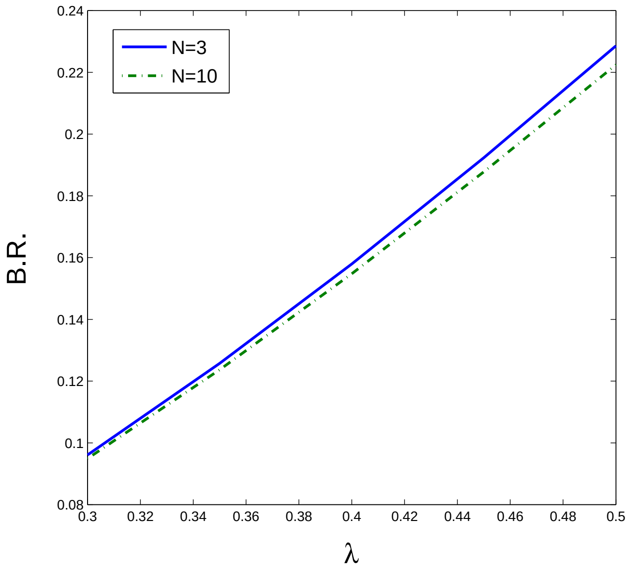

Figure 5. Effect of λ on $B.R.$

The effect of $\rho(=\lambda/\mu^*)$ on W_s is presented in Figure 4 for three inter-arrival time distributions. The three inter-arrival time distributions are taken as exponential, Erlang-2 (E_2) and hyper exponential (HE_2) distributions. The PH representations of the above inter-arrival time distributions are taken as (i) $\boldsymbol{\alpha}= (1.0)$, $\mathbf{T} = [-\gamma]$ with $\lambda = \gamma$, (ii) $\boldsymbol{\alpha}= (1.0, 0.0)$, $\mathbf{T} = \begin{bmatrix} -\gamma & \gamma \\ 0.0 & -\gamma \end{bmatrix}$ with $\lambda = \gamma/2$ and (iii) $\boldsymbol{\alpha}= (0.149883, 0.850117)$, $\mathbf{T} = \begin{bmatrix} -\gamma_1 & 0.0 \\ 0.0 & -\gamma_2 \end{bmatrix}$ with $1/\lambda = 0.149883/\gamma_1 + 0.850117/\gamma_2$. By suitably varying the values of γ, γ_1 and γ_2, various values of ρ can be obtained. The MSP representation is taken as in case (ii) of Figure 1. From the figure, it can be seen that HE_2 distribution yields higher W_s compared to exponential and E_2 inter-arrival time distributions. Further, for any inter-arrival time distribution W_s increases with ρ.

The impact of λ on $B.R.$ in $E_3/MMPP/1$ queue is presented in Figure 5 for $N = 3$ and 10. The PH representation of E_3 distribution is taken as $\boldsymbol{\alpha}= (1.0, 0.0, 0.0)$, $\mathbf{T} = \begin{bmatrix} -\gamma & \gamma & 0.0 \\ 0.0 & -\gamma & \gamma \\ 0.0 & 0.0 & -\gamma \end{bmatrix}$ with $\lambda = \gamma/3$. By suitably varying the values of γ, one can obtain various values of λ. The service time is taken as $MMPP$ whose infinitesimal generator is

$$\mathbf{R} = \begin{bmatrix} -0.1 & 0.1 & 0.0 \\ 0.003 & -0.183 & 0.18 \\ 0.02 & 0.38 & -0.4 \end{bmatrix}$$

and $\boldsymbol{\Upsilon} = diag(0.3, 0.569, 0.4)$. One may note that $\mathbf{L} = \mathbf{R} - \boldsymbol{\Upsilon}$, $\mathbf{M} = \boldsymbol{\Upsilon}$ and $\mathbf{f} = (0.6, 0.3, 0.1)$ with $\mu^* = 0.5$. The average balking rate increases with the increase of λ, for fixed N. Further, $B.R.$ decreases with the increase of N. However, the decrease in $B.R.$ is insignificant for $N > 10$.

6. Conclusions

In this paper, we have carried out an analysis of a finite buffer $GI/MSP/1$ queue with balking. The distributions of the system length at pre-arrival and arbitrary epochs are obtained. The former are obtained using embedded Markov chain technique while the latter are obtained using supplementary variable technique. Computational experiences with a variety of numerical results in the form of tables and graphs are discussed to display the effect of the system parameters on the performance measures. The present model can be generalized to finite buffer $GI/MSP/1$ queue with N-policy and balking using the procedure discussed in this paper which is left for future investigation.

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