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Stability Analysis from Fourth order Nonlinear Evolution Equations for Two Capillary Gravity Wave Packets in the Presence of Wind Flowing over Water

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Abstract. Asymptotically exact and nonlocal fourth order nonlinear evolution equations are derived for two coupled fourth order nonlinear evolution equations have been derived in deep water for two capillary-gravity wave packets propagating in the same direction in the presence of wind flowing over water.We have used a general method, based on Zakharov integral equation.On the basis of these evolution equations,the stability analysis is made for a uniform capillary gravity wave train in the presence of another wave train having the same group velocity. Instability condition is obtained and graphs are plotted for maximum growth rate of instability and for wave number at marginal stability against wave steepness for some different values of dimensionless wind velocity.

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1. Introduction

There has been considerable interest in the stability of finite amplitude gravity wave in deep water.Much of this interest has been focused on the instability of a uniform wave train to modulational perturbations.

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For small but finite amplitude,the successful approach to studying the stability of finite amplitude gravity waves in deep water is through the application of the lowest order nonlinear evolution equation,which is the nonlinear Schrödinger equation.This analysis is suitable for small wave steepness and for long-wavelength perturbations.But for wave steepness greater than 0.15 predictions from the nonlinear Schrödinger equation do not agree with the result of Longuet-Higgins[18, 19].Dysthe[10] has shown that a stability analysis made from a fourth-order nonlinear evolution equation that is one order higher than the nonlinear Schrödinger equation gives results consistent with the exact results of Longuet-Higgins[18, 19] and with the experimental results of Benjamin and Feir[2] for wave steepness up to 0.25.The fourth-order effects give a surprising improvement compared to ordinary nonlinear Schrödinger effects in many respects,and some of these points have been elaborated by Janssen [15].The dominant new effect that comes in the fourth order is the influence of wave-induced mean flow and this produces a significant deviation in the stability character.From these it can be concluded that a fourth-order evolution equation is a good starting point for studying nonlinear effects in surface waves.

Fourth order nonlinear evolution equation for deep water surface waves in different contexts and stability analysis made from them were derived by Dhar and Das[5, 6], Stiassnie [23] and Hogan[14],Hara and Mei[11, 12],Bhattacharyya and Das[3],Debsarma and Das[4].

All these analyses made by the above mentioned authors are for a single wave.Stability analysis of a surface gravity wave in deep water in the presence of a second wave has been made by Roskes[22] based on the lowest-order nonlinear Schrödinger equations. In his investigation modulational perturbation is restricted to a direction along which group velocity projections of the two waves overlap and it is argued that the modulation will grow at a faster rate along this direction when $0 < \theta < 70.5^0$,where θ is the angle between the two propagation directions of two waves.

Dhar and Das[7] made the same analysis of Roskes[22] making use of two coupled fourth-order nonlinear evolution equations that they derived for two wave packets having the same characteristic wave number. The same analysis including the effect of capillarity was later made by Dhar and Das[8] and S. Ahmadi [1] using the multiple scale method. They observed significant deviations from the results obtained from coupled cubic nonlinear Schrödinger equations.

Pierce and Knobloch[21] derived third order evolution equations for counterpropagating capillary- gravity wave trains having equal characteristic wave number and frequency propagating over finite depth water.The resulting equations are asymptotically exact and nonlocal.Later on, Dhar and Mondal [9]have derived asymptotically exact and nonlocal fourth order nonlinear evolution equations in deep water for two counterpropagating gravity wave packets in the presence of wind flowing over water.

In the present paper two coupled fourth order nonlinear evolution equations are derived in deep water for two capillary-gravity wave packets propagating in the same direction with unequal wave numbers in the presence of wind flowing over water.Here we have used a general method,based on Zakharov integral equation.Unlike Dhar and Das[7, 8],the evolution equations are derived here using Zakharov integral equation. Stiassnie[23] and Hogan[14] also used the Zakharov integral equation for the derivation of fourth order nonlinear evolution equations for a surface gravity wave packet and capillary-gravity wave packet respectively.In deriving the two coupled evolution equations,we make an extension of the paper by Hogan et al.[13],who derived the change in phase speed of one capillary-gravity wave train in the presence of another starting from the Zakharov integral equation.The expression for the change in phase speed for the case of gravity waves was first obtained by Longuet-Higgins and Phillips[17] by the perturbation method. Onorato et al.[20] also derived third-order evolution equations to study the problem of interaction of two wave systems in deep water with equal characteristic wave number and propagating in two different directions. They found that the introduction of a second wave results in an increase of the instability growth rates and causes enlargement of the instability region.

In our paper the relative changes in phase speed of each uniform wave train in the presence of another one have been derived.On the basis of two coupled nonlinear Schrödinger equations,the stability analysis is made of a uniform surface gravity wave train in the presence of a uniform capillary-gravity wave train,when the group velocities of the two wave trains coincide. The instability condition and an expression for the growth rate of instability are derived for a uniform gravity wave train in the presence of a capillary-gravity wave train.Stable-unstable regions and the growth rate of instability against perturbation wave number have been plotted for two different sets of values of wave numbers and for different values of wind velocity.

2. Basic Concept and Basic Equations

The common horizontal interface between air and water in the undisturbed state is considered as $z = 0$ plane.Here x and y are the horizontal coordinates and z is the vertical coordinate which is taken positive in the upward direction.In the undisturbed state air flows over water with a velocity v in a direction that is taken as the x- axis. We take $z = \beta(x, t)$ as the equation of the common interface at any time t in the perturbed state.

The perturbed velocity potentials $\phi(x, z, t)$ and $\phi'(x, z, t)$ of water and air respectively satisfy the following Laplace equations

$$\nabla^2\phi = 0 \qquad in \quad -\infty < z < \beta \tag{1}$$

$$\nabla^2\phi' = 0 \qquad in \quad \beta < z < \infty \tag{2}$$

The kinematic boundary condition for water is given by

$$\frac{\partial\phi}{\partial z} - \frac{\partial\beta}{\partial t} = \frac{\partial\phi}{\partial x}\frac{\partial\beta}{\partial x} \qquad at\ z = \beta \tag{3}$$

which gives a necessary condition for equality of water velocity at the interface normal to it to the normal velocity of the interface.

Similar condition for air is the following

$$\frac{\partial\phi'}{\partial z} - \frac{\partial\beta}{\partial t} - v\frac{\partial\beta}{\partial x} = \frac{\partial\phi'}{\partial x}\frac{\partial\beta}{\partial x} \qquad \text{at } z = \beta \tag{4}$$

The condition of continuity of pressure at the interface gives

$$\frac{\partial \phi}{\partial t} - \gamma \frac{\partial \phi'}{\partial t} + (1-\gamma) g\beta - \gamma v \frac{\partial \phi'}{\partial x} = -\frac{1}{2}(\nabla\phi)^2 + \frac{\gamma}{2}(\nabla\phi')^2 + s\frac{\partial^2 \beta}{\partial x^2} / \{1 + (\frac{\partial \beta}{\partial x})^2\}^{\frac{3}{2}} \quad \text{at } z = \beta \tag{5}$$

where $\gamma = \rho'/\rho$ is the ratio of densities of air to water, g is the acceleration due to gravity and s is the ratio of surface tension coefficient T_s to the water density ρ.

Also ϕ and ϕ' should satisfy the following conditions at infinity

$$\phi \rightarrow 0 \ as\ z \rightarrow -\infty, \phi' \rightarrow 0 \ as\ z \rightarrow \infty \tag{6}$$

We look for solutions of the above equations in the form

$$P = P_{00} + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [P_{mn} \ \exp\ i(m\psi_1 + n\psi_2) + P^*_{mn}\ exp\ -i(m\psi_1 + n\psi_2)] \tag{7}$$

where $\psi_1 = k_1 x - \omega_1 t$, $\psi_2 = k_2 x - \omega_2 t$ and P stands for ϕ, ϕ' and β. In the above summation on the right hand side of equation (7), $(m, n) \neq (0, 0)$. The Fourier coefficients $\phi_{00}, \phi'_{00}, \phi_{mn}, \phi'_{mn}, \phi^*_{mn}, \phi'^*_{mn}$ are functions of z , $x_1 = \epsilon x$, $t_1 = \epsilon t$ and $\beta_{00}, \beta_{mn}, \beta^*_{mn}$ are functions of x_1, t_1. Here * denotes complex conjugate, ϵ is a small ordering parameter measuring the weakness of wave steepness and ω, k satisfy the following linear dispersion relation for capillary gravity waves

$$(1+\gamma)\omega^2 - 2\gamma\omega k v + \gamma k^2 v^2 - (1-\gamma) g k - s k^3 = 0. \tag{8}$$

3. Derivation of Evolution Equations

In this section, we derive the two coupled nonlinear evolution equations using Zakharov's integral equation given by

$$i\frac{\partial A(\mathbf{k}, t)}{\partial t} = \int\int\int_{-\infty}^{\infty} T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) A^*(\mathbf{k}_1, t) A(\mathbf{k}_2, t) A(\mathbf{k}_3, t) \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \exp[i\{\omega(\mathbf{k}) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3)\}t] d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \tag{9}$$

where $A(\mathbf{k}, t)$ is related to the free surface elevation $\beta(\boldsymbol{x}, t)$ by

$$\beta(\boldsymbol{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\frac{|\mathbf{k}|}{2\omega(\mathbf{k})}\}^{\frac{1}{2}} \{A(\mathbf{k}, t) exp[i\{\mathbf{k}.\boldsymbol{x} - \omega(\mathbf{k})t\}] + \ c.c.\ \} d\mathbf{k} \tag{10}$$

In the above $\mathbf{k} = (k, l)$ is the wave vector, $\boldsymbol{x} = (x, y)$ is the horizontal spatial vector, c.c. denotes complex conjugate and the kernel $T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is a scalar function used by Krasitskii[16].

The linearized wave frequency $\omega(\mathbf{k})$ connected to $\mathbf{k}$ through the following linear dispersion relation given by

$$\omega(\mathbf{k}) = [\frac{|\mathbf{k}|^{\frac{1}{2}}}{1+\gamma}\{\gamma|\mathbf{k}|^{\frac{1}{2}}v + [(1-\gamma^2)g + (1+\gamma)s|\mathbf{k}|^2 - \gamma|\mathbf{k}|v^2\ ]^{\frac{1}{2}}\}] \tag{11}$$

The resonance condition for the four wave vectors is given by

$$\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 = \mathbf{0} \tag{12}$$

Now we consider two narrow capillary-gravity wave packets centered around the wave vectors $\mathbf{k}_p$ and $\mathbf{k}_q$, called the first and second wave packet respectively. With $\mathbf{k} = \mathbf{k}_p$, the condition (12) is satisfied for two waves with wave vectors $\mathbf{k}_p$ and $\mathbf{k}_q$ in the following three cases: (a) $\mathbf{k}_1 = \mathbf{k}_q, \mathbf{k}_2 = \mathbf{k}_q, \mathbf{k}_3 = \mathbf{k}_p$ (b) $\mathbf{k}_1 = \mathbf{k}_3 = \mathbf{k}_q, \mathbf{k}_2 = \mathbf{k}_p$ (c) $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k}_p$

For obtaining the evolution equation of the first wave packet we take $\mathbf{k} = \mathbf{k}_p + \mathbf{e}$ in equation (9) and introducing new variables $B_1(\mathbf{e}, t)$ and $B_2(\mathbf{e}, t)$ defined by

$$\begin{aligned} B_1(\mathbf{e}, t) &= A(\mathbf{k}_p + \mathbf{e}, t) exp[-i\{\omega(\mathbf{k}_p + \mathbf{e}) - \omega(\mathbf{k}_p)\}t] \\ B_2(\mathbf{e}, t) &= A(\mathbf{k}_q + \mathbf{e}, t) exp[-i\{\omega(\mathbf{k}_q + \mathbf{e}) - \omega(\mathbf{k}_q)\}t] \end{aligned} \tag{13}$$

equation (9) can be written as

$$\begin{aligned} & i\frac{\partial B_1(\mathbf{e}, t)}{\partial t} - B_1(\mathbf{e}, t)[\omega(\mathbf{k}_p + \mathbf{e}) - \omega(\mathbf{k}_p)] \\ & = \int\int\int_{-\infty}^{\infty} T(\mathbf{k}_p + \mathbf{e}, \mathbf{k}_q + \mathbf{e}_1, \mathbf{k}_q + \mathbf{e}_2, \mathbf{k}_p + \mathbf{e}_3) B_2^*(\mathbf{e}_1, t) B_2(\mathbf{e}_2, t) B_1(\mathbf{e}_3, t) \\ & \qquad \times \delta(\mathbf{e} + \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) d\mathbf{e}_1 d\mathbf{e}_2 d\mathbf{e}_3 \\ & + \int\int\int_{-\infty}^{\infty} T(\mathbf{k}_p + \mathbf{e}, \mathbf{k}_q + \mathbf{e}_1, \mathbf{k}_p + \mathbf{e}_2, \mathbf{k}_q + \mathbf{e}_3) B_2^*(\mathbf{e}_1, t) B_1(\mathbf{e}_2, t) B_2(\mathbf{e}_3, t) \\ & \qquad \times \delta(\mathbf{e} + \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) d\mathbf{e}_1 d\mathbf{e}_2 d\mathbf{e}_3 \\ & + \int\int\int_{-\infty}^{\infty} T(\mathbf{k}_p + \mathbf{e}, \mathbf{k}_p + \mathbf{e}_1, \mathbf{k}_p + \mathbf{e}_2, \mathbf{k}_p + \mathbf{e}_3) B_1^*(\mathbf{e}_1, t) B_1(\mathbf{e}_2, t) B_1(\mathbf{e}_3, t) \\ & \qquad \times \delta(\mathbf{e} + \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) d\mathbf{e}_1 d\mathbf{e}_2 d\mathbf{e}_3 \end{aligned} \tag{14}$$

in which we replace $\mathbf{k}_1 = \mathbf{k}_q + \mathbf{e}_1$, $\mathbf{k}_2 = \mathbf{k}_q + \mathbf{e}_2$, $\mathbf{k}_3 = \mathbf{k}_p + \mathbf{e}_3$ for the first triple integral;$\mathbf{k}_1 = \mathbf{k}_q + \mathbf{e}_1$, $\mathbf{k}_2 = \mathbf{k}_p + \mathbf{e}_2$, $\mathbf{k}_3 = \mathbf{k}_q + \mathbf{e}_3$ for the second and finally $\mathbf{k}_1 = \mathbf{k}_p + \mathbf{e}_1$,$\mathbf{k}_2 = \mathbf{k}_p + \mathbf{e}_2$, $\mathbf{k}_3 = \mathbf{k}_p + \mathbf{e}_3$ for the third. The surface elevations $\eta_1(\boldsymbol{x}, t)$ and $\eta_2(\boldsymbol{x}, t)$ for the first and second wave packets respectively for the new variables become

$$\begin{aligned} \eta_1(\boldsymbol{x}, t) &= \sqrt{\frac{1+\gamma}{2}}\frac{1}{2\pi}\ exp\ i[\mathbf{k}_p.\boldsymbol{x} - \omega(\mathbf{k}_p)t].\int_{-\infty}^{\infty} B_1(\mathbf{e}, t)\ \exp\ i(\mathbf{e}.\boldsymbol{x}) \\ & \times [\frac{|\mathbf{k}_p + \mathbf{e}|^{\frac{1}{4}}}{[\gamma v|\mathbf{k}_p + \mathbf{e}|^{\frac{1}{2}} + \{(1-\gamma^2)g + (1+\gamma)s|\mathbf{k}_p + \mathbf{e}|^2 - \gamma v^2|\mathbf{k}_p + \mathbf{e}|\}^{\frac{1}{2}}]^{\frac{1}{2}}}]d\mathbf{e} + c.c. \end{aligned} \tag{15}$$

$$\eta_2(\boldsymbol{x},t) = \sqrt{\frac{1+\gamma}{2}}\frac{1}{2\pi}\ \exp\ i[\mathbf{k}_q.\boldsymbol{x} - \omega(\mathbf{k}_q)t]\int_{-\infty}^{\infty} B_2(\mathbf{e},t)\ \exp\ i(\mathbf{e}.\boldsymbol{x})$$
$$\times[\frac{|\mathbf{k}_q+\mathbf{e}|^{\frac{1}{4}}}{[\gamma v|\mathbf{k}_q+\mathbf{e}|^{\frac{1}{2}} + \{(1-\gamma^2)g + (1+\gamma)s|\mathbf{k}_q+\mathbf{e}|^2 - \gamma v^2|\mathbf{k}_q+\mathbf{e}|\}^{\frac{1}{2}}]^{\frac{1}{2}}}]d\mathbf{e} +\ c.c. \tag{16}$$

We now consider two wave packets having wave numbers k_1 and k_2 where $k_1 > k_2$,both propagating along the x axis and we take two wave vectors $\mathbf{k}_p$ and $\mathbf{k}_q$ as

$$\mathbf{k}_p \equiv \mathbf{k}_1 = k_1\hat{\boldsymbol{x}}, \mathbf{k}_q \equiv \mathbf{k}_2 = k_2\hat{\boldsymbol{x}} \tag{17}$$

where $\hat{\boldsymbol{x}}$ is a unit vector along the x axis.Also we shall consider modulational perturbation only along the x axis, so that we can take $\mathbf{e} = e\hat{\boldsymbol{x}}$. Using $\mathbf{k}_p = k_1\hat{\boldsymbol{x}}$ and $\mathbf{e} = e\hat{\boldsymbol{x}}$ in equations(15),(16) and considering Taylor-expansion in powers of e and keeping only the linear terms in e,we get the following expressions for $\eta_1(\boldsymbol{x},t)$ and $\eta_2(\boldsymbol{x},t)$

$$\eta_1(\boldsymbol{x},t) = \alpha_1(\boldsymbol{x},t)exp[i\{\mathbf{k}_1.\boldsymbol{x} - \omega(\mathbf{k}_1).t\}] + c.c. \tag{18}$$

$$\eta_2(\boldsymbol{x},t) = \alpha_2(\boldsymbol{x},t)exp[i\{\mathbf{k}_2.\boldsymbol{x} - \omega(\mathbf{k}_2).t\}] + c.c. \tag{19}$$

where

$$\alpha_1(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda_1(\mathbf{e},t)exp(i\mathbf{e}.\boldsymbol{x})d\mathbf{e} \tag{20}$$

$$\alpha_2(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda_2(\mathbf{e},t)exp(i\mathbf{e}.\boldsymbol{x})d\mathbf{e} \tag{21}$$

and

$$\lambda_1(\mathbf{e},t) = \sqrt{\frac{1+\gamma}{2}}.\ \frac{k_1^{\frac{1}{4}}}{b_1^{\frac{1}{4}}}(1+b_1^{'}e)\ B_1(\mathbf{e},t) \tag{22}$$

$$\lambda_2(\mathbf{e},t) = \sqrt{\frac{1+\gamma}{2}}.\ \frac{k_2^{\frac{1}{4}}}{b_2^{\frac{1}{4}}}(1+b_2^{'}e)\ B_2(\mathbf{e},t) \tag{23}$$

In equations (22) and (23) $b_1,\ b_1,\ b_2,\ b_2$ are given by

$$b_1 = [\{(1-\gamma^2)g + (1+\gamma)sk_1^2 - \gamma v^2 k_1\}^{\frac{1}{2}} + \gamma v k_1^{\frac{1}{2}}]$$

$$b_1^{'} = \frac{1}{4k_1} - \frac{1}{4}[\frac{\gamma v}{\sqrt{k_1}} + \frac{2(1+\gamma)sk_1 - \gamma v^2}{\{(1-\gamma^2)g + (1+\gamma)sk_1^2 - \gamma v^2 k_1\}}]$$

$$\times \frac{1}{\{(1-\gamma^2)g + (1+\gamma)sk_1^2 - \gamma v^2 k_1\}^{\frac{1}{2}} + \gamma v k_1^{\frac{1}{2}}}$$

$$b_2 = [\{(1-\gamma^2)g + (1+\gamma)sk_2^2 - \gamma v^2 k_2\}^{\frac{1}{2}} + \gamma v k_2^{\frac{1}{2}}].$$

$$b_2' = \frac{1}{4k_2} - \frac{1}{4}[\frac{\gamma v}{\sqrt{k_2}} + \frac{2(1+\gamma)sk_2 - \gamma v^2}{\{(1-\gamma^2)g + (1+\gamma)sk_2^2 - \gamma v^2 k_2\}}]$$

$$\times \frac{1}{\{(1-\gamma^2)g + (1+\gamma)sk_2^2 - \gamma v^2 k_2\}^{\frac{1}{2}} + \gamma v k_2^{\frac{1}{2}}}$$

Integrating equation(14) with respect to **e** and using the relations (18),(20) and (22) the evolution equation (14) assumes the following form

$$\frac{i}{2\pi}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\lambda_1(\mathbf{e},t)exp(i\mathbf{e}.\boldsymbol{x})d\mathbf{e} - \frac{1}{2\pi}\int_{-\infty}^{\infty}[\omega(\mathbf{k}_1+\mathbf{e}) - \omega(k_1)]\lambda_1(\mathbf{e},t)exp(i\mathbf{e}.\boldsymbol{x})d\mathbf{e}$$
$$= \sqrt{\frac{1+\gamma}{2}}\frac{1}{2\pi}[\int\int\int_{-\infty}^{\infty}T(\mathbf{k}_1+\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1, \mathbf{k}_2+\mathbf{e}_1, \mathbf{k}_2+\mathbf{e}_2, \mathbf{k}_1+\mathbf{e}_3)\lambda_2^*(\mathbf{e}_1,t)\lambda_2(\mathbf{e}_2,t)\lambda_1(\mathbf{e}_3,t)$$
$$\times \frac{\omega_2}{k_2}\{1 - b_1(e_1-e_2) - b_2(e_1+e_2)\}exp\{i(\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1).\boldsymbol{x}\}d\mathbf{e}_1 d\mathbf{e}_2 d\mathbf{e}_3$$
$$+ \int\int\int_{-\infty}^{\infty}T(\mathbf{k}_1+\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1, \mathbf{k}_2+\mathbf{e}_1, \mathbf{k}_1+\mathbf{e}_2, \mathbf{k}_2+\mathbf{e}_3)\lambda_2^*(\mathbf{e}_1,t)\lambda_1(\mathbf{e}_2,t)\lambda_2(\mathbf{e}_3,t)$$
$$\times \frac{\omega_2}{k_2}\{1 - b_1(e_1-e_3) - b_2(e_1+e_3)\}exp\{i(\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1).\boldsymbol{x}\}d\mathbf{e}_1 d\mathbf{e}_2 d\mathbf{e}_3$$
$$+ \int\int\int_{-\infty}^{\infty}T(\mathbf{k}_1+\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1, \mathbf{k}_1+\mathbf{e}_1, \mathbf{k}_1+\mathbf{e}_2, \mathbf{k}_1+\mathbf{e}_3)\lambda_1^*(\mathbf{e}_1,t)\lambda_1(\mathbf{e}_2,t)\lambda_1(\mathbf{e}_3,t)$$
$$\times \frac{\omega_1}{k_1}\{1 - 2b_1e_1\}exp\{i(\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1).\boldsymbol{x}\}d\mathbf{e}_1 d\mathbf{e}_2 d\mathbf{e}_3] \quad (24)$$

Writing $\mathbf{k}_1 + \mathbf{e} = (k_1 + e)\hat{\boldsymbol{x}}$ in $\omega(\mathbf{k}_1 + \mathbf{e})$ and expanding in powers of e up to third degree, we get the following expression of left hand side of equation(24) after evaluation of Fourier inversion integrals.

$$i\frac{\partial\alpha_1}{\partial t} + i\frac{\omega_1}{k_1(1+\gamma)}Q_1\frac{\partial\alpha_1}{\partial x} - \frac{\omega_1}{k_1^2(1+\gamma)}Q_2\frac{\partial^2\alpha_1}{\partial x^2} - i\frac{\omega_1}{k_1^3(1+\gamma)}Q_3\frac{\partial^3\alpha_1}{\partial x^3} \quad (25)$$

where

$$Q_1 = \frac{1}{2}[\gamma v + \frac{(1-\gamma^2)g - 2\gamma v^2 k_1 + 3(1+\gamma)sk_1^2}{\gamma v + (1-\gamma^2)g - 2\gamma v^2 k_1 + 3(1+\gamma)sk_1^2}]$$

$$Q_2 = \frac{1}{8}[\gamma v + \frac{(1-\gamma^2)g - 2\gamma v^2 k_1 + 3(1+\gamma)sk_1^2}{\gamma v + (1-\gamma^2)g - 2\gamma v^2 k_1 + 3(1+\gamma)sk_1^2}]$$

$$Q_3 = \frac{3}{16}[\gamma v + \frac{(1-\gamma^2)g - 2\gamma v^2 k_1 + 3(1+\gamma)sk_1^2}{\gamma v + (1-\gamma^2)g - 2\gamma v^2 k_1 + 3(1+\gamma)sk_1^2}]$$

As modulational perturbations are considered to take place along the x-axis only, we set $\mathbf{e}_i = e_i\hat{\boldsymbol{x}}$ $(i = 1, 2, 3)$ in the arguments of T appearing on the right hand side of equation(24).Now we make Taylor expansions of them in powers of e_i $(i = 1, 2, 3)$ up to first degree in these variables in which we have used the following notations

$$s(k) = \frac{sk^2}{g}, \quad s_i = s(k_i), \quad \omega_i = \omega(k_i), \quad (i = 1, 2), \quad m = \frac{k_2}{k_1}, \quad n = \frac{\omega_2}{\omega_1}, \quad c_g = \frac{d\omega(k)}{dk}$$

The right hand side of equation (24) now becomes

$$\frac{\sqrt{1+\gamma}}{2}\frac{1}{2\pi}[\int\int\int_{-\infty}^{\infty}\frac{\omega_2}{k_2}T_1\ \lambda_2^*(\mathbf{e}_1,t)\lambda_2(\mathbf{e}_2,t)\lambda_1(\mathbf{e}_3,t)\ \exp\{i(\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1).\boldsymbol{x}\}d\mathbf{e}_1d\mathbf{e}_2d\mathbf{e}_3$$
$$+\int\int\int_{-\infty}^{\infty}\frac{\omega_2}{k_2}T_2\ \lambda_2^*(\mathbf{e}_1,t)\lambda_1(\mathbf{e}_2,t)\lambda_2(\mathbf{e}_3,t)\ \exp\{i(\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1).\boldsymbol{x}\}d\mathbf{e}_1d\mathbf{e}_2d\mathbf{e}_3$$
$$+\int\int\int_{-\infty}^{\infty}\frac{\omega_1}{k_1}T_3\ \lambda_1^*(\mathbf{e}_1,t)\lambda_1(\mathbf{e}_2,t)\lambda_1(\mathbf{e}_3,t)\ exp\{i(\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1).\boldsymbol{x}\}d\mathbf{e}_1d\mathbf{e}_2d\mathbf{e}_3] \tag{26}$$

where

$$T_1 = T(\mathbf{k}_1+\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1, \mathbf{k}_2+\mathbf{e}_1, \mathbf{k}_2+\mathbf{e}_2, \mathbf{k}_1+\mathbf{e}_3)[1-p_1(e_1-e_2)-p_2(e_1+e_2)]$$

$$T_2 = T(\mathbf{k}_1+\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1, \mathbf{k}_2+\mathbf{e}_1, \mathbf{k}_1+\mathbf{e}_2, \mathbf{k}_2+\mathbf{e}_3)[1-p_1(e_1-e_3)-p_2(e_1+e_3)]$$

$$T_3 = T(\mathbf{k}_1+\mathbf{e}_2+\mathbf{e}_3-\mathbf{e}_1, \mathbf{k}_1+\mathbf{e}_1, \mathbf{k}_1+\mathbf{e}_2, \mathbf{k}_1+\mathbf{e}_3)[1-2p_1e_1]$$

and

$$p_1 = \frac{1}{4k_1} - \frac{1}{4}[\frac{\gamma v}{\sqrt{k_1}} + \frac{2(1+\gamma)\frac{s_1}{k_1} - \gamma v^2}{\{(1-\gamma^2)+(1+\gamma)s_1 - \gamma v^2 k_1\}}]$$

$$\times\frac{1}{\{(1-\gamma^2)+(1+\gamma)s_1-\gamma v^2k_1\}^{\frac{1}{2}}+\gamma v k_1^2}$$

$$p_2 = \frac{1}{4k_2} - \frac{1}{4}[\frac{\gamma v}{\sqrt{k_2}} + \frac{2(1+\gamma)\frac{s_2}{k_2} - \gamma v^2}{\{(1-\gamma^2)+(1+\gamma)s_2 - \gamma v^2 k_2\}}]$$

$$\times\frac{1}{\{(1-\gamma^2)+(1+\gamma)s_2-\gamma v^2k_2\}^{\frac{1}{2}}+\gamma v k_2^2}$$

We next introduce the following dimensionless variables

$$\alpha_1^{'} = k_1\alpha_1, \quad \alpha_2^{'} = k_2\alpha_2, \quad x^{'} = k_2 x, \quad t^{'} = \omega_2 t, \quad u^{'} = \sqrt{k_2/g}\; v \tag{27}$$

in the expressions (25) and (26). Now deleting the primes and taking the Fourier inversion integrals of expression (26) we get the following nonlinear evolution equation for the first wave packet in the presence of second wave packet.

$$\begin{aligned} & i(\frac{\partial\alpha_1}{\partial t} + \gamma_1^{(1)}\frac{\partial\alpha_1}{\partial x}) + \gamma_2^{(1)}\frac{\partial^2\alpha_1}{\partial x^2} + i\gamma_3^{(1)}\frac{\partial^3\alpha_1}{\partial x^3} = \delta_1^{(1)}\alpha_1^2\alpha_1^* + i\delta_2^{(1)}\alpha_1\alpha_1^*\frac{\partial\alpha_1}{\partial x} \\ & + i\delta_3^{(1)}\alpha_1^2\frac{\partial\alpha_1^*}{\partial x} + \delta_4^{(1)}\alpha_1 H[\frac{\partial}{\partial x}(\alpha_1\alpha_1^*)] + \zeta_1^{(1)}\alpha_1\alpha_2\alpha_2^* + i\zeta_2^{(1)}\alpha_2\alpha_2^*\frac{\partial\alpha_1}{\partial x} \\ & + i\zeta_3^{(1)}\alpha_1\alpha_2^*\frac{\partial\alpha_2}{\partial x} + i\zeta_4^{(1)}\alpha_1\alpha_2\frac{\partial\alpha_2^*}{\partial x} + \zeta_5^{(1)}\alpha_1 H[\frac{\partial}{\partial x}(\alpha_2\alpha_2^*)] \end{aligned} \tag{28}$$

where H is the Hilbert transform operator given by

$$H(\theta) = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\theta(\xi)}{\xi - x}d\xi \tag{29}$$

The above coefficients $\gamma_i^{(1)}$ $(i = 1, 2, 3), \delta_i^{(1)}$ $(i = 1, 2, 3, 4)$ and $\zeta_i^{(1)}$ $(i = 1, 2, 3, 4, 5)$ appearing in equation(28) are given in the Appendix. In the absence of the second wave and for $u = 0, \gamma = 0$ the coupled equations reduce to a single equation (2.20) of Hogan[14]. Also in the absence of the second wave and for $s = 0$, we recover the evolution equation (34) of Dhar and Das[5] for a single gravity wave train.

Proceeding in the same way and making an interchange between the suffixes p and q in the evolution equation (14), we obtain the following nonlinear evolution equation for the second wave packet in the presence of first wave packet.

$$\begin{aligned} & i(\frac{\partial\alpha_2}{\partial t} + \gamma_1^{(2)}\frac{\partial\alpha_2}{\partial x}) + \gamma_2^{(2)}\frac{\partial^2\alpha_2}{\partial x^2} \\ & + i\gamma_3^{(2)}\frac{\partial^3\alpha_2}{\partial x^3} = \delta_1^{(2)}\alpha_2^2\alpha_2^* + i\delta_2^{(2)}\alpha_2\alpha_2^*\frac{\partial\alpha_2}{\partial x} \\ & + i\delta_3^{(2)}\alpha_2^2\frac{\partial\alpha_2^*}{\partial x} + \delta_4^{(2)}\alpha_2 H[\frac{\partial}{\partial x}(\alpha_2\alpha_2^*)] + \zeta_1^{(2)}\alpha_2\alpha_1\alpha_1^* + i\zeta_2^{(2)}\alpha_1\alpha_1^*\frac{\partial\alpha_2}{\partial x} \\ & + i\zeta_3^{(2)}\alpha_2\alpha_1^*\frac{\partial\alpha_1}{\partial x} + + i\zeta_4^{(2)}\alpha_2\alpha_1\frac{\partial\alpha_1^*}{\partial x} + \zeta_5^{(2)}\alpha_2 H[\frac{\partial}{\partial x}(\alpha_1\alpha_1^*)] \end{aligned} \tag{30}$$

where the coefficients $\gamma_i^{(2)}$ $(i = 1, 2, 3)$, $\delta_i^{(2)}$ $(i = 1, 2, 3, 4)$ and $\zeta_i^{(2)}$ $(i = 1, 2, 3, 4, 5)$ are given in the Appendix.

4. Stability Analysis

The uniform wave train solutions of the coupled equations (28) and (30) are given by

$$\alpha_1 = \alpha_{01}\ \exp(-i\Delta\omega_1 t) \tag{31}$$

$$\alpha_2 = \alpha_{02}\ exp(-i\Delta\omega_2 t) \tag{32}$$

where α_{01} and α_{02} are real constants.Substituting equations (31) and (32) in equations (28) and (30)respectively,the amplitude dependent nonlinear frequency shifts of the two waves $\Delta\omega_1$ and $\Delta\omega_2$ are

$$\Delta\omega_1 = \delta_1^{(1)}\alpha_{01}^2 + \zeta_1^{(1)}\alpha_{02}^2 \tag{33}$$

$$\Delta\omega_2 = \delta_1^{(2)}\alpha_{02}^2 + \zeta_1^{(2)}\alpha_{01}^2 \tag{34}$$

The dimensionless wave numbers of the first and second wave are k_1/k_2 and 1 respectively.Therefore the amplitude dependent shifts in phase speeds Δc_1 and Δc_2 of the two waves are the following

$$\Delta c_1 = \frac{\Delta\omega_1}{k_1/k_2} = m(\delta_1^{(1)}\alpha_{01}^2 + \zeta_1^{(1)}\alpha_{02}^2) \tag{35}$$

$$\Delta c_2 = \Delta\omega_2 = \delta_1^{(2)}\alpha_{02}^2 + \zeta_1^{(2)}\alpha_{01}^2 \tag{36}$$

To study modulational stability of two wave trains we introduce the following perturbations in the uniform solutions

$$\alpha_1 = \alpha_{01}[1 + \alpha_1'(\xi, t)]exp(-i\Delta\omega_1 t) \tag{37}$$

$$\alpha_2 = \alpha_{02}[1 + \alpha_2'(\xi, t)]exp(-i\Delta\omega_2 t) \tag{38}$$

where $\alpha_1'(\xi, t)$, $\alpha_2'(\xi, t)$ are small perturbations of amplitudes α_1 and α_2 respectively.

Inserting equations(37) and (38) in two evolution equations (28) and (30) respectively and then linearizing with respect to α_1' and α_2', we obtain the following two equations

$$\begin{aligned}
& i(\frac{\partial\alpha_1'}{\partial t} + \gamma_1^{(1)}\frac{\partial\alpha_1'}{\partial x}) + \gamma_2^{(1)}\frac{\partial^2\alpha_1'}{\partial x^2} + i\gamma_3^{(1)}\frac{\partial^3\alpha_1'}{\partial x^3} \\
& = \delta_1^{(1)}\alpha_1'^2\alpha_1'^* + i\delta_2^{(1)}\alpha_1'\alpha_1'^*\frac{\partial\alpha_1'}{\partial x} + i\delta_3^{(1)}\alpha_1'^2\frac{\partial\alpha_1'^*}{\partial x} \\
& + \delta_4^{(1)}\alpha_1' H[\frac{\partial}{\partial x}(\alpha_1'\alpha_1'^*)] + \zeta_1^{(1)}\alpha_1'\alpha_2'\alpha_2'^* + i\zeta_2^{(1)}\alpha_2'\alpha_2'^*\frac{\partial\alpha_1'}{\partial x} \\
& + i\zeta_3^{(1)}\alpha_1'\alpha_2'^*\frac{\partial\alpha_2'}{\partial x} + i\zeta_4^{(1)}\alpha_1'\alpha_2'\frac{\partial\alpha_2'^*}{\partial x} + \zeta_5^{(1)}\alpha_1' H[\frac{\partial}{\partial x}(\alpha_2'\alpha_2'^*)]
\end{aligned} \tag{39}$$

$$
\begin{aligned}
&i(\frac{\partial \alpha_2^{'}}{\partial t}+\gamma_1^{(2)}\frac{\partial \alpha_2^{'}}{\partial x})+\gamma_2^{(2)}\frac{\partial^2 \alpha_2^{'}}{\partial x^2}+i\gamma_3^{(2)}\frac{\partial^3 \alpha_2^{'}}{\partial x^3}\\
&=\delta_1^{(2)}\alpha_2^{'2}\alpha_2^{'*}+i\delta_2^{(2)}\alpha_2^{'}\alpha_2^{'*}\frac{\partial \alpha_2^{'}}{\partial x}+i\delta_3^{(2)}\alpha_2^{'2}\frac{\partial \alpha_2^{'*}}{\partial x}\\
&+\delta_4^{(2)}\alpha_2^{'}H[\frac{\partial}{\partial x}(\alpha_2^{'}\alpha_2^{'*})]+\zeta_1^{(2)}\alpha_2^{'}\alpha_1^{'}\alpha_1^{'*}+i\zeta_2^{(2)}\alpha_1^{'}\alpha_1^{'*}\frac{\partial \alpha_2^{'}}{\partial x}\\
&+i\zeta_3^{(2)}\alpha_2^{'}\alpha_1^{'*}\frac{\partial \alpha_1^{'}}{\partial x}+i\zeta_4^{(2)}\alpha_2^{'}\alpha_1^{'}\frac{\partial \alpha_1^{'*}}{\partial x}+\zeta_5^{(2)}\alpha_2^{'}H[\frac{\partial}{\partial x}(\alpha_1^{'}\alpha_1^{'*})]
\end{aligned}
\tag{40}
$$

Setting $\alpha_1^{'}=\alpha_r^{(1)}+i\alpha_i^{(1)}$ and $\alpha_2^{'}=\alpha_r^{(2)}+i\alpha_i^{(2)}$ in the above two equations (39) and (40) respectively where $\alpha_r^{(1)},\alpha_i^{(1)},\alpha_r^{(2)},\alpha_i^{(2)}$ are real and then assuming the space time dependence of $\alpha_r^{(1)},\ \alpha_r^{(2)},\ \alpha_i^{(1)},\ \alpha_i^{(2)}$ is of the form exp $i(\mu x-\Omega t)$ and finally equating real and imaginary parts on both sides of each equation we get the following four coupled equations,

$$A_1\alpha_r^{(1)}+B_1\alpha_r^{(2)}-i(\Omega-C_1^{(+)})\alpha_i^{(1)}-D_1^{(-)}\alpha_i^{(2)}=0 \tag{41}$$

$$i(\Omega-C_1^{(-)})\alpha_r^{(1)}+iD_1^{(+)}\alpha_r^{(2)}+E_1\alpha_i^{(1)}=0 \tag{42}$$

$$A_2\alpha_r^{(2)}+B_2\alpha_r^{(1)}-i(\Omega-C_2^{(+)})\alpha_i^{(2)}-iD_2^{(-)}\alpha_i^{(1)}=0 \tag{43}$$

$$i(\Omega-C_2^{(-)})\alpha_r^{(2)}+iD_2^{(+)}\alpha_r^{(1)}+E_2\alpha_i^{(2)}=0 \tag{44}$$

where A_j,B_j,C_j,D_j,E_j $(j=1,2)$ are given by

$$
\begin{aligned}
&A_1=\gamma_2^{(1)}\mu^2+2(\delta_1^{(1)}-\delta_4^{(1)}|\mu|)\alpha_{01}^2\\
&B_1=2(\zeta_1^{(1)}-\zeta_5^{(1)}|\mu|)\alpha_{01}^2\\
&C_1^{\pm}=\gamma_1^{(1)}\mu-\gamma_3^{(1)}\mu^3-\delta_2^{(1)}\mu\alpha_{01}^2\pm\delta_3^{(1)}\mu\alpha_{01}^2-\zeta_2^{(1)}\mu\alpha_{02}^2\\
&D_1^{\pm}=(\zeta_3^{(1)}\pm\zeta_4^{(1)})\mu\alpha_{02}^2\\
&E_1=\gamma_2^{(1)}\mu^2\\
&A_2=\gamma_2^{(2)}\mu^2+2(\delta_1^{(2)}-\delta_4^{(2)}|\mu|)\alpha_{02}^2\\
&B_1=2(\zeta_1^{(2)}-\zeta_5^{(2)}|\mu|)\alpha_{01}^2\\
&C_2^{\pm}=\gamma_1^{(2)}\mu-\gamma_3^{(2)}\mu^3-\delta_2^{(2)}\mu\alpha_{01}^2\pm\delta_3^{(2)}\mu\alpha_{01}^2-\zeta_2^{(2)}\mu\alpha_{02}^2\\
&D_2^{\pm}=(\zeta_3^{(2)}\pm\zeta_4^{(2)})\mu\alpha_{02}^2\\
&E_2=\gamma_2^{(2)}\mu^2
\end{aligned}
\tag{45}
$$

Neglecting higher order terms,the condition for the existence of a nontrivial solution to the above four algebraic equations gives the following nonlinear dispersion relation

$$[(\Omega-C_1)^2-A_1E_1][(\Omega-C_2)^2-A_2E_2] = G(\Omega-C_1)(\Omega-C_2)-F_1(\Omega-C_1)-F_2(\Omega-C_2)+G^{'} \quad (46)$$

where

$$\begin{aligned}
C_1 &= \gamma_1^{(1)}\mu - \gamma_3^{(1)}\mu^3 - \delta_2^{(1)}\mu\alpha_{01}^2 - \zeta_2^{(1)}\mu\alpha_{02}^2 \\
C_2 &= \gamma_1^{(2)}\mu - \delta_2^{(2)}\mu\alpha_{02}^2 - \zeta_2^{(2)}\mu\alpha_{01}^2 \\
F_1 &= 2\gamma_2^{(2)}\{\zeta_1^{(2)}(\zeta_1^{(2)} + \zeta_4^{(1)}) + \zeta_1^{(1)}(\zeta_3^{(2)} - \zeta_4^{(2)})\}\mu^3\alpha_{01}^2\alpha_{02}^2 \\
F_2 &= 2\gamma_2^{(1)}\{(\zeta_1^{(1)} + \zeta_4^{(2)}) + \zeta_1^{(2)}(\zeta_3^{(1)} - \zeta_4^{(1)})\}\mu^3\alpha_{01}^2\alpha_{02}^2 \\
G &= 2(\zeta_3^{(1)}\zeta_3^{(2)} + \zeta_4^{(1)}\zeta_4^{(2)})\mu^2\alpha_{01}^2\alpha_{02}^2 \\
G^{'} &= 4\gamma_2^{(1)}\gamma_2^{(2)}\{\zeta_1^{(1)}\zeta_1^{(2)} - (\zeta_1^{(1)}\zeta_5^{(2)} + \zeta_1^{(2)}\zeta_5^{(1)})\mu\}\mu^4\alpha_{01}^2\alpha_{02}^2.
\end{aligned} \quad (47)$$

We restrict our stability analysis to the case of nearly equal group velocities of the two waves i.e, we assume $\gamma_1^{(1)} \approx \gamma_1^{(2)}$. Now it can be shown from two evolution equations that

$$\Omega - \gamma_1^{(1)}\mu = O(\epsilon^2)$$

and

$$\Omega - \gamma_1^{(2)}\mu = O(\epsilon^2)$$

where ϵ is a small ordering parameter, the smallness of α_{01}, α_{02} and μ. The nonlinear dispersion relation(46) at fourth order can be solved for the second wave train in the presence of the first wave train as follows:

$$\begin{aligned}
&[(\Omega - C_2) + 0.5F_2/\{(\Omega^{(2)} - C_2)^2 - A_1E_1\}]^2 \\
&= A_2E_2 + \{G^{'} - F_1(\Omega^{(2)} - \gamma_1^{(1)}\mu)\}/\{(\Omega^{(2)} - C_2)^2 - A_1E_1\}
\end{aligned} \quad (48)$$

where $\Omega^{(1)}$ and $\Omega^{(2)}$ are the solutions of the dispersion relation (46) for the first and second wave trains at the lowest order given by

$$\Omega^{(j)} = \gamma_1^{(j)}\mu \pm \{\gamma_2^{(j)}\mu^2(\gamma_2^{(j)}\mu^2 + 2\delta_1^{(j)})\alpha_{0j}^2\}^{\frac{1}{2}}\ , \quad (j = 1, 2) \quad (49)$$

The condition of instability from equation (48) of the second wave train in the presence of first wave train is the following

$$A_2E_2+\{G^{'}-F_1(\Omega^{(2)}-\gamma_1^{(1)}\mu)\}/\{(\Omega^{(2)}-C_2)^2-A_1E_1\} < 0 \quad (50)$$

The above instability condition in the absence of first wave train becomes

$$A_2E_2 < 0$$

that is,

$$\gamma_2^{(2)}\mu^2\{\gamma_2^{(2)}\mu^2 + 2(\delta_1^{(2)} - \delta_4^{(2)}|\mu|)\alpha_{02}^2\} < 0 \quad (51)$$

which is similar to the instability condition of single wave packet.The instability condition (51) in the absence of capillarity is identical with the instability condition (57) of Dhar and Das[5].Also for $u = 0,\ \gamma = 0$ and in the absence of capillarity, the above instability condition reduces to equation (3.8) of Dysthe[10].

From equation (50) the stable-unstable regions of the second wave train in the presence of the first wave train are shown in figures 1 and 2 for two different sets of values of wave numbers $(k_1, k_2) = (3.0008, 0.6289)$ and $(k_1, k_2) = (5.0001, 0.3394)$ respectively.In the said figures, we have plotted marginal stability curves for the second wave train of smaller wave numbers assuming an amplitude of the first wave train of larger wave number to be $a_{01} = 0.2$.We have also plotted in figures 1 and 2 the similar curve of the second wave train in the absence of the first wave train. From the figures it is found that the instability region of the second wave train expands due to the presence of the first wave train. We also observe that the instability region is shortened slightly with the inclusion of fourth order terms.Further with the increase of wind velocity,the instability region is again shortened.

The growth rate of instability I of the second wave train of longer wavelength is given by

$$I = [-A_2E_2 - \frac{H - F_1(\Omega^{(2)} - \gamma_1^{(1)}\mu)}{(\Omega^{(2)} - C_1)^2 - A_1E_1}]^{\frac{1}{2}} \tag{52}$$

From these figures, it is found that the growth rate of instability of the second wave train increases due to the presence of the first wave train and it increases with the increase of the amplitude of the first wave train.We also observe that the influence of fourth order term is to increase the growth rate of instability.Further the growth rate of instability increases with the increase of wind velocity.

We have plotted in figures 3-6 the growth rate of instability I from equation (52)of the second wave train against the perturbation wave number for two values of the amplitude of the first wave train and for two different values of wind velocity. These curves are shown in figures 3,4 and in figures 5,6 for two sets of values of wave numbers $(k_1, k_2) = (3.0008, 0.6289)$ and $(k_1, k_2) = (5.0001, 0.3394)$ respectively.In the said figures,we have plotted the growth rate of instability curves for the second wave train in the absence of the first wave train and also have plotted the corresponding curves that can be obtained from third order evolution equations.

In the figures 1-6 for the wave numbers k_1 and k_2 that we have considered here, the first and second waves fall in the categories of capillary-gravity wave and gravity wave respectively.Therefore, these figures show stable-unstable regions and growth rate of instabilities of a surface-gravity wave train in the presence of capillary gravity wave train.

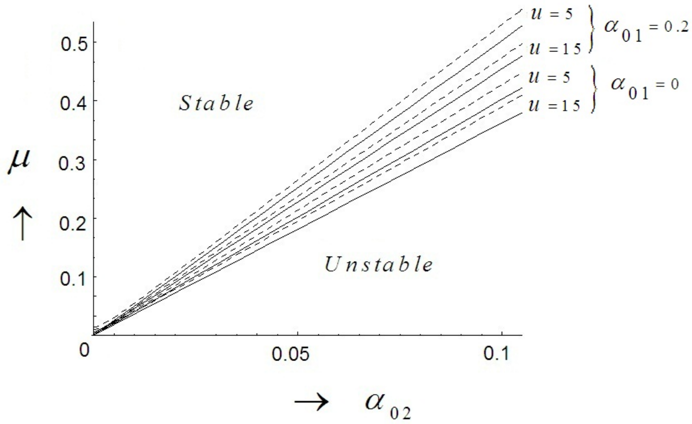

Figure 1. Stable-unstable regions of the second wave train for some different values of dimensionless wind velocity u written on the graphs.Here $(k_1, k_2) = (3.0008, 0.6289)$. ———— represents fourth order results and .... represents third order results.

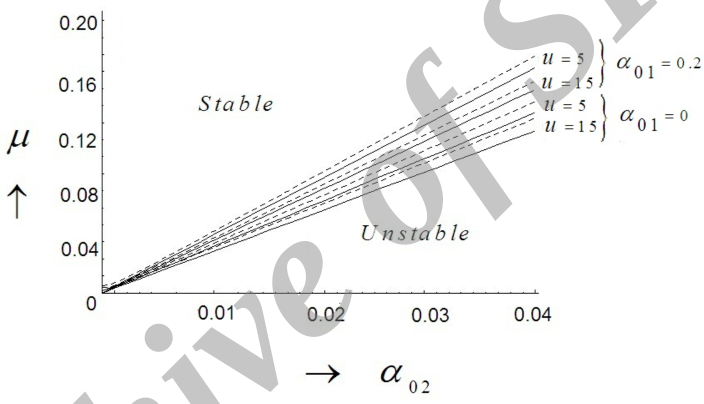

Figure 2. Stable-unstable regions of the second wave train for some different values of dimensionless wind velocity u written on the graphs.Here $(k_1, k_2) = (5.0001, 0.3394)$. ———— represents fourth order results and .... represents third order results.

5. Discussion and Conclusion

The reason for starting from fourth order nonlinear evolution equation is motivated by the fact , as shown by Dysthe[10],that a fourth order nonlinear evolution equation is a good starting point for making stability analysis of a uniform wave train in deep water. Here we have used a general method,based on Zakharov integral equation for the derivation of evolution equations. The instability condition is obtained of a wave of greater wavelength in the presence of a wave of shorter wavelength. The two evolution equations are then used to investigate the stability analysis of a uniform surface gravity wave train in the presence of another capillary gravity wave train having the same group velocity. In this paper we have derived analytically two coupled fourth order nonlinear evolution equations in deep water for two capillary gravity wave packets propagated in the same direction in the pres-

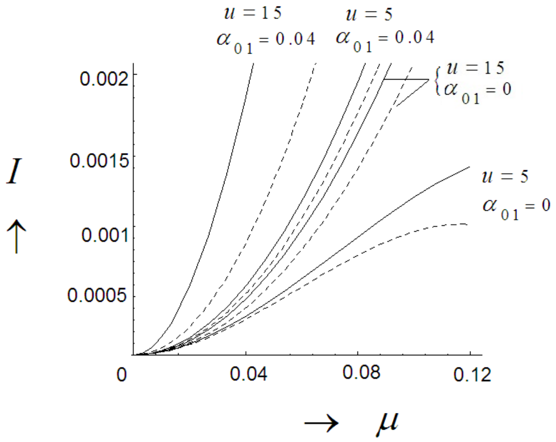

Figure 3. Growth rate of instability I of the second wave train against the perturbation wave number μ for some different values of dimensionless wind velocity u written on the graphs. Here $(k_1, k_2) = (3.0008, 0.6289), \alpha_{02} = 0.05$. ———— represents fourth order results and .... represents third order results.

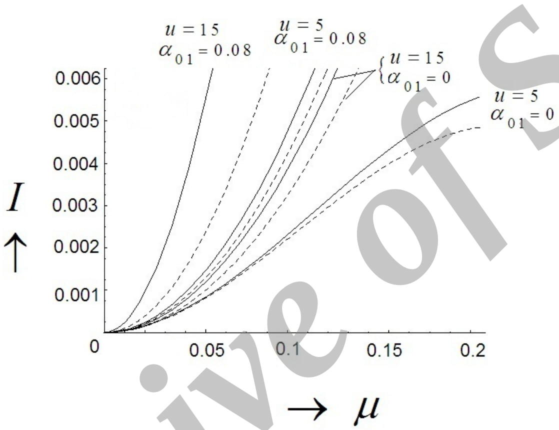

Figure 4. Growth rate of instability I of the second wave train against the perturbation wave number μ for some different values of dimensionless wind velocity u written on the graphs. Here $(k_1, k_2) = (3.0008, 0.6289), \alpha_{02} = 0.1$. ————- represents fourth order results and .... represents third order results.

ence of wind flowing over water. Instability regions are plotted in figures 1 and 2 for two sets of values of wave numbers and for different values of non-dimensional wind velocity. It is found that the instability regions for a surface gravity wave train in the presence of a capillary gravity wave train expand with the increase of wave steepness of the capillary gravity wave train for fixed value of the wind velocity. Again with the increase of the wind velocity, the instability regions are shortened for fixed value of wave steepness of the capillary gravity wave train. The growth rate of instability of a uniform wave train with smaller wave number has been plotted in figures 3-6 against perturbation wave number for two values of the amplitude of the wave train of greater wave number and for different values of wind velocity. From the figure it is observed that the presence of a wave train of smaller wavelength increases the growth rate of instability of a uniform wave train of larger wavelength for fixed value of the wind velocity. Further the growth

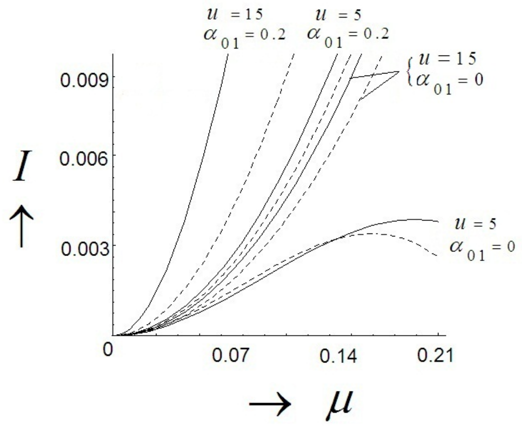

Figure 5. Growth rate of instability I of the second wave train against the perturbation wave number μ for some different values of dimensionless wind velocity u written on the graphs. Here $(k_1, k_2) = (5.0001, 0.3394), \alpha_{02} = 0.05$. ——————— represents fourth order results and .... represents third order results.

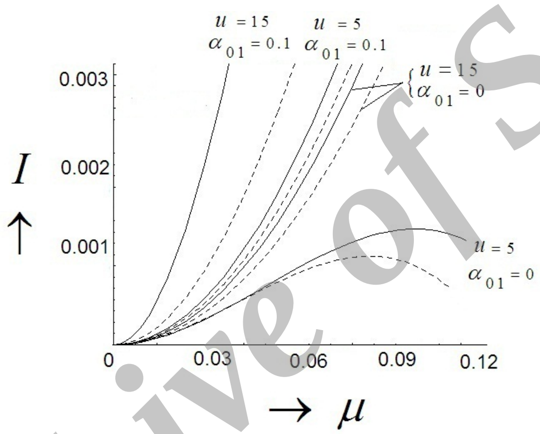

Figure 6. Growth rate of instability I of the second wave train against the perturbation wave number μ for some different values of dimensionless wind velocity u written on the graphs. Here $(k_1, k_2) = (5.0001, 0.3394), \alpha_{02} = 0.1$. ——————— represents fourth order results and .... represents third order results.

rate of instability increases with the increase of wind velocity for fixed value of the amplitude of the first wave train. In the figures 1-6, for the stable-unstable region and for growth rate of instability, we also find a comparison between third and fourth order, results . We observe that at fourth order instability regions get reduced slightly and the growth rate of instability are increased slightly.

6. Appendices

$$\gamma_1^{(1)} = \frac{m}{2(1+\gamma)n}[\gamma u + \frac{R_2^{(1)}}{(\gamma u + R_1^{(1)})}]$$

$$\gamma_2^{(1)} = \frac{-m^2}{8(1+\gamma)n}[\frac{4R_1^{(1)}\{3(1+\gamma)s_1 - \gamma u^2\} - R_2^{(1)2}}{R_1^{(1)2}}]$$

$$\gamma_3^{(1)} = \frac{-3m^3}{16(1+\gamma)n}[\frac{-4R_1^{(1)}R_2^{(1)}\{3(1+\gamma)s_1 - \gamma u^2\} + 8(1+\gamma)R_1^{(1)2}}{R_1^{(1)3}}]$$

$$\gamma_1^{(2)} = \frac{m}{2(1+\gamma)n}[\gamma u + \frac{R_2^{(2)}}{(\gamma u + R_1^{(2)})}]$$

$$\gamma_2^{(2)} = \frac{-m^2}{8(1+\gamma)n}[\frac{4R_1^{(2)}\{3(1+\gamma)s_2 - \gamma u^2\} - R_2^{(2)2}}{R_1^{(2)2}}]$$

$$\gamma_3^{(2)} = \frac{-3m^3}{16(1+\gamma)n}[\frac{-4R_1^{(2)}R_2^{(2)}\{3(1+\gamma)s_2 - \gamma u^2\} + 8(1+\gamma)R_1^{(2)2}}{R_1^{(2)3}}]$$

$$\delta_1^{(1)} = \frac{\sqrt{1+\gamma}}{16n}[\frac{\gamma u + 7(1-\gamma^2) + R_1^{(1)} + \gamma u^2 + 2(1+\gamma)s_1^2}{R_1^{(1)}(\gamma u + 1 - \gamma^2 - 2(1+\gamma)s_1}]$$

$$\delta_2^{(1)} = \frac{-3\sqrt{1+\gamma}\ m}{16n}[\frac{\gamma u + 7(1-\gamma^2) + R_1^{(1)} - (1+\gamma)\{2s_1 + 9s_1^2 - 4s_1^3 - 4s_1^4\}}{R_1^{(1)2}\{\gamma u + 1 - \gamma^2 - 2(1+\gamma)s_1\}^2}]$$

$$\delta_3^{(1)} = \frac{-\sqrt{1+\gamma}\ m}{32n}[\frac{\{\gamma u + 2(1-\gamma^2) - R_1^{(1)}\}\{\gamma u + 7(1-\gamma^2) + R_1^{(1)} + \gamma u^2 + 2(1+\gamma)s_1^2\}}{R_1^{(1)2}\{\gamma u + 1 - \gamma^2 - 2(1+\gamma)s_1\}}]$$

$$\delta_4^{(1)} = \frac{\sqrt{1+\gamma}\ m}{2n}[1 + 2\gamma u - \gamma^2]$$

$$\delta_1^{(2)} = \frac{\sqrt{1+\gamma}}{16n}[\frac{\gamma u + 7(1-\gamma^2) + R_1^{(2)} + \gamma u^2 + 2(1+\gamma)s_2^2}{R_1^{(2)}(\gamma u + 1 - \gamma^2 - 2(1+\gamma)s_2}]$$

$$\delta_2^{(2)} = \frac{-3\sqrt{1+\gamma}\ m}{16n}[\frac{\gamma u + 7(1-\gamma^2) + R_1^{(2)} - (1+\gamma)\{2s_2 + 9s_2^2 - 4s_1^3 - 4s_2^4\}}{R_1^{(2)2}\{\gamma u + 1 - \gamma^2 - 2(1+\gamma)s_2\}^2}]$$

$$\delta_3^{(2)} = \frac{-\sqrt{1+\gamma}\ m}{32n}[\frac{\{\gamma u + 2(1-\gamma^2) - R_1^{(2)}\}\{\gamma u + 7(1-\gamma^2) + R_1^{(2)} + \gamma u^2 + 2(1+\gamma)s_2^2\}}{R_1^{(2)2}\{\gamma u + 1 - \gamma^2 - 2(1+\gamma)s_2\}}]$$

$$\delta_4^{(2)} = \frac{\sqrt{1+\gamma}}{2}[1+2\gamma u-\gamma^2]$$

$$\zeta_1^{(1)} = \frac{\sqrt{1+\gamma}}{8m^3}[8\gamma u+\gamma u^2+4m^2-3ns_2-2m(1-m)/n]$$

$$\zeta_2^{(1)} = \frac{\sqrt{1+\gamma}}{16m^2}[2\gamma u-3\gamma u^2+mns_1+2m(1-m)(1-2s_1)n-4mn-16m^2$$

$$+4m(1+m)n^2+3ns_2(5-\gamma u^2-2s_1)]$$

$$\zeta_3^{(1)} = \frac{\sqrt{1+\gamma}}{16m^2}[5\gamma u-7\gamma u^2-2m^2(3-\gamma u^2-2s_1)+3ns_2(3-\gamma u^2-2s_1)$$

$$-2m(3+2\gamma u+2s_2)+6ns_2+(\gamma u-3u^2)/5]$$

$$\zeta_4^{(1)} = \frac{\sqrt{1+\gamma}}{8m^2}[\gamma u-3\gamma u^2+m(1-m)(1-2s_1)-m^2(3-\gamma u^2-2s_1)+\frac{3}{2}ns_2(3-\gamma u-2\gamma u^2-2s_1)$$

$$+m(3+2\gamma u^2+2s_2)-\frac{n}{m}(3s_2+\gamma u^2-2m^2n)]$$

$$\zeta_5^{(1)} = \frac{\sqrt{1+\gamma}}{2m}(1-2\gamma u)$$

$$\zeta_1^{(2)} = \frac{\sqrt{1+\gamma}}{8m^3}[8\gamma u+\gamma u^2+4m^2-3ns_1-2m(1-m)/n]$$

$$\zeta_2^{(2)} = \frac{\sqrt{1+\gamma}}{16m^2}[2\gamma u-3\gamma u^2+mns_2+2m(1-m)(1-2s_2)n$$

$$-4mn-16m^2+4m(1+m)n^2+3ns_1(5-\gamma u^2-2s_2)]$$

$$\zeta_3^{(2)} = \frac{\sqrt{1+\gamma}}{16m^2}[5\gamma u-7\gamma u^2-2m^2(3-\gamma u^2-2s_2)$$

$$+3ns_1(3-\gamma u^2-2s_2)-2m(3+2\gamma u+2s_1)+6ns_1+(\gamma u-3u^2)/5]$$

$$\zeta_4^{(2)} = \frac{\sqrt{1+\gamma}}{8m^2}[\gamma u-3\gamma u^2+m(1-m)(1-2s_2)-m^2(3-\gamma u^2-2s_2)+\frac{3}{2}ns_1(3-\gamma u-2\gamma u^2-2s_2)$$

$$+m(3+2\gamma u^2+2s_1)-\frac{n}{m}(3s_1+\gamma u^2-2m^2n)]$$

$$\zeta_5^{(2)}=\frac{\sqrt{1+\gamma}\ m^2}{2n}(1-2\gamma u)$$

where

$$R_1^{(1)}=(1-\gamma^2)-\gamma u^2k_1+(1+\gamma)s_1,\qquad R_2^{(1)}=(1-\gamma^2)-2\gamma u^2k_1+3(1+\gamma)s_1$$

$$R_1^{(2)}=(1-\gamma^2)-\gamma u^2k_2+(1+\gamma)s_2,\qquad R_2^{(2)}=(1-\gamma^2)-2\gamma u^2k_2+3(1+\gamma)s_2$$

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