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Effect of Counterpropagating Capillary Gravity Wave Packets on Third Order Nonlinear Evolution Equations in the Presence of Wind Flowing Over Water

A. K. Dhar[a] and J. Mondal[b,*]

[a] *Department of Mathematics,Indian Institute of Engineering Science and Technology, P.O. Botanical Garden, Howrah - 711103, West Bengal, India;*
[b] *Department of Mathematics,Kandi Raj High School, Murshidabad-742137, West Bengal, India.*

Abstract. Asymptotically exact and nonlocal third order nonlinear evolution equations are derived for two counterpropagating surface capillary gravity wave packets in deep water in the presence of wind flowing over water.From these evolution equations stability analysis is made for a uniform standing surface capillary gravity wave trains for longitudinal perturbation.Instability condition is obtained and graphs are plotted for maximum growth rate of instability and for wave number at marginal stability against wave steepness for some different values of dimensionless wind velocity.

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1. Introduction

One approach to studying the stability of finite amplitude surface waves in deep water is through the application of the lowest order nonlinear evolution equation,which is the nonlinear Schrödinger equation.Zakharov's [12] study is along this line, allowing for finite amplitude wave trains to be subjected to modulational

*Corresponding author. Email: joydev.71@gmail.com

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perturbations in two horizontal directions both along and perpendicular to the direction of the wave train. Benney and Newell [1] and Hasimoto and Onto [10] derived a single equation describing long-time evolution of the envelope of one dimensional surface-gravity wave packet on the surface of water of finite depth. Devey and Stewartson [3],M. Matinfar [9] S. Ahmadi [5] extended this for a two dimensional wave packet and showed that the nonlinear evolution equation in this case is governed by two coupled equations. These equations including the effect of capillarity were derived by Djordjevic and Redekopp [6] which give the nonlinear evolution equation of a two dimensional capillary gravity wave packet. The corresponding equation for a one dimensional wave packet was obtained by Kawahara [8].

The third order nonlinear evolution equations have been derived by Pierce and Knobloch[11] for two counterpropagating capillary gravity wave packets on the surface of water of finite depth.The resulting equations are asymptotically exact and nonlocal and generalize the equations derived by Djordjevic and Redekopp [6] for counterpropagating waves. In the present paper third order nonlinear evolution equations are derived for two counterpropagating capillary gravity wave packets in the surface water of infinite depth in the presence of wind flowing over water. So this paper is an extension of the evolution equations derived by Pierce and knobloch [11] for an infinite depth water and in the presence of wind flowing over water. These evolution equations remain valid when the dimensionless wind velocity is less than a critical velocity. This critical velocity is defined by the fact that a wave becomes linearly unstable if the wind velocity exceeds this critical velocity. From these evolution equations stability analysis is investigated for a uniform standing surface capillary gravity wave trains with respect to longitudinal perturbation.The expressions for the maximum growth rate of instability and the wave number at marginal stability are derived.Graphs are plotted for maximum growth rate of instability and for wave number at marginal stability against wave steepness for some different values of dimensionless wind velocity. It is observed that in the third order analysis the maximum growth rate of instability increases steadily with the increase of wave steepness.The growth rate is found to be appreciably much higher for dimensionless wind velocity approaching its critical value. The wave number at marginal stability has also been plotted against wave steepness for some different values of dimensionless wind velocity.

2. Basic Concept and Basic Equations

We take the common horizontal interface between water and air in the undisturbed state as $z = 0$ plane and assume that air flows over water with a velocity u in a direction that is taken as the x- axis. We take $z = \zeta(x, y, t)$ as the equation of the common interface at any time t in the perturbed state.We introduce the dimensionless quantities $\widetilde{\phi}$, $\widetilde{\phi}\,'$, $\widetilde{\zeta}$, $(\widetilde{x}, \widetilde{y}, \widetilde{z})$, $\widetilde{t}$, $\widetilde{v}$, $\widetilde{\gamma}$ and $\widetilde{s}$ which are respectively,the perturbed velocity potential in water, perturbed velocity potential in air, surface elevation of the water-air interface,space coordinates,time,air flow velocity,the ratio of the densities of air to water and surface tension.

These dimensionless quantities are related to the corresponding dimensional quantities by the following relations

$$\widetilde{\phi} = \sqrt{k_0^3/g}\ \phi, \quad \widetilde{\phi}\,' = \sqrt{k_0^3/g}\ \phi\,', \quad (\widetilde{x}, \widetilde{y}, \widetilde{z}) = (k_0x, k_0y, k_0z),$$

$$\widetilde{\zeta} = k_0\zeta, \quad \widetilde{t} = \omega t, \quad \widetilde{v} = \sqrt{k_0/g}\; u, \quad \widetilde{\gamma} = \rho\,'/\rho, \quad \widetilde{s} = Tk_0^2/g,$$

where k_0 is some characteristic wave number, g is the acceleration due to gravity, ρ and ρ' are the densities of water and air respectively and T is the dimensional surface tension. In the future, all the quantities will be written in their dimensionless form with their tilde sign ($\sim$) dropped.

The perturbed velocity potentials ϕ and ϕ' satisfy the following Laplace equations

$$\nabla^2\phi = 0 \qquad \text{in} \quad -\infty < z < \zeta \tag{1}$$

$$\nabla^2\phi' = 0 \qquad \text{in} \quad \zeta < z < \infty \tag{2}$$

The kinematic boundary condition for water is

$$\left(\frac{\partial\phi}{\partial z} - \frac{\partial\zeta}{\partial t} = \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}\right), \qquad \text{when} \quad \mathrm{z} = \zeta \tag{3}$$

which gives a necessary condition for equality of water velocity at the interface normal to it to the normal velocity of the interface.

Similar condition for air is the following

$$\left(\frac{\partial\phi\,'}{\partial z} - \frac{\partial\zeta}{\partial t} - v\frac{\partial\zeta}{\partial x} = \frac{\partial\phi\,'}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi\,'}{\partial y}\frac{\partial\zeta}{\partial y}\right), \qquad \text{when} \quad \mathrm{z} = \zeta \tag{4}$$

The condition of continuity of pressure at the interface gives

$$\left\{\frac{\partial\phi}{\partial t} - \gamma\frac{\partial\phi\,'}{\partial t}\right\} + (1-\gamma)\zeta - \gamma v\frac{\partial\phi\,'}{\partial x} = -\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial\, x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}$$
$$+\frac{\gamma}{2}\left\{\left(\frac{\partial\phi\,'}{\partial\, x}\right)^2 + \left(\frac{\partial\phi\,'}{\partial y}\right)^2 + \left(\frac{\partial\phi\,'}{\partial\, z}\right)^2\right\} + s\left\{1 + \left(\frac{\partial\zeta}{\partial\, x}\right)^2 + \left(\frac{\partial\zeta}{\partial\, y}\right)^2\right\}^{-\frac{3}{2}}$$
$$\left\{\left(\frac{\partial\zeta}{\partial\, x}\right)^2\frac{\partial^2\zeta}{\partial\, y^2} + \left(\frac{\partial\zeta}{\partial\, y}\right)^2\frac{\partial^2\zeta}{\partial\, x^2} - 2\frac{\partial\zeta}{\partial\, x}\frac{\partial\zeta}{\partial\, y}\frac{\partial^2\zeta}{\partial\, x\partial\, y} + \frac{\partial^2\zeta}{\partial\, x^2} + \frac{\partial^2\zeta}{\partial\, y^2}\right\} \tag{5}$$

$$\text{when} \quad z = \zeta$$

Finally ϕ and ϕ' should satisfy the following boundary conditions at infinity

$$\frac{\partial\phi}{\partial z} \to 0 \quad \text{when} \quad \mathrm{z} \to -\infty \tag{6}$$

$$\frac{\partial\phi'}{\partial z} \to 0 \quad \text{when} \quad \mathrm{z} \to +\infty \tag{7}$$

Since the disturbance is assumed to be a progressive wave we look for solutions of the equations(1)–(7) in the following form

$$P = P_{00} + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left\{ P_{mn}\ \exp i(m\psi_1 + n\psi_2) + P^*_{mn} \exp -i(m\psi_1 + n\psi_2) \right\}, \tag{8}$$

where P stands for ϕ, ϕ' and ζ; $\psi_1 = kx - \omega t$, $\psi_2 = kx + \omega t$. In the summation on the right of equation (8), $(m, n) \neq (0, 0)$. Here $\phi_{00}, \phi_{mn}, \phi^*_{mn}, \phi\ '_{00}, \phi\ '_{mn}, \phi'^*_{mn}$ are functions of z, $x_1 = \epsilon x, y_1 = \epsilon y, t_1 = \epsilon t$; $\zeta_{00}, \zeta_{mn}, \zeta^*_{mn}$ are functions of x_1, y_1, t_1. ϵ is a small parameter measuring the weakness of wave steepness, which is the product of wave amplitude and wave number and the sign * denotes complex conjugate.

The linear dispersion relation determining ω

$$(1+\gamma)\omega^2 - 2\gamma\omega v + \gamma v^2 - (1-\gamma) - s = 0 \tag{9}$$

which gives two values of ω given by

$$\omega_{\pm} = \left(\gamma v \pm \sqrt{1 - \gamma^2 - \gamma v^2 + s(1-\gamma)} \right) / (1+\gamma) \tag{10}$$

which corresponds to two modes and we designate these two modes as positive and negative modes. The positive mode moves in the positive direction of the x-axis with a frequency $\{\sqrt{1 - \gamma^2 - \gamma v^2 + s(1-\gamma)} + \gamma v\}/(1+\gamma)$ while the negative mode moves in the negative direction of the x-axis with a frequency $\left(\sqrt{1 - \gamma^2 - \gamma v^2 + s(1-\gamma)} - \gamma v\right)/(1+\gamma)$. If v is replaced by $-v$ the frequency of the positive mode becomes equal to the frequency of the negative mode. So the results for the negative mode can be obtained from those for the positive mode by replacing v by $-v$. Therefore we have made a nonlinear analysis for the positive mode only and then we have obtained the results for the negative mode by replacing v by $-v$.

From the expression (10) for $\omega_{\pm}$ we find that for linear stability v should satisfy the condition

$$|v| < \sqrt{\left\{1 - \gamma^2 + s(1-\gamma)\right\}/\gamma} \tag{11}$$

So our present analysis will remain valid as long as the dimensionless flow velocity of the wind becomes less than the critical value $\sqrt{\left\{1 - \gamma^2 + s(1-\gamma)\right\}/\gamma}$. For air flowing over water $\gamma = 0.00129$ and this critical value becomes 27.87 for $s = 0.075$.

3. Derivation of evolution equations

On substituting the expansions (8) in equations (1),(2),(6),(7) and then equating the coefficients of $\exp i(m\psi_1 + n\psi_2)$ for $\{(m,n)=(1,0),(0,1),(2,0),(0,2),(1,1),(-1,1)\}$ we get the following equations:

$$\left(\frac{\partial^2}{\partial z^2} - \triangle^2_{mn} \right) \phi_{mn} = 0 \tag{12}$$

$$\left(\frac{\partial^2}{\partial z^2} - \triangle^2_{mn} \right) \phi'_{mn} = 0 \tag{13}$$

$$\frac{\partial \phi_{mn}}{\partial z} \to 0 \quad \text{at} \quad z \to -\infty \tag{14}$$

$$\frac{\partial \phi'_{mn}}{\partial z} \to 0 \quad \text{at} \quad z \to +\infty \tag{15}$$

where Δ_{mn} is the operator given by

$$\Delta^2_{mn} = \left\{ (m+n) - i\epsilon \frac{\partial}{\partial x_1} \right\}^2 - \epsilon^2 \frac{\partial^2}{\partial y_1^2} \tag{16}$$

The solutions of equations (12) and (13) satisfying boundary conditions (14) and (15) respectively can be put in the following forms

$$\phi_{mn} = \exp\left(\triangle_{mn} z\right) A_{mn} \tag{17}$$

$$\phi'_{mn} = \exp\left(-\triangle_{mn} z\right) A'_{mn} \tag{18}$$

where $A_{mn},\ A'_{mn}$ are functions of x_1, y_1 and t_1.

On substituting the expansions (8) in the Taylor expanded forms of equations (3)–(5) about $z = 0$ and then equating the coefficients of $\exp i(m\psi_1 + n\psi_2)$ for $\{(m,n)=$ $(1,0)$,$(0,1)$,$(2,0)$,$(0,2)$,$(1,1)$,$(-1,1)\}$ on both sides,we get the following equations

$$\left(\frac{\partial \phi_{mn}}{\partial z}\right)_{z=0} + i\left\{ (m-n)\omega + i\epsilon \frac{\partial}{\partial t_1} \right\} \zeta_{mn} = a_{mn} \tag{19}$$

$$\left(\frac{\partial \phi'_{mn}}{\partial z}\right)_{z=0} + i\left\{ (m-n)\omega + i\epsilon \frac{\partial}{\partial t_1} \right\} \zeta_{mn} - iv\left\{ (m+n) - i\epsilon \frac{\partial}{\partial x_1} \right\} \zeta_{mn} = b_{mn} \tag{20}$$

$$-i\left\{ (m-n)\omega + i\epsilon \frac{\partial}{\partial t_1} \right\} (\phi_{mn})_{z=0} + i\gamma \left\{ (m-n)\omega + i\epsilon \frac{\partial}{\partial t_1} \right\} (\phi'_{mn})_{z=0}$$
$$+ s\Delta^2_{mn}\zeta_{mn} + (1-\gamma)\zeta_{mn} - i\gamma v \left\{ (m+n) - i\epsilon \frac{\partial}{\partial x_1} \right\} (\phi'_{mn})_{z=0} = \quad c_{mn} \tag{21}$$

where a_{mn}, b_{mn}, c_{mn} are contributions from nonlinear terms and $(\ )_{z=0}$ implies the value of the quantity inside parentheses at $z = 0$. Now for the above six values of (m,n) we obtain six sets of equations,in which we substitute the solutions for ϕ_{mn}, ϕ'_{mn} given by (17) and (18). The sets of equations corresponding to $\{(m,n)$=(1,0),(0,1)$\}$,$\{(m,n)$=$(2,0)$,$(0,2)$,$(1,1)$, $(-1,1)\}$ will be called, respectively,the first and second sets.

To solve the above three sets of equations we make the following perturbation expansions for the quantities A_{mn}, A'_{mn}, ζ_{mn} for the above values of (m, n):

$$
\begin{aligned}
F_{mn} &= \sum_{p=1}^{\infty} \epsilon^p F_{mn}^{(p)}, \qquad \text{for } (m,n) = (1,0),(0,1) \\
&= \sum_{p=2}^{\infty} \epsilon^p F_{mn}^{(p)}, \qquad \text{for } (m,n) = (2,0),(0,2),(1,1),(-1,1)
\end{aligned}
\tag{22}
$$

where F_{mn} stands for $A_{mn},, A'_{mn}$ and ζ_{mn}.

On substituting the expansions (22) in the above three sets of equations and then equating coefficients of various powers of ϵ on both sides,we obtain a sequence of equations. From the first order (that is lowest order) and second order equations corresponding to (19)and (20) of the first set of equations we obtain solutions for $A_{10}^{(1)}, A_{10}^{'(1)}, A_{10}^{(2)}, A_{10}^{'(2)}$ and $A_{01}^{(1)}$, $A_{01}^{'(1)}, A_{01}^{(2)}, A_{01}^{'(2)}$ respectively. Next, from the second order equation corresponding to (19),(20) and (21) of the second set of equations,we obtain solutions for $(A_{20}^{(2)}, A_{20}^{'(2)}, \zeta_{20}^{(2)})$,$(A_{02}^{(2)}, A_{02}^{'(2)}, \zeta_{02}^{(2)})$,$(A_{11}^{(2)}, A_{11}^{'(2)}, \zeta_{11}^{(2)})$, $(A_{-11}^{(2)}, A_{-11}^{'(2)}, \zeta_{-11}^{(2)})$ respectively. Following Pierce and Knobloch [11] we use the following transformations of all perturbed quantities in slow space coordinates and time

$$
\xi_+ = x_1 - c_g t_1, \;\; \xi_- = x_1 + c_g t_1, \;\; \zeta = y_1, \; \tau = \epsilon t_1, \tag{23}
$$

where c_g is the group velocity given by $c_g = (d\omega/dk)_{k=1}$. The equations corresponding to (21) for $\{(m,n)$=$(1,0)$,$(0,1)\}$ of the first set of equations,which has not been used in obtaining the above perturbation solutions can be put in the following convenient forms after eliminating $A_{mn}^{(p)}, A_{mn}^{'(p)}$

$$
\left[\omega_1^2 + \gamma(\omega_1 - vk)^2 - (1-\gamma)\Delta_{10}\right]\zeta_{10} = -i\omega_1 a_{10} - i\gamma(\omega_1 - vk)b_{10} - \Delta_{10}c_{10} \tag{24}
$$

$$
\left[\omega_1^2 + \gamma(\omega_1 - vk)^2 - (1-\gamma)\Delta_{01}\right]\zeta_{01} = -i\omega_1 a_{01} - i\gamma(\omega_1 - vk)b_{01} - \Delta_{01}c_{01} \tag{25}
$$

where $a_{10}, b_{10}, c_{10}, a_{01}, b_{01}, c_{01}$ are contributions from nonlinear terms.

From equations (24) and (25) we get the following equations in two successive orders starting from the lowest order two.

$O(\epsilon^2)$:

$$
\frac{\zeta_{10}^{(1)}}{\partial \xi_-} = 0\ , \tag{26}
$$

$$
\frac{\zeta_{01}^{(1)}}{\partial \xi_+} = 0. \tag{27}
$$

which shows that $\zeta_{10}^{(1)}$ and $\zeta_{01}^{(1)}$ are independent of ξ_- and ξ_+ respectively. $O(\epsilon^3)$:

$$i\frac{\partial\zeta_{10}^{(1)}}{\partial\tau}+i\gamma_0\frac{\partial\zeta_{10}^{(2)}}{\partial\xi_-}+\gamma_1\frac{\partial^2\zeta_{10}^{(1)}}{\partial\xi_+^2}+\gamma_2\frac{\partial^2\zeta_{10}^{(1)}}{\partial\zeta^2}=\delta_1\zeta_{10}^{(1)^2}\zeta_{10}^{(1)^*}+\delta_2\zeta_{10}^{(1)}\zeta_{01}^{(1)}\zeta_{01}^{(1)^*}, \tag{28}$$

$$-i\frac{\partial\zeta_{01}^{(1)}}{\partial\tau}+i\gamma_0\frac{\partial\zeta_{01}^{(2)}}{\partial\xi_+}+\gamma_1\frac{\partial^2\zeta_{01}^{(1)}}{\partial\xi_-^2}+\gamma_2\frac{\partial^2\zeta_{01}^{(1)}}{\partial\zeta^2}=\delta_1\zeta_{01}^{(1)^2}\zeta_{01}^{(1)^*}+\delta_2\zeta_{01}^{(1)}\zeta_{10}^{(1)}\zeta_{10}^{(1)^*}. \tag{29}$$

Using equations (26) and (27) in equations (28) and (29) respectively we obtain the following third order nonlinear evolution equations for two counterpropagating waves:

$$\begin{aligned}i\frac{\partial\zeta_{10}^{(1)}}{\partial\tau}+i\gamma_0\frac{\partial\zeta_{10}^{(2)}}{\partial\xi_-}+\gamma_1\big(\frac{\partial^2}{\partial\xi_+^2}+\frac{\partial^2}{\partial\xi_-^2}\big)\zeta_{10}^{(1)}+\gamma_2\frac{\partial^2\zeta_{10}^{(1)}}{\partial\eta^2}\\ =\delta_1|\zeta_{10}^{(1)}|^2\zeta_{10}^{(1)}+\delta_2|\zeta_{01}^{(1)}|^2\zeta_{10}^{(1)}\end{aligned} \tag{30}$$

$$\begin{aligned}-i\frac{\partial\zeta_{01}^{(1)}}{\partial\tau}+i\gamma_0\frac{\partial\zeta_{01}^{(2)}}{\partial\xi_+}+\gamma_1\big(\frac{\partial^2}{\partial\xi_+^2}+\frac{\partial^2}{\partial\xi_-^2}\big)\zeta_{01}^{(1)}+\gamma_2\frac{\partial^2\zeta_{01}^{(1)}}{\partial\eta^2}\\ =\delta_1|\zeta_{01}^{(1)}|^2\zeta_{01}^{(1)}+\delta_2|\zeta_{10}^{(1)}|^2\zeta_{01}^{(1)}\end{aligned} \tag{31}$$

where the coefficients γ_0 , γ_1 , γ_2 and δ_1 , δ_2 are given in the Appendix.

In equation (30),if we restrict to the nonlinear evolution of unidirectional wave train propagating in the positive direction of x-axis, that is if we set $\zeta_{01}=0$ and assume that ζ_{10} is independent of ξ_-, then we recover the third-order nonlinear evolution equation for a capillary gravity waves in the presence of wind flowing over water. This reduced equation in the absence of capillarity is found to be same as equation (34) of Dhar and Das [4] after neglecting the fourth order terms. This reduced equation for $v=0$, $\gamma=0$ and $s=0$ has also been verified to be equivalent to equation (2) of Janssen [7] if we consider the third order terms only.

As each of the left and right propagating waves sees the counterpropagating wave only through its mean square amplitude,the nonlocal mean field equations suitable for stability analysis obtained from (30) and (31) by applying the averaging procedure of Pierce and Knobloch [11]. Therefore,following them we define the average of a function of two variables ξ_+ and ξ_- with respect to any one of these two variables by

$$\langle h\rangle_\pm=\frac{1}{p_\pm}\int_{-(\frac{1}{2})p_\pm}^{(\frac{1}{2})p_\pm} h\, d\xi_\pm \tag{32}$$

where p_+ and p_- are the periods of the function h with respect to ξ_+ and ξ_- respectively. If h is not periodic, then by $\langle h\rangle_\pm$ we shall mean

$$\langle h\rangle_\pm=\int_{-\infty}^{\infty} h\, d\xi_\pm \tag{33}$$

provided the above integral exists.

Taking the average of equation (30) with respect to ξ_- over the period of ζ_{10} we get the following third order nonlocal mean-field equation for ζ_{10}:

$$i\frac{\partial \zeta_{10}^{(1)}}{\partial \tau} + \gamma_1 \frac{\partial^2 \zeta_{10}^{(1)}}{\partial \xi_+^2} + \gamma_2 \frac{\partial^2 \zeta_{10}^{(1)}}{\partial \eta^2} = \delta_1 |\zeta_{10}^{(1)}|^2 \zeta_{10}^{(1)} + \delta_2 |\zeta_{01}^{(1)^2}| \zeta_{10}^{(1)}, \tag{34}$$

Similarly taking the average of equation (31) with respect to ξ_+ over the period of ζ_{01} we get the following third order nonlocal mean-field evolution equation for ζ_{01}:

$$-i\frac{\partial \zeta_{01}^{(1)}}{\partial \tau} + \gamma_1 \frac{\partial^2 \zeta_{01}^{(2)}}{\partial \xi_-^2} + \gamma_2 \frac{\partial^2 \zeta_{01}^{(1)}}{\partial \eta^2} = \delta_1 |\zeta_{01}^{(1)}|^2 \zeta_{01}^{(1)} + \delta_2 |\zeta_{10}^{(1)}|^2 \zeta_{01}^{(1)}, \tag{35}$$

In equations (34) and (35), if we put $v = 0, \gamma = 0$ then we get nonlocal mean field evolution equations in the third order (lowest order) for infinite depth water. These reduced equations become the same as equations (1b) of Pierce and Knobloch [11] when we proceed to the limit as $h \to \infty$.

4. Stability Analysis

Equations (34) and (35) admit the solution

$$\zeta_{10} = \zeta_{10}^{(0)} = \alpha_0 \ \exp\bigl(i\Delta\omega\tau\bigr), \qquad \zeta_{01} = \zeta_{01}^{(0)} = \alpha_0 \ \exp\bigl(-i\Delta\omega\tau\bigr) \tag{36}$$

where α_0 is real constant and the nonlinear frequency shift

$$\Delta\omega = -\bigl(\delta_1 + \delta_2\bigr)\alpha_0^2 \tag{37}$$

To study modulational stability of these wave trains we introduce the following perturbations

$$\zeta_{10} = \zeta_{10}^{(1)} + \epsilon\zeta_{10}^{(2)} = \zeta_{10}^{(0)}\bigl(1 + R_{10}\bigr) \tag{38}$$

$$\zeta_{01} = \zeta_{01}^{(1)} + \epsilon\zeta_{01}^{(2)} = \zeta_{01}^{(0)}\bigl(1 + R_{01}\bigr) \tag{39}$$

where

$$R_{10} = R_{10}\bigl(\xi_+, \zeta, \tau\bigr), \qquad R_{01} = R_{01}\bigl(\xi_-, \zeta, \tau\bigr).$$

Now we substitute (38) and (39) in equations (34) and (35) respectively and linearize and finally separate into real and imaginary parts to obtain the following equations in the lowest order

$$-\frac{\partial R_{10}^i}{\partial \tau} + \gamma_1 \frac{\partial^2 R_{10}^r}{\partial \xi_+^2} + \gamma_2 \frac{\partial^2 R_{10}^r}{\partial \eta^2} = 2\delta_1 \alpha_0^2 R_{10}^r + 2\delta_2 \alpha_0^2 R_{01}^r. \tag{40}$$

$$\frac{\partial R_{10}^{r}}{\partial \tau}+\gamma_1\frac{\partial^2 R_{10}^{i}}{\partial \xi_+^2}+\gamma_2\frac{\partial^2 R_{10}^{i}}{\partial \eta^2}=0 \tag{41}$$

where superscripts r and i indicate real and imaginary parts of the associated variables. In the transverse direction we consider the following uniform perturbations

$$\begin{aligned} R_{10}^{r}&=p_{10}+r_{10}e^{i\lambda\xi_+}+r_{10}^{*}e^{-i\lambda\xi_+}, \qquad & R_{10}^{i}&=q_{10}+s_{10}e^{i\lambda\xi_+}+s_{10}^{*}e^{-i\lambda\xi_+},\\ R_{01}^{r}&=p_{01}+r_{01}e^{i\lambda\xi_-}+r_{01}^{*}e^{-i\lambda\xi_-}, \qquad & R_{01}^{i}&=q_{01}+s_{01}e^{i\lambda\xi_-}+s_{01}^{*}e^{-i\lambda\xi_-}, \end{aligned} \tag{42}$$

where p, q, r, s are functions of τ only.

We have assumed the dependence on τ to be of the form $\exp(-i\Omega\tau)$. Now introducing perturbation relations (42) in equations (40) and (41) and equating coefficient of $e^{i\lambda\xi_+}$, on both sides we obtain the flowing equations from the lowest order equations (40) and (41)

$$\left(\gamma_1\lambda^2+2\delta_1\alpha_0^2\right)r_{10}-i\Omega_1 s_{10}=0 \tag{43}$$

$$i\Omega_1 r_{10}+\gamma_1\lambda^2 s_{10}=0 \tag{44}$$

The nontrivial solution of (43) and (44) is given by

$$\begin{aligned} &\Omega^2=\gamma_1\lambda^2\left(\gamma_1\lambda^2+2\delta_1\alpha_0^2\right)\\ &\text{or}\ \ \Omega=\left\{\gamma_1\lambda^2\left(\gamma_1\lambda^2+2\delta_1\alpha_0^2\right)\right\}^{\frac{1}{2}} \end{aligned} \tag{45}$$

From relation (45), we observe that instability occurs when $\gamma_1\delta_1 < 0$ for long wavelengths that is for $\lambda \to 0^+$. When instability condition is fulfilled, the growth rate of instability Γ is given by

$$\Gamma=\left[-\gamma_1\lambda^2\left(\gamma_1\lambda^2+2\delta_1\alpha_0^2\right)\right]^{\frac{1}{2}} \tag{46}$$

For $\lambda^2=-\delta_1\alpha_0^2/\gamma_1$, we obtain the following expression of the maximum growth rate of instability

$$\Gamma_m=|\delta_1|\alpha_0^2 \tag{47}$$

At marginal stability

$$\gamma_1\lambda^2+2\delta_1\alpha_0^2=0$$

and the wave number λ at marginal stability is given by

$$\lambda=\frac{\sqrt{2}\delta_1\alpha_0}{\sqrt{|\gamma_1\delta_1|}} \tag{48}$$

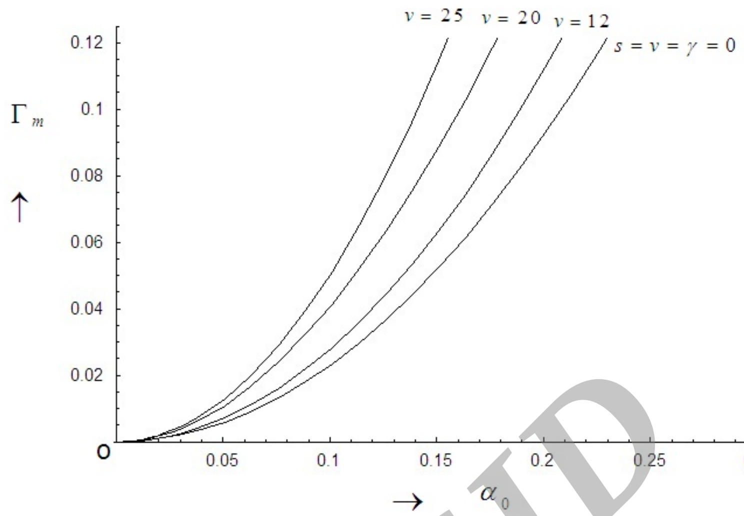

Figure 1. Maximum growth rate of instability Γ_m against wave steepness α_0 for some different values of dimensionless wind velocity v. Here $\gamma = 0.00129.$ and $s = 0.075$ for all the graphs.

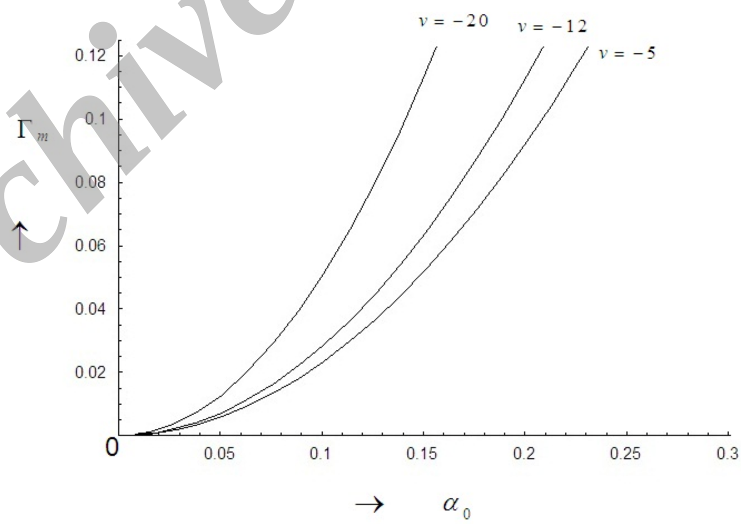

Figure 2. Maximum growth rate of instability Γ_m against wave steepness α_0 for some different values of dimensionless wind velocity v. Here $\gamma = 0.00129.$ and $s = 0.075$ for all the graphs.

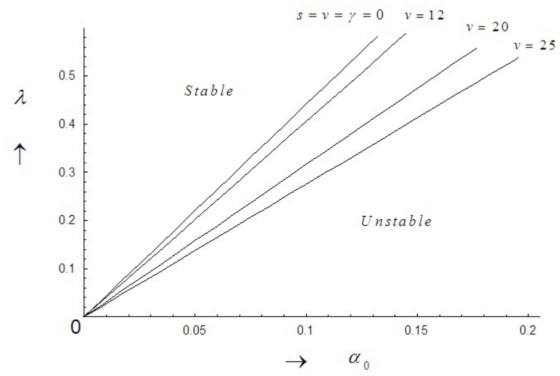

Figure 3. Wave number λ at marginal stability against wave steepness α_0 for some different values of dimensionless wind velocity v. Here $\gamma = 0.00129$ and $s = 0.075$ for all the graphs except for the one with $s = v = \gamma = 0$ written on the graph.

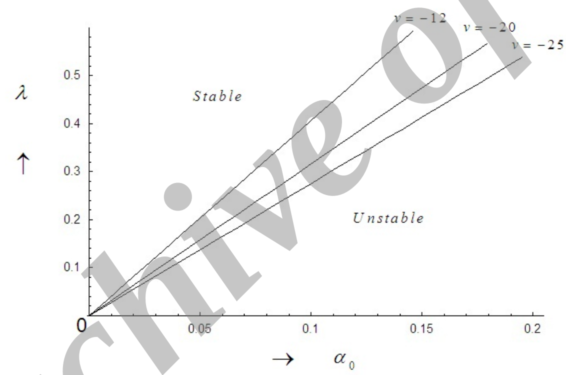

Figure 4. Wave number λ at marginal stability against wave steepness α_0 for some different values of dimensionless wind velocity v. Here $\gamma = 0.00129$ and $s = 0.075$ for all the graphs.

In Figures 1 and 2 the maximum growth rate Γ_m of instability which can be obtained from equation (47) has been plotted against wave steepness α_0 for some different values of dimensionless wind velocity v and for $s = 0.075$. From these graphs it is found that for waves with sufficiently small waves numbers the maximum growth rate of instability Γ_m increases steadily with the increase of wave steepness α_0. The maximum growth rate also increases with the increase of dimensionless wind velocity v . The growth rate is found to be appreciably much higher for dimensionless wind velocity approaching its critical value. Again in Figures 3 and 4 the wave number λ at marginal stability which can be obtained from equation (48) has been plotted against wave steepness α_0 for some different values of dimensionless wind velocity v. From these graphs it is observed that the instability regions are shortened with the increase of the absolute value of the wind velocity.

5. Conclusions

The third order nonlinear evolution equations have been derived by Pierce and Knobloch [11] for two counterpropagating capillary gravity wave packets on the surface of water of finite depth.The resulting equations are asymptotically exact and nonlocal and generalize the equations derived by Djordjevic and Redekopp [6] for counterpropagating waves.Our paper is an extension of the evolution equations derived by Pierce and knobloch [11] for an infinite depth water and in the presence of wind flowing over it. From these evolution equations instability condition is obtained and graphs are plotted showing maximum growth rate of instability Γ_m against wave steepness α_0 for some different values of dimensionless wind velocity v. From the graphs it is found that the maximum growth rate of instability Γ_m increases steadily with the increase of wave steepness α_0. The maximum growth rate also increases with the increase of dimensionless wind velocity v. The growth rate of instability is found to be appreciably much higher for dimensionless wind velocity approaching its critical value. Graphs are also plotted for the wave number λ at marginal stability against wave steepness α_0 for some different values of dimensionless wind velocity v. From the graphs it is observed that the instability regions are shortened with the increase of the absolute value of the wind velocity.

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6. Appendices

Coefficients of the evolution equations (3.33) and (3.34)

$$\gamma_0 = \frac{2\gamma v\omega - 2\gamma v^2 + (1-\gamma) + 3s}{(1+\gamma)\omega^2 - \gamma v\omega}, \qquad \gamma_1 = \frac{2\gamma v c_g - (1+\gamma)c_g^2 - \gamma v^2 + 3s}{2(1+\gamma)\omega^2 - 2\gamma v\omega},$$

$$\gamma_2 = \frac{(1-\gamma) - 2\gamma v c_g + 3s}{4(1+\gamma)\omega^2 - 4\gamma v\omega},$$

$$\delta_1 = [(2\omega^4 + 6\omega^2 - 9s) + \gamma\{\frac{21}{2}(\omega^2 + v^2) + 2(2+p_1)(\omega - v)(\omega - v - 2\omega^2)$$

$$-(1+2p_1)\omega + 15\omega v\} + \gamma v(\omega - v)(6p_1 + 9)]/[12\omega^2 - 8\omega^4 - \gamma(\omega - v)^2],$$

$$\delta_2 = [31\omega^4 - 23\omega^2 + s^2(1-\gamma) - 8s + 8\gamma(\omega - v)^2,$$

where $$c_g = \frac{2\gamma v\omega - 2\gamma v^2 + (1-\gamma) + 3s}{2(1+\gamma)\omega - 2\gamma v}.$$

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