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A Recursive Formula for the Number of Fuzzy Subgroups of Finite Cyclic Groups

Mahdi Imanparast [1]✉, Hamid Darabi [2]

(1) Department of Computer Science, University of Bojnord, Bojnord, Iran
(2) Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
m.imanparast@ub.ac.ir; darabi@iauesf.ac.ir

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Abstract

Fuzzy groups and fuzzy theory have a lot of applications in several sciences such as mathematics, computer science, computer and electrical engineering. Hence, counting the number of fuzzy subgroups of finite groups to classify them is an important issue in fuzzy theory. The Main goal of this paper is to give an explicit formula for the number of fuzzy subgroups of a finite cyclic group $G = Z_{p_1} \times Z_{p_2} \times Z_{p_3} \times \ldots \times Z_{p_k}$*, where* $p_1, p_2, \ldots, p_k$ *are distinct prime numbers. We introduce a very simple recursive formula to count the number of subgroups of G.*

Keywords: *Fuzzy groups, Lattice, Finite cyclic groups, Chains*

1. Introduction

One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group [1]. Several papers have treated the particular case of finite abelian group. Laszlo [2] studied the construction of fuzzy subgroups of groups of the orders one to six. Zhang and Zou [3] have determined the number of fuzzy subgroups of cyclic groups of the order p^n where p is a prime number. Murali and Makamba in [4] and [5], considering a similar problem, found the number of fuzzy subgroups of abelian groups of the order $p^n q^m$ where p and q are different primes. In [6], Tarnauceanu and Bentea established the recurrence relation verified by the number of fuzzy subgroups of finite cyclic groups. Their result is the improving of Murali's works in [4] and [5]. Finally, authors in [8] establish an idea to count the number of fuzzy subgroups in the particular case of finite cyclic groups, namely $Z_p \times Z_q \times Z_r \times Z_s$ where p, q, r and s are distinct prime numbers.

Finding the number of fuzzy subgroups is a common problem in fuzzy groups to classify them. This paper discusses a particular case of finite cyclic groups $G = Z_{p_1} \times Z_{p_2} \times \ldots \times Z_{p_k}$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers. However, the result is a generalization of [8] approach in a general case.

2. Preliminaries

First of all, some basic notions and results of fuzzy theory which are required later in this paper from [10] and [11], are presented. In this section, group G is assumed to be a finite group.

Let $(G,.,e)$ be a group (e denotes the identity of G) and $\mu: G \to [0,1]$ a fuzzy subset of G. We say that μ is a fuzzy subgroup of G if it satisfies the following two conditions:

(a) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in G$.

(b) $\mu(x^{-1}) \geq \mu(x)$, for any $x \in G$.

In this situation we have $\mu(x^{-1}) = \mu(x)$, for any $x \in G$, and $\mu(e) = \max \mu(G)$ [9].

Theorem 2.1 [7] A fuzzy subset μ of G is a fuzzy subgroup of G if and only if there is a chain of subgroups of G, $P_1(\mu) \leq P_2(\mu) \leq \cdots \leq P_n(\mu) = G$, where μ can be written as:

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1(\mu) \\ \theta_2, & x \in P_2(\mu) \\ \vdots & \\ \theta_n, & x \in P_n(\mu) \end{cases} \tag{1}$$

Definition 2.1 [7] Let μ, γ be fuzzy subgroups of G of the form

$$\mu(x) = \begin{cases} \theta_1, & x \in A_1 \\ \theta_2, & x \in A_2 \\ \vdots & \\ \theta_m, & x \in A_m \end{cases} \quad and \quad \gamma(x) = \begin{cases} \delta_1, & x \in B_1 \\ \delta_2, & x \in B_2 \\ \vdots & \\ \delta_n, & x \in B_n \end{cases}$$

with $\theta_i, \delta_i \in [0,1]$, $\theta_k > \theta_l$, $\delta_k > \delta_l$ for $k < l$ and $\cup_{1 \leq i \leq m} A_i = G$, $\cup_{1 \leq j \leq n} B_j = G$, then μ and γ are equivalent if $m = n$ and $A_i = B_i$, $\forall i \in \{1, 2, \cdots, n\}$.

Lemma 2.1 The number of fuzzy subgroups of G is equal to the number of chain on the lattice subgroups of G.

2.1 Previous Works

There are several attempts to solve this problem. This section describes two important works in [1] and [8].

2.1.1 Tarnauceanu and Bentea Results

Let G be a finite subgroup, then G has a direct decomposition as follows:

$$G \cong Z_{p_1^{\alpha_1}} \times Z_{p_2^{\alpha_2}} \times Z_{p_3^{\alpha 3}} \times \ldots \times Z_{p_k^{\alpha_k}},$$

where p_i, $i = 1, \ldots, k$ are distinct primes and $\alpha_i \in N$, $i = 1, \ldots, k$. Also, the number of distinct fuzzy subgroups of G coincides with the number of all chains of subgroups of G that ends in G, say $g_k(\alpha_1, \alpha_2, \ldots, \alpha_k)$ (obviously, this value depends only on $\alpha_1, \alpha_2, \ldots, \alpha_k$ and not on $p_1, p_2, \ldots, p_k$). Let ς be the set consisting of all chains of subgroups of G that ends in G and $H_1, H_2, ..., H_k$ the minimal subgroups of G. We denote, the set of all chains of the lattice interval $[G/H_r] = \{H \in L(G) \mid H_r \subseteq H \subseteq G\}$ that end in G, by ς_r, ($r = 1, \cdots, k$). Clearly, we have:

$$g_k(\alpha_1, \alpha_2, ..., \alpha_k) = |\varsigma| = 2 \,|\bigcup_{r=1}^{k} \varsigma_r|.$$

Now, by applying the well-known Inclusion-Exclusion principle, one obtains that:

$$g_k(\alpha_1, \alpha_2, ..., \alpha_k) = 2\sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \le i_1 < i_2 < .... < i_r \le k} |\varsigma_{i_1} \cap \varsigma_{i_2} \cap ... \cap \varsigma_{i_r}| \quad (2)$$

the next theorem is a consequence of the above relation [1].

Theorem 2.2 Let G be a finite cyclic group and $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ the decomposition of /G/ as a product of prime factors, then the number $g_k(\alpha_1, \alpha_2, ..., \alpha_k)$ of all distinct fuzzy subgroups of G is given by

$$g_k(\alpha_1, \alpha_2, ..., \alpha_k) = 2^{\sum_{r=1}^{k} \alpha_r} \sum_{i_2=0}^{\alpha_2} \sum_{i_2=0}^{\alpha_2} \cdots \sum_{i_2=0}^{\alpha_2} (-\frac{1}{2})^{\sum_{r=2}^{k} i_r} \prod_{r=2}^{k} \binom{\alpha_r}{i_r} \binom{\alpha_1 + \sum_{m=2}^{r} (\alpha_m - i_m)}{\alpha_r} \quad (3).$$

where the above iterated sums are equal to 1 for $k = 1$.

2.1.2 Sulaiman and Abd Ghafur Results

This subsection deals with some results and theorems due to [8], without any proof. Considering (1), they denote the number of fuzzy subgroups of G as $o(F_G)$, while the number of fuzzy subgroups μ of G with $P_1(\mu) = H$ is denoted by $o(F_{P_1=H})$. From Theorem 2.1 we have

$$o(F_G) = \sum_{H \le G} o(F_{P_1=H}) \quad (4)$$

and

$$o(F_{P_1=G}) = 1. \quad (5)$$

Theorem 2.3 Let H be a subgroup of G, and the set of all subgroups of G, which contain H (but are not equal to H) be $\{H_1, H_1, \cdots, H_n = G\}$. Then: $o(F_{P_1=H}) = \sum_{1 \le i \le n} o(F_{P_1=H_i})$.

Theorem 2.4 Let e be identity element of a group G. Then $o(F_G) = 2.o(F_{P_1=\{e\}})$.

Theorem 2.5 Let $G = Z_p \times Z_q$, where p and q are distinct primes. Then, the number of fuzzy subgroup of G is 6.

Theorem 2.6 Let $G = Z_p \times Z_q \times Z_r$, where p, q and r are distinct primes. Then, the number of fuzzy subgroup of G is 26.

3. Main Results

A recursive formula to count the number of finite cyclic fuzzy subgroups $G = Z_{p_1} \times Z_{p_2} \times Z_{p_3} \times \ldots \times Z_{p_k}$ (that is called G_k in this section), is presented in this section.

Before that, we compute some initial numbers of cyclic fuzzy subgroup of G_k with complicated formula (3) in theorem 2.2.

That formula defined for

$$G \cong Z_{p_1^{\alpha_1}} \times Z_{p_2^{\alpha_2}} \times Z_{p_3^{\alpha 3}} \times \ldots \times Z_{p_k^{\alpha_k}},$$

where $p_i, (i = 1,...,k)$ are distinct primes and $\alpha_i \in N$, $i = 1,...,k$.

Setting $\alpha_i = 1;\ \forall i = 1,2,...,k$, in (3) results $G_k = Z_{p_1} \times Z_{p_2} \times \ldots \times Z_{p_k}$ and:

$$o(F_{G_k}) = g_k(\alpha_1, \alpha_2, ..., \alpha_k) \qquad ; \alpha_i = 1;\ \forall\, i = 1,...,k$$

$$= g_k(1,1,...,1) = 2^{\sum_{r=1}^{k}(1)} \sum_{i_2=0}^{1} \sum_{i_3=0}^{1} \cdots \sum_{i_k=0}^{1} (-\frac{1}{2})^{\sum_{r=2}^{k} i_r} \prod_{r=2}^{k} \binom{1}{i_r} \binom{1 + \sum_{m=2}^{r}(1 - i_m)}{1}$$

$$= 2^k \sum_{i_2=0}^{1} \sum_{i_3=0}^{1} \cdots \sum_{i_k=0}^{1} (-\frac{1}{2})^{\sum_{r=2}^{k} i_r} \prod_{r=2}^{k} (1) \left(1 + \sum_{m=2}^{r}(1 - i_m)\right); \quad \binom{1}{0} = \binom{1}{1} = 1, \ \binom{a}{1} = a$$

$$= 2^k \sum_{i_2=0}^{1} \sum_{i_3=0}^{1} \cdots \sum_{i_k=0}^{1} (-\frac{1}{2})^{\sum_{r=2}^{k} i_r} \prod_{r=2}^{k} \left(1 + \sum_{m=2}^{r}(1 - i_m)\right) \tag{6}$$

The relation (6) can be applied to compute number of subgroups of G_k. For example in k=2, $G_2 = Z_{p_1} \times Z_{p_2}$ and

$$o(F_{G_2}) = 2^2 \sum_{i_2=0}^{1} (-\frac{1}{2})^{\sum_{r=2}^{2} i_r} \prod_{r=2}^{2} \left(1 + \sum_{m=2}^{2}(1 - i_m)\right)$$

$$= 4 \sum_{i_2=0}^{1} (-\frac{1}{2})^{i_2} \prod_{r=2}^{2} \left(1 + (1 - i_2)\right) = 4 \sum_{i_2=0}^{1} (-\frac{1}{2})^{i_2} \left(2 - i_2\right)$$

$$= 4((-\frac{1}{2})^0 \left(2 - 0\right) + (-\frac{1}{2})^1 \left(2 - 1\right)) = 4((1)(2) + (-\frac{1}{2})(1)) = 8 - 2 = 6$$

By applying some hard computations, one can find other numbers as listed in Table 1. But for $k \geq 3$ this computation is a little tiresome. The following theorem presents a simple recursive formula for this problem.

Table 1. Compute $o(F_{G_k})$ by formula (6)

k	$o(F_{G_k})$
2	6
3	26
4	150
5	1082

Theorem 3.1 Let G_k be a finite cyclic group of form $G_k = Z_{p_1} \times Z_{p_2} \times Z_{p_3} \times \ldots \times Z_{p_k}$, where $p_1, p_2, ..., p_k$ are distinct prime numbers. Then number of fuzzy subgroups of G_k is:

$$o(F_{G_k}) = 2.o(F_{P_1=\{e\}_k});$$

$$o(F_{P_1=\{e\}_k}) = 1 + \sum_{i=1}^{k-1} \binom{k}{i} . o(F_{\{e\}_i}) \ ; \qquad o(F_{P_1=\{e\}_1}) = 1. \tag{7}$$

when the $\{ e_i \}$; $i = 1, 2, \cdots, k$ denote the identity of G_i.

Proof: we focused to determine a relation between $o(F_{G_k})$ and lattice diagram. For $G_2 = Z_{p_1} \times Z_{p_2} = Z_p \times Z_q$, order of G_2 is *pq.* Hence, every nontrivial subgroups of G_2 must be of order *p* or *q.* We have only two nontrivial subgroups of G_2, namely $H_1 = Z_p \times \{0\}$ and $H_2 = \{0\} \times Z_q$. The lattice subgroups of G_2 are shown in Figure 1.

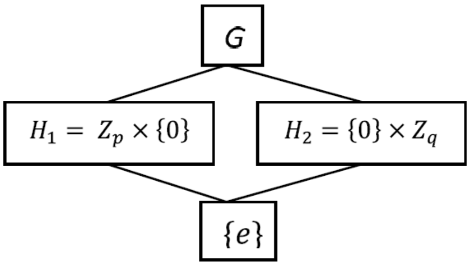

Figure 1. Lattice subgroup of $G_2 = Z_p \times Z_q$

According to the relation (5) we have $o(F_{P_1=G}) = 1$ and using Theorem 2.3 we have

$$o(F_{P_1=H_1}) = o(F_{P_1=H_2}) = 1,$$

and $o(F_{P_1=\{e\}_2}) = 3$.

Now, Let $G_3 = Z_{p_1} \times Z_{p_2} \times Z_{p_3} = Z_p \times Z_q \times Z_r$, where the order of G_3 is *pqr.* Hence, every nontrivial subgroup of G_3 must be of order *p, q, r, pq, pr* or *rq.* We have six nontrivial subgroups of G_3:

$$H_1 = Z_p \times \{0\} \times \{0\} \qquad H_2 = \{0\} \times Z_q \times \{0\}$$
$$H_3 = \{0\} \times \{0\} \times Z_r \qquad H_4 = Z_p \times Z_q \times \{0\}$$
$$H_5 = Z_p \times \{0\} \times Z_r \qquad H_6 = \{0\} \times Z_q \times Z_r$$

The lattice subgroups of G_3 are shown in Figure 2.

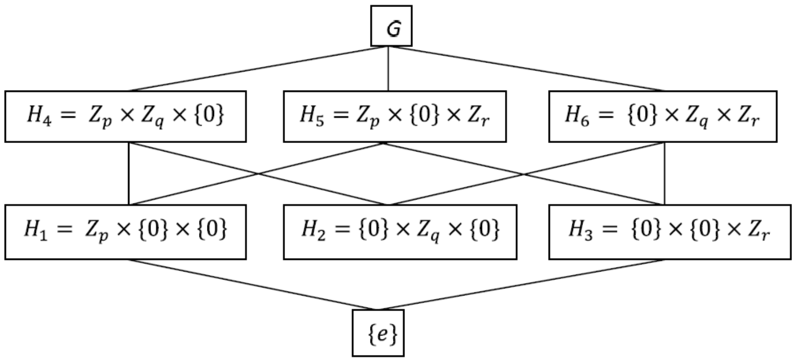

Figure 2. Lattice subgroup of $G_3 = Z_p \times Z_q \times Z_r$

According to relation (5) we have $o(F_{P_1=G}) = 1$ and using Theorem 2.3 we have:

$$o(F_{P_1=H_4}) = o(F_{P_1=H_5}) = o(F_{P_1=H_6}) = 1,$$
$$o(F_{P_1=H_1}) = o(F_{P_1=H_2}) = o(F_{P_1=H_3}) = 3,$$
$$\text{and } \; o(F_{P_1=\{e\}_3}) = 13\,.$$

This means:

$$o(F_{P_1=\{e\}_3}) = 1 + 3(1) + 3(3) = 13\,,$$

the sum of all $o(F_{P_1=H_i})$ and $o(F_G)$, which come in upper levels of $\{e\}_3$ in lattice diagram. Similarly, this approach can be used to draw diagrams for other G_k (k=1111). This is the key to find the formula.

First, we compute number of all H_i that appear from nontrivial subgroups of G_k and classifying them to k+1 levels. We can classify nontrivial subgroups of G_k to k+1 set (ϕ_i ; $i = 0, 1, \cdots, n$) as follows:

0) Show no letter of $p_1, p_1, ..., p_k$: $\phi_0 = \{\{\{0\} \times \{0\} \times \cdots \times \{0\}\}\} = \{e\}$

1) Show only one letter of $p_1, p_1, ..., p_k$:

$$\phi_1 = \{\left\{Z_{p_1} \times \{0\} \times \cdots \times \{0\}\right\}, \cdots, \left\{\{0\} \times \cdots \times \{0\} \times Z_{p_k}\right\}\}$$

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k-1) There are k-1 letters of $p_1, p_1, ..., p_k$:

$$\phi_{k-1} = \{\left\{Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_{k-1}} \times \{0\}\right\}, \cdots, \left\{\{0\} \times Z_{p_2} \times \cdots \times Z_{p_{k-1}} \times Z_{p_k}\right\}\}$$

k) There are all letters of $p_1, p_1, ..., p_k$:

$$\varphi_k = \{Z_{p_1} \times Z_{p_2} \times Z_{p_3} \times \cdots \times Z_{p_k}\} = \{G\}$$

Each of these k+1 levels generates elements of each level of lattice diagram. We can find the number of all elements in each level of the lattice diagram (namely $|\varphi_i|$) by: $\binom{k}{i}$.

Let H_L be the set of all subgroups of G_k in level L ($L=0,1,\cdots,k$). From theorem 2.3 we can see the number of $o(F_{P_1=H_{Lj}})$; $H_{Lj}\in H_L$, in each level can compute from relative subgroups that appear in upper levels of it. This number is equal for all subgroups in same level of the diagram.

By summing all number of upper levels, we can count $o(F_{P_1=\{e\}_k})$ in level k. Then, we have:

$$o(F_{P_1=\{e\}_k})=o(F_{P_1=G})+\sum_{L=1}^{k-1}|\varphi_L|.o(F_{H_{L_j}})$$

when at $o(F_{H_{L_j}})$, H_{L_j} is one of the subgroups in level L. If we consider each H_{L_j} in level L, it can be considered equivalent to $\{e\}_L$. So, $o(F_{H_{L_j}})=o(F_{\{e\}_L})$.

By $|\varphi_i|=\binom{k}{i}$ and $o(F_{P_1=G})=1$, we have:

$$\begin{aligned}o(F_{P_1=\{e\}_k})&=o(F_{P_1=G})+\sum_{L=1}^{k-1}|\varphi_L|.o(F_{H_{L_j}})\\&=1+\sum_{L=1}^{k-1}\binom{k}{L}.o(F_{\{e\}_L})\end{aligned}$$

Finally, from theorem 2.4 we have:

$$o(F_{G_k})=2.o(F_{P_1=\{e\}_k}).\qquad\square$$

Figures 3 and 4 show the steps of this approach in each level of cyclic groups G_4 and G_5.

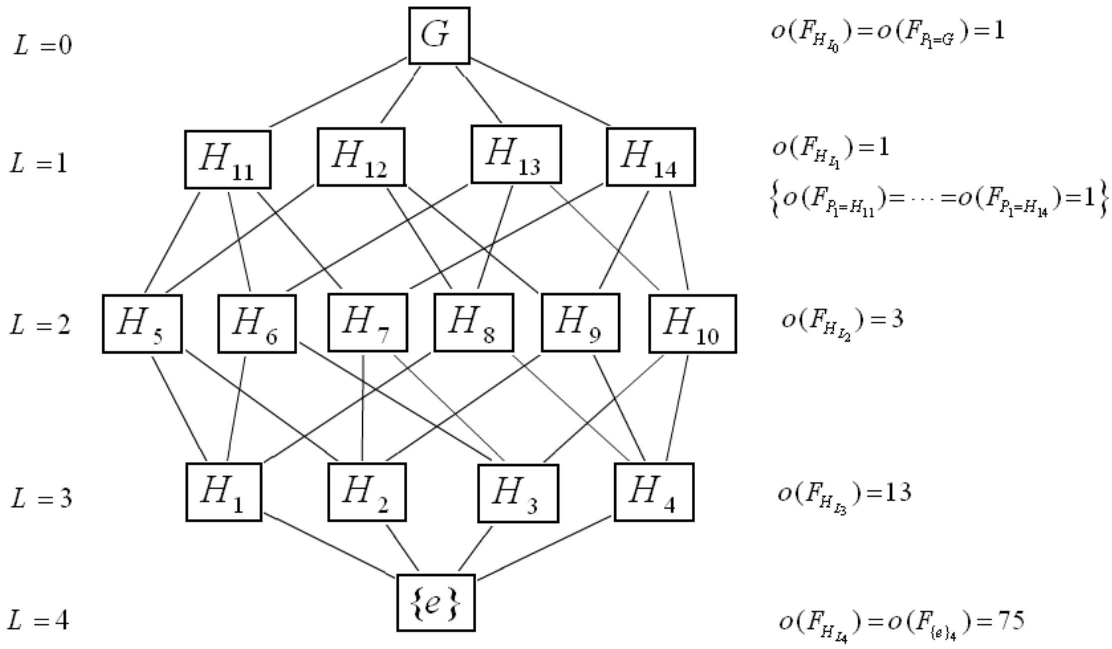

Figure 3. Lattice subgroup of $G_4 = Z_{p_1} \times Z_{p_2} \times Z_{p_3} \times Z_{p_4}$

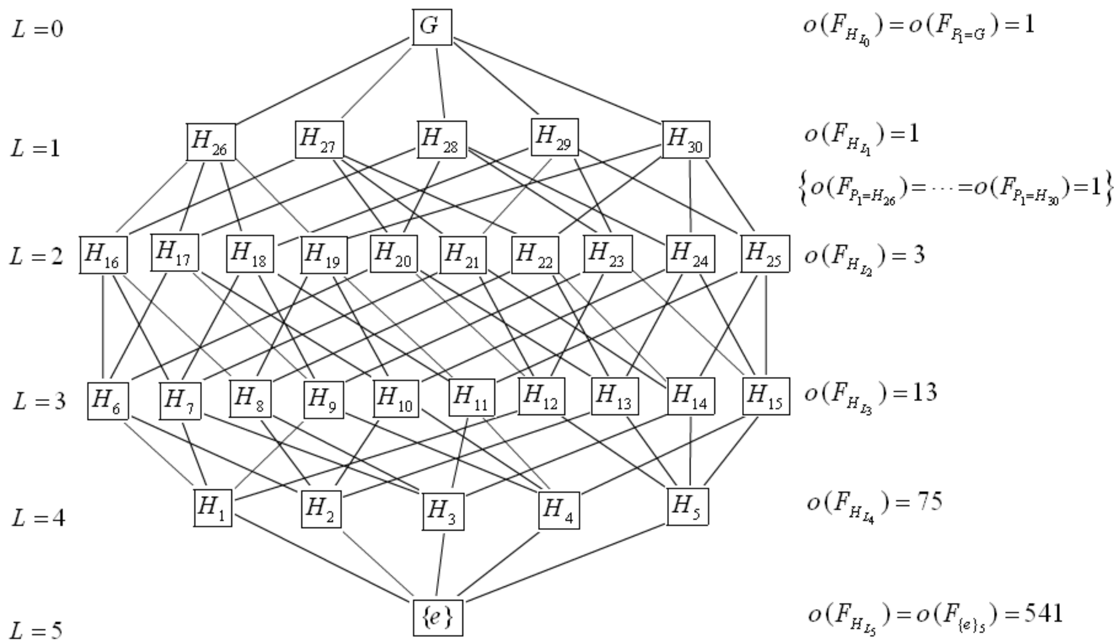

Figure 4. Lattice subgroup of $\boldsymbol{G_5 = Z_{p_1} \times Z_{p_2} \times Z_{p_3} \times Z_{p_4} \times Z_{p_5}}$

4. Conclusion

One of the most important problems in fuzzy group theory is concerned with classifying the fuzzy subgroups of a finite group. This subject has enjoyed a quick evolution in the last few years. Several papers have considered the particular case of finite cyclic groups.

In this paper, we considered some approaches about the number of fuzzy subgroups of finite cyclic groups and presented an explicit recursive formula for the number of fuzzy subgroups of a finite cyclic group $G = Z_{p_1} \times Z_{p_2} \times Z_{p_3} \times \ldots \times Z_{p_k}$, where $p_1, p_2,..., p_k$ are distinct prime numbers. We determined all subgroups and drew the diagram of subgroups lattice of G. The number of fuzzy subgroups of G could be determined by counting the number of chains on the lattice.

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6. References

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