Journal of Advances in Computer Research
Quarterly pISSN: 2345-606x eISSN: 2345-6078
Sari Branch, Islamic Azad University, Sari, I.R.Iran
(Vol. 5, No. 2, May 2014), Pages: 51-64
www.jacr.iausari.ac.ir

Time-Distance Optimal Trajectory Planning For Mobile Robots On Straight And Circular Paths

Hossein Barghi Jond[1 *], Adel Akbarimajd[2], Nurhan Gursel Ozmen[3]

(1) Young Researchers and Elite Club, Ahar Branch, Islamic Azad University, Ahar, Iran
(2) Department of Electrical and Computer Engineering, University of Mohaghegh Ardabili, Ardabil, Iran
(3) Department of Mechanical Engineering, Karadeniz Technical University, Trabzon, Turkey

h-barghi@iau-ahar.ac.ir; akbarimajd@uma.ac.ir; gnurhan@ktu.edu.tr

Received: 2013/12/01; Accepted: 2014/01/18

Abstract

Trajectories are generally used to describe the space and time required to perform a desired motion task for a mobile robot or manipulator system. In this paper, we considered a cubic polynomial trajectory for the problem of moving a mobile robot from its initial position to a goal position in over a continuous set of time. Along the path, the robot requires to observe a certain acceleration profile. Then, we formulated an optimization approach to generate optimal trajectory profiles for the mobile robot in the cases of maximum-distance and minimum-time problems. The optimization problem presented to find the trajectory strategy that would give the robot time-distance optimality to move from a start point to an end point where the robot should stay inside its acceleration limits all the time. The problem solved analytically because as it is well known, numerical solutions and iterative methods are time-consuming, therefore, our closed-form solutions demand low computation time. Finally, the results are verified by simulations.

Keywords: *Mobile Robots, Trajectory Planning, Constrained Optimization, Acceleration limits*

1. Introduction

In motion planning, trajectories comprise to the position, velocity and acceleration profiles that are generated correspond robot's initial to goal configurations over a time history [1]. Trajectories are generally used to describe the space and time required to perform a desired motion task for a mobile robot or manipulator system. Therefore, by generating of fit trajectories, our aim is to specify motion profiles to navigating the robot from its current configuration to a desired final point. Usually, it is need some point descriptions along the path be known to the user such as initial, interval and final points profiles. Another important issue in trajectory generation functions is that they need to provide smooth and continuous motions in time for the robot. At least, a desirable function has second derivative. Smooth and continuous functions could keep away the robot from rough and jerky motions that cause of vibrations and unsightly behaviour [1].

As a quick literature review, we refer to some related works. In the manipulator based trajectory planning, a planning mode of trajectory for serial-link robots using higher-degree polynomials developed in [2], and in [3] the PSO used to search the global time-optimal trajectory. In the mobile robots, a method for generating acceleration-based optimal smooth piecewise trajectories are proposed in [4] where smoothness of velocity profile generation were considered. An optimal trajectory plan to minimize the tracking error for a differential driving mobile robot is proposed in [5]. Trajectory planning for non-holonomic wheeled mobile considering obstacle avoidance and constraints like bounded velocities, accelerations and torques is studied in [6]. Satisfying the initial and final postures/velocities, battery voltage and obstacle avoidance factors, a near-time-optimal trajectory finding procedure are proposed in [7]. Time optimal trajectory planning with a differential evolution-fuzzy inference system and neural networks is proposed in [8] and [9], respectively. In [10], a sequential convex programming (SCP) for finding optimum time trajectory planning is used. A method in [11] is proposed to generate minimum-time optimal velocity profiles considering acceleration limits. A switching time computation (STC) method is presented in [12] to generate time-optimal collision-free trajectory planning.

Also, other important works include time minimizing in the spline curve path [13], presenting a model of polynomial s-curve trajectory profiles in the recursive form with minimum time consideration [14], using three path planning primitives, namely straight-line segments, circular segments, and continuous-curvature turns in the path planning [15]. Trajectory planning based on a new formulation of the dynamic potential function with the aim of optimizing the trajectory and some trajectory planning solutions was also proposed in [16]. As a close issue to over work, in [17] the problem of finding an optimal velocity profile to traverse the path in shortest time is described, but the method developed on a dynamic model of mobile robots.

In this paper, we considered a cubic polynomial trajectory for the problem of moving a mobile robot from its initial position to a goal position in over a continuous set of time. The initial position and velocity profiles of the mobile robot is known. Also, velocity profile is presented in final position. Along the path, the robot requires to observe a certain acceleration profile. For describing such a motion, we require a function that generates smooth position, velocity and acceleration profiles, so that satisfy all mentioned constraints. There are many functions that could interpolate initial and final positions and satisfy all conditions, but we are interested to find a one that generates the optimal values for position, velocity and acceleration profiles, whereas there are no other functions with better generated profiles. We found these optimal functions by analytical methods for a cubic polynomial trajectory and we prove its optimality in profile generations by simulations. Before introducing analytical method, we propose an optimization form problem such that it describes optimality conditions for a third-order polynomial function. Then, we solve it by an analytical method.

The rest of this paper organized as the sequel. The next section describes how we formulate optimization problems and then, used analytical methods to solve it and to find optimal trajectory planning of the robot. The simulation results and discussions are provided in the Third Section. The last section includes conclusions.

2. Trajectory Planning Strategy

We devise trajectory planning of a mobile robot in two basic motions including linear and circular ones. Given initial and final positions and velocities, it is required that the trajectory equation at least has four parameters. To achieve smooth and continuous trajectories, it is used a third-order polynomial. The problem is determining the polynomial coefficients to get optimal trajectories considering starting and end points positions, velocity constraints, and acceleration limits.

2.1 Trajectories of Straight Path

It is assumed that the robot travels along a straight (linear) path (in x-direction) starting from the rest at the origin. We describe the position of the robot in x-direction as the next equation

$$x(t) = l_1 t^3 + l_2 t^2 \tag{1}$$

By considering the acceleration limits and zero endpoint velocities, the solution of this problem with two different objective functions is presented in the following.
Maximum Distance Problem: Assume that the final time of the trajectory t_f is fixed, it is desired to find an optimum λ_1, λ_2 such that the robot will cover the maximum distance. This problem can be formulated as

$$\max_{l_1, l_2} x(t_f) = l_1 t_f^3 + l_2 t_f^2 \tag{2}$$

subject to

$$v(t_f) = 0 \tag{2a}$$

$$|a| \leq \Phi \tag{2b}$$

where a is the robot's acceleration parameter and the F is a constant that specify allowed maximum and minimum accelerations. Here, equation (2a) is a constraint which implies on zero velocity at the end of the trajectory and inequality (2b) is the limited acceleration constraint.
As the acceleration is linear, if inequality (2b) satisfied at $t=0$ and $t=t_f$, then it would be satisfied for all time durations between $0<t<t_f$. Therefore, we can replace inequality (2b) with the following equations

$$|6 l_1 t_f + 2 l_2| \leq \Phi \qquad , t = t_f \tag{2c}$$

$$|2 l_2| \leq \Phi \qquad , t = 0 \tag{2d}$$

Finding t_f from equation (2a) and substituting it in equations (2c) and (2d), we conclude that the maximum value of equation (2) is achieved if we take λ_1 and λ_2 as below

$$l_2 = \frac{1}{2}\Phi, \; l_1 = \frac{-1}{3t_f}\Phi \tag{3}$$

Accordingly, from equation (1), optimal plan for the robot's trajectory in *x*-direction would be

$$x(t)=\Phi\left(\frac{-1}{3}\frac{t^3}{t_f}+\frac{1}{2}t^2\right) \tag{4}$$

Then, the maximum value of covered distance $x(t_f)$ is given by $\frac{1}{6}t_f^2\Phi$.

Minimum Time Problem: Assume that the robot has to cover a fixed distance of $x(t_f)=L$ in minimum time. The problem can be represented by the following objective function

$$\min_{l_1,l_2}(t_f)\ ,\ \ l_1t_f^3+l_2t_f^2=L \tag{5}$$

The constraints for this problem are the same equation (2a) and inequality (2b). By obtaining λ_1 from equation (2a) and substituting it in the cost function equation (5), we get

$$\min_{l_2}(t_f)\ ,\quad \frac{1}{3}t_f^2 l_2=L \qquad or \qquad \min_{l_2}(\sqrt{\frac{3L}{l_2}}) \tag{6}$$

By the same rational for the maximum distance case, we will have λ_1 and λ_2 as are given in equation (3). Therefore the minimum time is obtained as $t_{f(\min)}=(\Phi)^{-\frac{1}{2}}\sqrt{6L}$.

Finally, by substituting $t_{f\,(min)}$, λ_1 and λ_2 in equation (1), optimal plan for the robot's trajectory in the minimum time case is given by

$$x(t)=\Phi\left(-\frac{1}{3}\Phi^{\frac{1}{2}}(6L)^{-\frac{1}{2}}t^3+\frac{1}{2}t^2\right) \tag{7}$$

2.2 Trajectories of Circular Path

It is assumed that the robot travels from an initial angular position $\theta(0)$ to the final $\theta(t_f)$ along a circle with radius c. The length of arc or distance traveled by the robot is

$$s(t_f)=(q(t_f)-q(0))c \tag{8}$$

In the planning of the robot trajectory in this circular path, we just have to determine the angular position over time. To this aim, assuming the robot moves from rest, we use a third order polynomial as below to describing it's angular position.

$$q(t)=l_1t^3+l_2t^2 \tag{9}$$

Now, the problem is to find λ_1, λ_2 in order to get an optimum angular trajectory according to two different objective functions of maximum distance and minimum time problems. Here, acceleration of the robot composed of centripetal (a_c) and tangential components (a_t) (see Figure 1).

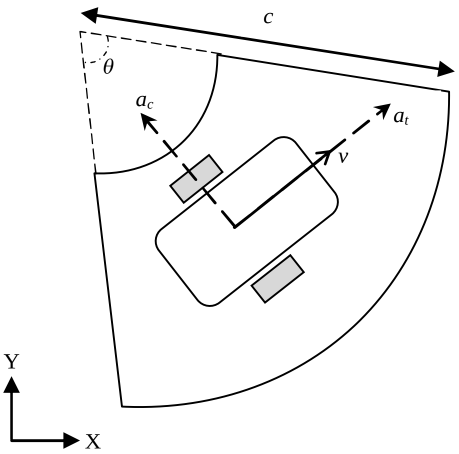


Figure 1. The robot following a circular path.

Then, the limited acceleration constraint could be written as

$$\left|\vec{a}_c + \vec{a}_t\right| < \Phi \tag{10}$$

Equation (10) shows a non-linear inequality constraint that satisfies limited acceleration condition in the case of trajectory at circular paths.
Maximum Distance Problem: The issue is represented with the following optimization problem

$$\max_{l_1, l_2} \quad s(t_f) = c(l_1 t_f^3 + l_2 t_f^2) \tag{11}$$

subject to equation (2a) and inequality (10) where the latter can be rewritten as

$$\left|(6 l_1 t + 2 l_2)^2 + (3 l_1 t^2 + 2 l_2 t)^4\right| < \Phi^2 c^{-2} \qquad ,0 < t < t_f \tag{11a}$$

Finding λ_1 from equation (2a) and substituting in inequality (11a), the problem yields

$$\max_{l_1} \quad s(t_f) = \frac{1}{3} c\, l_2 t_f^2 \tag{12}$$

subject to

$$\left|\left(\frac{2l_2}{t_f}\right)^2 (t_f - 2t)^2 + \left(\frac{2l_2}{t_f}\right)^4 (t^2 - t_f t)^4\right| < \Phi^2 c^{-2} \qquad ,0 < t < t_f \tag{12a}$$

Differentiating inequality (12a) with respect to time, only one local maximum point at $t = 0.5t_f$ is obtained. Then, if inequality (12a) held at $t = 0.5t_f$ and $t = t_f$, it would be held in all instances of interval $0 < t < t_f$. Therefore, we can rewrite it as

$$\left(\frac{l_2 t_f}{2}\right)^4 < \Phi^2 c^{-2} \quad (12b)$$

$$4 l_2^2 < \Phi^2 c^{-2} \quad (12c)$$

Inequalities (12b) and (12c) yields to

$$-\frac{2}{t_f}\Phi^{\frac{1}{2}} c^{-\frac{1}{2}} < l_2 < \frac{2}{t_f}\Phi^{\frac{1}{2}} c^{-\frac{1}{2}} \quad (12d)$$

$$-\frac{1}{2}\Phi c^{-1} < l_2 < \frac{1}{2}\Phi c^{-1} \quad (12e)$$

Maximum value of cost function equation (12) is satisfied with maximum value of λ_2. Then the solution of the optimization problem is

$$l_2 = \min\left(\frac{2}{t_f}\Phi^{\frac{1}{2}} c^{-\frac{1}{2}}, \frac{1}{2}\Phi c^{-1}\right) \quad (13a)$$

From equation (2a), also we obtain λ_1

$$l_1 = -\frac{2}{3t_f}\min\left(\frac{2}{t_f}(\Phi^{\frac{1}{2}} c^{-\frac{1}{2}}, \frac{1}{2}\Phi c^{-1}\right) \quad (13b)$$

As a result, the maximum angular trajectory (of position) is obtained by using the following equation

$$q(t) = \min\left(\frac{2}{t_f}\Phi^{\frac{1}{2}} c^{-\frac{1}{2}}, \frac{1}{2}\Phi c^{-1}\right)(-\frac{2}{3t_f}t^3 + t^2) \quad (14)$$

and the corresponding maximum distance is

$$\max s(t_f) = \frac{1}{3} c t_f^2 \min\left(\frac{2}{t_f}\Phi^{\frac{1}{2}} c^{-\frac{1}{2}}, \frac{1}{2}\Phi c^{-1}\right) \quad (15)$$

Minimum Time Problem: Assume that the robot covers a given arc-length $s(t_f)=s_f$, the problem is represented as

$$\min_{l_1, l_2}(t_f) \quad , \quad c(l_1 t_f^3 + l_2 t_f^2) = s_f \quad (16)$$

The constraints are the same as in the maximum distance problem of circular paths. Using equations (2a) and (16), we can reduce the problem into

$$\min_{l_2}(t_f) = \sqrt{\frac{3s_f}{c\,l_2}} \tag{17}$$

subject to

$$-\frac{4}{3s_f}\Phi < l_2 < \frac{4}{3s_f}\Phi \tag{17a}$$

$$-\frac{1}{2}\Phi c^{-1} < l_2 < \frac{1}{2}\Phi c^{-1} \tag{17b}$$

The minimum value for t_f obtains by the maximum attainable value of λ_2 that is

$$l_2 = \min\left(\frac{4}{3s_f}\Phi, \frac{1}{2}\Phi c^{-1}\right) \tag{18a}$$

and from equation (2a), we have λ_1

$$l_1 = -\frac{2}{3t_f}\min\left(\frac{4}{3s_f}\Phi, \frac{1}{2}\Phi c^{-1}\right) \tag{18b}$$

After substituting λ_1 and λ_2 into equation (16), minimum time is obtained as $t_{f(\min)} = \frac{3}{2}(\Phi c^{-1})^{-\frac{1}{2}} s_f c^{-1}$.

Finally, optimal plan for the robot's angular position in the minimum time problem is given by

$$q(t) = \min\left(\frac{4}{3s_f}\Phi, \frac{1}{2}\Phi c^{-1}\right)\left(-\frac{2}{3t_{f(\min)}}t^3 + t^2\right) \tag{19}$$

2.3 Motion of Wheels

In Motion planning of a wheeled mobile robot, trajectory equations of robot's wheels should be determined. The simplest control method for a robot is differential driving technique. These types of mobile robots generally have two independent analogous DC motors along the same wheel axle. If we assume that the center of mass (COM) is below the wheel's axle, therefore we could use all of previously generated trajectory equations for describing the trajectory of the COM point of a robot over time. The differential drive robot with only two motorized wheels, when moves in *x* direction at straight paths, the motion equations of its wheels is obtained as below

$$w_r(t) = w_l(t) = \frac{1}{r}x(t) \tag{20a}$$

also, while the robot follows an arc-path of a circle, we could get motion equation of the wheels as

$$w_r(t) = \frac{1}{r}(c + \frac{d}{2})q(t) \ , \quad w_l(t) = \frac{1}{r}(c - \frac{d}{2})q(t) \qquad (20b)$$

where w_r and w_l denotes traveled trajectory equations of robot's right and left of the active wheels, respectively. The radius of active wheels is r and the distance between them is d (see Figure 2). The robot's configuration (position and orientation) is denoted by vector $\mathbf{q}=(x\ y\ q)^T$.

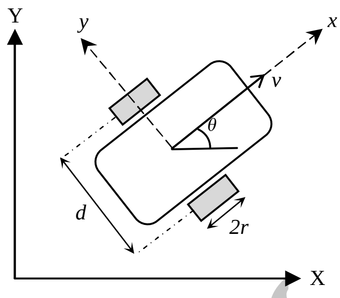


Figure 2. Robot posture from top view.

Kinematic and dynamic modeling of these types of mobile robots are described in [18].

3. Simulations and Discussions

In the previous section, considering maximum distance and minimum time problems, we have found closed-form solutions holding optimal trajectories for the robot. As it is well known, constrained optimization problems may have numerical solutions and solved by different optimization techniques, but generally they are iterative and time-consuming. Therefore, the main advantage of analytical solving methods and closed-form solution is their low computation time, as our proposed trajectory has it also.
In the simulations, we define the robot's accelerations is bounded by $(-0.8166, 0.8166) m/s^2$ over time history. In order the robot has to avoid from these values such that the positive accelerations the limit is $0.8166 m/s^2$ and in negative accelerations the limit is $-0.8166 m/s^2$. These limit values are valid for linear trajectory and by multiplying in c^{-1}, limit accelerations for circular trajectory is obtained (see inequality (11a)).

3.1 Simulations of Trajectories in Straight Path

Maximum Distance Trajectories: Maximum distance solution of the case study of $t_f=20sec$ is obtained as

$$x(t) = \frac{0.8166}{60}\left(-t^3 + 30t^2\right) \qquad (21)$$

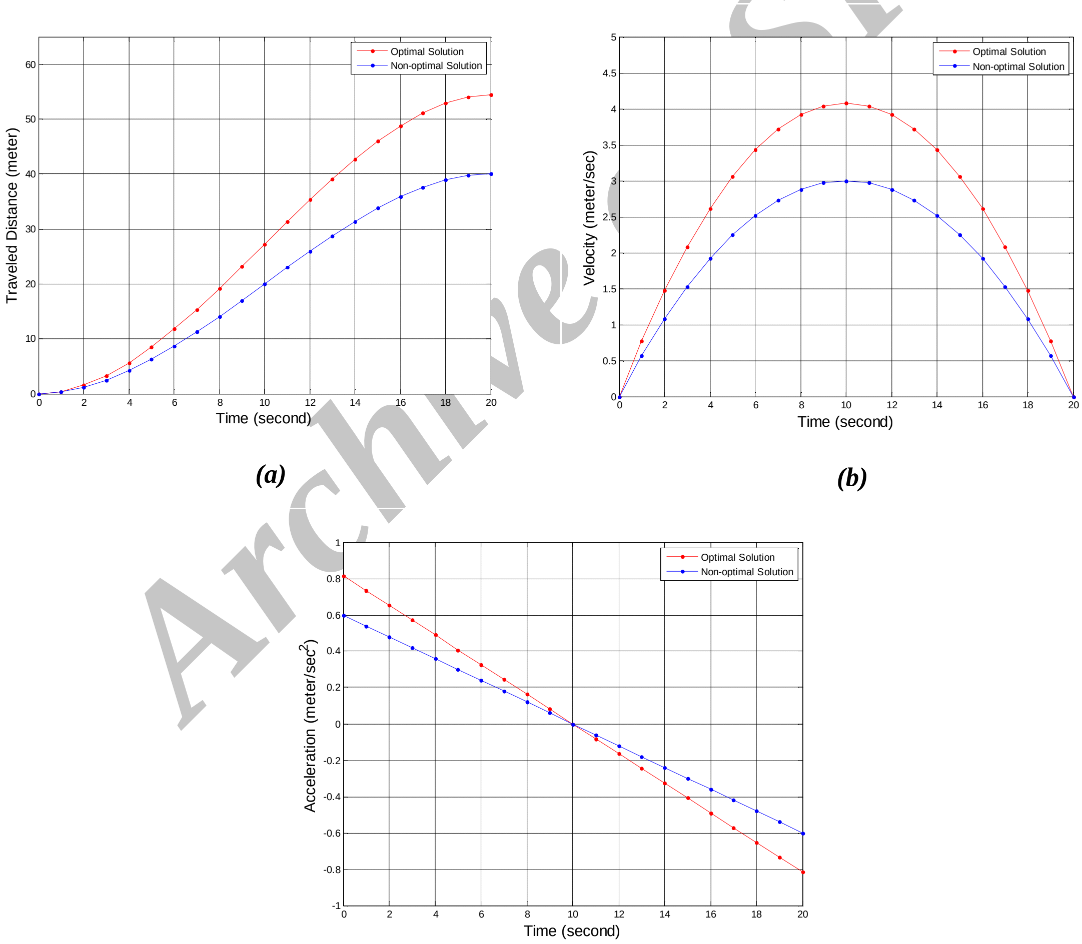
Simulation results of the above trajectory in the form of traveled distance, velocity and acceleration of the robot is given in Figure 3. The maximum traveled distance is *54.4m*.

To show that equation (21) is the optimal trajectory from the maximum distance point of view, the following trajectory by new bounded acceleration of *(-0.6,0.6)m/s²* is considered as below

$$x(t) = \frac{0.6}{60}\left(-t^3 + 30t^2\right) \tag{22}$$

According to the curves shown in Figure 3, limited acceleration constraint still holds but the covered distance is less than the value obtained from equation (21), then trajectory of (22) is a non-optimal solution despite it holds the problem's constraints.

Here and later, the term non-optimal trejectory refers to ones that are obtained randomly or generated by a numerical approximation method to track optimal properties of a motion. In other words, we used this non-optimal functions to prove that our solutions are optimal trajectories such that they are obtained analytically, and are not based on numerical and iterative methods.



(a) *(b)* *(c)*

Figure 3. Straight path-maximum distance plots, (a) traveled distance, (b) velocity, (c) acceleration.

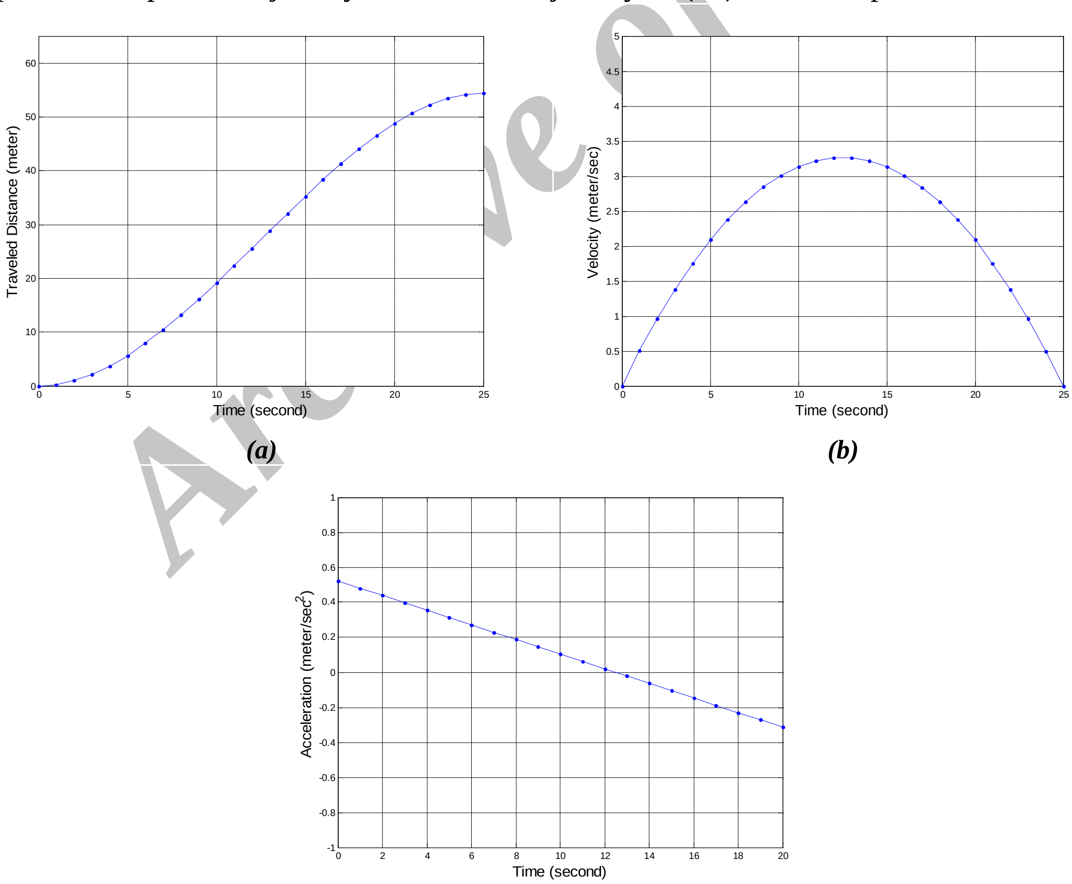
Minimum Time Trajectories: Moreover, the minimum time solution of the case study of $L=54.4m$, is obtained as $t_{f(min)}=20sec$. This means that the maximum distance values is the solution of the assumed minimum time problem as well. An immediate conclusion is that the optimal trajectory of the maximum distance problem with an assumed t_f and the solution $max(x(t_f))$ is the same as the optimal trajectory of the minimum time problem when $x(t_f)$ assumed to be $max(x(t_f))$ and t_f found. Finally, using equation (7) minimum time solution of the case study of $L=54.4m$ is approximated as

$$x(t) = \left(-\frac{1.32}{97.6} t^3 + \frac{8}{19.5} t^2 \right) \tag{23}$$

where the above trajectory is same with the maximum time solution that expressed in equation (21).
To show that equation (23) is the optimal trajectory from the minimum time point of view, a trajectory by the assuming of $L=54.4m$ and the new bounded acceleration $(-0.52,0.52)m/s^2$ is considered below

$$x(t) = \left(-\frac{8.5}{1220} t^3 + \frac{1}{4} t^2 \right) \tag{24}$$

Figure 4. illustrates that the covered distance is same with optimal trajectory and limited acceleration constraint still held but the trajectory time is bigger than the time value spend with optimal trajectory. Therefore, trajectory of (24) is a non-optimal solution.



(c)

Figure 4. Straight path-minimum time plots, (a) traveled distance, (b) velocity, (c) acceleration.

3.2 Simulations of Trajectories in Circular Path

As mentioned before, by multiplying the limited acceleration values to c^{-1} in the case of circular trajectory, new bounded accelerations in simulation's curves will be different from straight paths. Here, with assuming $c=0.5$, the acceleration threshold is twice respect to the straight path.

Maximum Distance Trajectories: Maximum distance solution of the case study of $t_f=5sec$ is approximated as

$$s(t) = \left(-\frac{1}{15}t^3 + \frac{1}{4}t^2\right) \tag{25}$$

that yields $s(t_f)=2.13m$ for maximum arc-length. Corresponding curves of equation (25) are shown in Figure 5. As a non-optimal trajectory we used the following equation

$$s(t) = \left(-\frac{1}{41.6}t^3 + \frac{1}{5.6}t^2\right) \tag{26}$$

to the robot and the results depicted in Figure 5. It can be seen that the limited acceleration still holds but the traveled arc-length is less than the optimal solution of equation (25). Note that the accelerations in Figure 5(c). generated corresponds to the nonlinear behaviour of the total acceleration in circular motion.

Minimum Time Trajectories: Similar to the straight path trajectory, the solution of the minimum time problem assuming $s(t_f)=2.13m$ yields $t_f=5$sec as minimum time. Therefore, the minimum time solution using equation (19) and with assuming $s_f=2.13m$ is obtained as

$$s(t) = (-\frac{1}{30.3}t^3 + \frac{1}{4}t^2) \tag{27}$$

where the above equation is obtained same with the maximum time solution in equation (25).

To show that equation (27) is the optimal trajectory from the minimum time point of view, as straight path simulations, the following trajectory by the assuming of $s_f=2.13m$ and the new bounded acceleration $(-0.52,0.52)m/s^2$ is considered

$$s(t) = \left(-\frac{1}{60.5}t^3 + \frac{1}{6.3}t^2\right) \tag{28}$$

Figure 6 clearly shows that the covered distance by trajectory of (28) is correspond with the optimal trajectory but the trajectory time is bigger.

Due to the simulation results, it can be concluded that the thoroughness of the proposed analytic solving approach, time-distance optimal motion is guaranteed in case of the cube-polynomials trajectory planners.

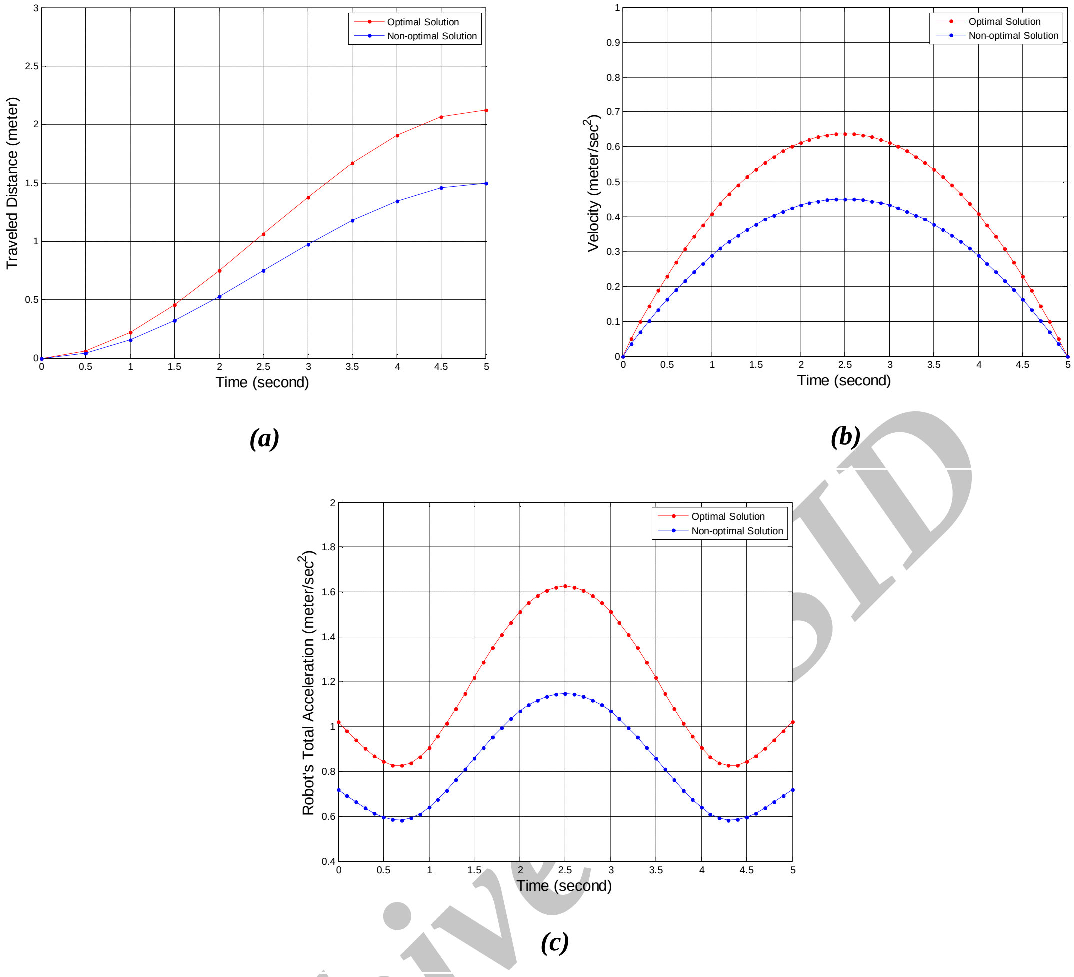


Figure 5. Circular path-maximum distance plots, (a) traveled arc-lengths, (b) velocity, (c) robot's total accelerations (resulting from linear and angular components).

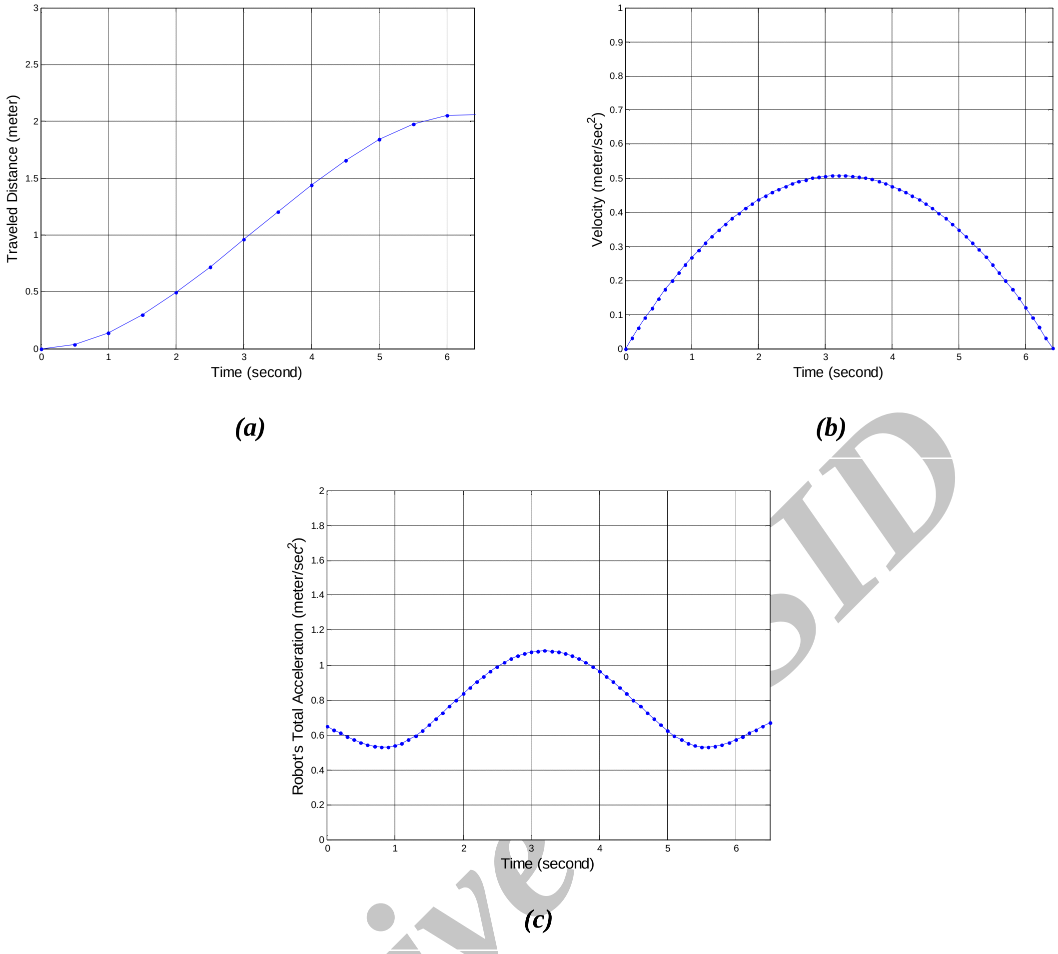


Figure 6. Circular path-minimum time plots, (a) traveled arc-lengths, (b) velocity, (c) robot's total accelerations.

4. Conclusion

We studied optimal trajectory planning of a mobile robot in two basic motions including linear and circular ones. Firstly, given some parameters from initial and final position and velocity, we used them in the formulation of an optimization problem, such that solving it was connected to determining the coefficients of a third-order polynomial. Then, by simulation results, the obtained third-order polynomial trajectory proved time-distance optimality.

References

[1] J. J. Craig, "Introduction to Robotics, Mechanics and Control," *Third Edition, Pearson Prentice Hall*, pp. 201, 2005.

[2] M. Boryga and A. Grabos, "Planning of manipulator motion trajectory with higher-degree polynomials use," *Mechanism and Machine Theory*, vol. 44, pp. 1400–1419, 2009.

[3] P. Huang and Y. Xu, "PSO-Based Time-Optimal Trajectory Planning for Space Robot with Dynamic Constraints," *Proceedings of the 2006 IEEE International Conference on Robotics and Biomimetics*, December 17 - 20, Kunming, China, 2006.

[4] A. Elnagar and A. Hussein, "On optimal constrained trajectory planning in 3D environments," *Robotics and Autonomous Systems*, Vol. 33, pp. 195–206, 2000.

[5] S. Han, B. S. Choi and J. M. Lee, "A precise curved motion planning for a differential driving mobile robot," *Mechatronics*, vol. 18, pp. 486–494, 2008.

[6] M. Haddad, T. Chettibi, S. Hanchi and H.E. Lehtihet, "A random-profile approach for trajectory planning of wheeled mobile robots," *European Journal of Mechanics A/Solids*, vol. 26, pp. 519–540, 2007.

[7] J. S. Choi and B. K. Kim, "Near-Time-Optimal Trajectory Planning for Wheeled Mobile Robots with Translational and Rotational Sections," *IEEE Transactions on Robotics and Automation*, Vol. 17, No. 1, 2001.

[8] S. Aydin and H. Temeltas, "Time-Optimal Trajectory Planning using A Smart Evolutionary Algorithm with Fuzzy Inference System," *Proceedings of the 2002 IEEE International Symposium on Intelligent Control*, Vancouver, Canada, October 27-30, 2002.

[9] S. Aydin and H. Temeltas, "Time Optimal Trajectory Planning for Mobile Robots by Differential Evolution Algorithm and Neural Networks," *IEEE International Symposium on Industrial Electronics*, Page(s) 352 – 357, vol. 1, 2003.

[10] Q. T. Dinh and M. Diehl, "An Application of Sequential Convex Programming to Time Optimal Trajectory Planning for a Car Motion," *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, P.R. China, December 16-18, 2009.

[11] E. Velenis and P. Tsiotras, "Optimal Velocity Profile Generation for Given Acceleration Limits: Theoretical Analysis," *American Control Conference*, Portland, OR, USA, June 8-10, 2005.

[12] Z. Han and S. Li "Switching-Time Computation for Time-optimal Trajectory Planning of Wheeled Mobile Robots," *Proceedings of the 8th World Congress on Intelligent Control and Automation*, Jinan, China, July 6-9, 2010.

[13] M. Lepetic, G. Klancar, I. Skrjanc, D. Matko, et. al. "Time optimal path planning considering acceleration limits," *Robotics and Autonomous Systems* , Vol. 45, pp. 199–210, 2003.

[14] K. D. Naguyen, I. M. Chen and T. Ch. Ng "Planning algorithms for s-curve trajectories," *Proc. IEEE/ASME International conf. on Advanced Intelligent Mechatronics*, Zurich, 2007.

[15] E. S. Kardos and B. Kiss "Continuous-curvature paths for mobile robots," *Periodica polytechnic Electrical Engineering*, Vol. 53, No 1-2, pp. 63-72, 2009.

[16] M. H. Korayem, M. Nazemizadeh and H. N. Rahimi, "Trajectory optimization of nonholonomic mobile manipulators departing to a moving target amidst moving obstacles," *Acta Mechanica*, Vol. 224, Issue 5, pp. 995-1008, 2013.

[17] M. Brezak and I. Petrovic, "Time-Optimal Trajectory Planning Along Predefined Path for Mobile Robots with Velocity and Acceleration Constraints," *2011 IEEE/ASME International Conference on Advanced Intelligent Mechatronics (AIM2011)*, Budapest, Hungary, July 3-7, 2011.

[18] R. Siegwart and R. Nourbakhsh, "Introduction to Autonomous Mobile Robots," *The MIT Press*, 2004.