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General Complex Fuzzy Transformation Semigroups in Automata

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Abstract

In this paper, we define the concepts of a complex fuzzy subset and a complex fuzzy finite state automaton. Then we extend the notion of a complex fuzzy finite state automaton and introduce the notion of a general complex fuzzy automaton. After that we define the concept of a max- min general complex fuzzy automaton and construct some equivalence relations and some congruence relations in a max-min general complex fuzzy automaton and obtain different types of monoids in a max-min general complex fuzzy automaton and define a homomorphism between them. Then we define the concepts of a general complex fuzzy transformation semi- group, a faithful general complex fuzzy transformation semigroup and a faithful general complex fuzzy transformation semigroup associated with a max-min general complex fuzzy automaton. Then we derive relationships between a max-min general complex fuzzy automaton and a general complex fuzzy transformation semigroup.

Keywords: *Fuzzy Automata, Semigroup, Equivalence Relation, Congruence Relation.*

1. Introduction

The concept of fuzzy automata was introduced by Wee in 1967 [14].

Automata have a long history both in theory and application. Automata are the prime example of general computational systems over discrete spaces. Among the conventional spectrum of automata (i.e. deterministic finite state automaton, non-deterministic finite state automaton, probabilistic automata and fuzzy finite state automata), deterministic finite state automata have been the most applied automata to different areas. Fuzzy automata not only provide a systematic approach to handle uncertainty in such systems, but also are able to handle continuous spaces. In general, fuzzy automata provide an attractive systematic way for generalizing discrete applications. Moreover, fuzzy automata are able to create capabilities which are hardly achievable by other tools.

A fuzzy finite state automaton (FFA) is a six-tuple denoted as $\tilde{F} = (Q, \sum, R, Z, \delta, \omega)$, where Q is a finite set of states, $\sum$ is a finite set of input symbols, R is the start state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \sum \times Q \to [0,1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol and $\omega : Q \to Z$ is the output function. The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted as $\delta(q_i, a_k, q_j)$.

Associated with each $\delta(q_i, a_k, q_j)$, there is a membership value in [0, 1]. We call this membership value the weight of the transition.

In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and introduced the notion of general fuzzy automata [1].

Definition 1.1. [7] Let X be a nonempty set. Then a function from $X \times X$ into X is called a binary operation on X. If $*$ is a binary operation on X, then the pair $(X,*)$ is called a mathematical system.

If $(X,*)$ is a mathematical system such that $\forall a,b,c \in X$,

$$(a*b)*c = a*(b*c) \tag{1}$$

then $*$ is called associative and $(X,*)$ is called a semigroup.

If $(X,*)$ is a mathematical system such that there exists $e \in X$ and $\forall a \in X$,

$$a*e = a = e*a \tag{2}$$

then e is called an identity of $(X,*)$ and $(X,*)$ is said to have an identity.

A semigroup $(X,*)$ is called a monoid if it has an identity.

Definition 1.2. [7] Let $\sum$ be a set. A word of $\sum$ is the product of a finite sequence of elements in $\sum$, Λ will denote the empty word and $\sum^*$ is the set of all words on $\sum$. The length $l(x)$ of the word $x \in \sum^*$ is the number of its letters, so $l(\Lambda) = 0$.

In this paper, we introduce several new concepts and construct some equivalence relations and some congruence relations and obtain different types of monoids in a max-min general complex fuzzy automaton.

Definition 1. 3. Let $C^* = \{c + di : c, d \in [0,1], i = \sqrt{-1}\}$. A complex fuzzy subset μ of X is a function of X into C^*. So if μ be a complex fuzzy subset of X, then $|\mu|$ is a fuzzy subset of X. If $\mu(x) = c + di$, then $\mu(x) = r\exp(i\theta)$, which $\theta = \arg \mu(x)$ is argument of $\mu(x)$ and $r = |\mu(x)|$. For a nonempty set X, $\tilde{P}(X)$ denoted the set of all complex fuzzy subsets on X.

Definition 1.4. A complex fuzzy finite state automaton (CFFA) is a six-tuple denoted as $\tilde{F} = (Q, \sum, R, Z, \delta, \omega)$, where Q is a finite set of states, $\sum$ is a finite set of input symbols, R is the start state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \sum \times Q \to C^*$ is the complex fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol and $\omega : Q \to Z$ is the output function.

The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted as $\delta(q_i, a_k, q_j)$.

Associated with each $|\delta(q_i, a_k, q_j)|$, there is a membership value in [0,1]. We call this membership value the weight of the transition. The set of all transitions of $\tilde{F}$ is denoted as Δ.

Definition 1.5. A general complex fuzzy automaton (GCFA) $\tilde{F}$ is an eight-tuple machine denote as $\tilde{F} = (Q, \sum, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where

(i) Q is a finite set of states, $Q = \{q_1, q_2, ..., q_n\}$,

(ii) $\sum$ is a finite set of input symbols, $\sum = \{a_1, a_2, ..., a_m\}$,

(iii) $\tilde{R}$ is the set of fuzzy start states,

(iv) Z is a finite set of output symbols, $Z=\{b_1,b_2,...,b_k\}$,
(v) $\omega: Q \to Z$ is the output function,
(vi) $\tilde{\delta}:(Q\times[0,1])\times\Sigma\times Q \to C^*$ is the augmented transition function,
(vii) $F_1:[0,1]\times[0,1]\to[0,1]$ is called membership assignment function,
(viii) $F_2:[0,1]^* \to [0,1]$ is called multi-membership resolution function.

The function $F_1(\mu,|\delta|)$ has two parameters μ and $|\delta|$, where μ is the membership value of a predecessor and $|\delta|$ is the weight of a transition.

In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as

$$\mu^{t+1}(q_j)=\left|\tilde{\delta}((q_i,\mu^t(q_i)),a_k,q_j)\right| = F_1(\mu^t(q_i),|\delta(q_i,a_k,q_j)|) \tag{3}$$

So $\tilde{\delta}((q_i,\mu^t(q_i)),a_k,q_j)=\mu^{t+1}(q_j)\exp(i\theta)$ such that θ is argument of $\delta(q_i,a_k,q_j)$.

This means that membership value of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

If $\tilde{\delta}((q_i,\mu^{t_i}(q_i)),a_j,q_{j+1}) = r_j\exp(i\theta_j)$, $j=1,2,...,n$, then we define

$$\bigvee_{j=1}^{n}\tilde{\delta}((q_i,\mu^{t_i}(q_i)),a_j,q_{j+1}) = r\exp(i\theta) \tag{4}$$

where $r=\max\{r_1,r_2,...,r_n\}$ and $\theta=\max\{\theta_1,\theta_2,...,\theta_n\}$.
Also we define

$$\bigwedge_{j=1}^{n}\tilde{\delta}((q_i,\mu^{t_i}(q_i)),a_j,q_{j+1}) = r\exp(i\theta) \tag{5}$$

where $r=\min\{r_1,r_2,...,r_n\}$ and $\theta=\min\{\theta_1,\theta_2,...,\theta_n\}$.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.
Let $Q_{act}(t_i)$ be the set of all active states at time $t_i, \forall i \geq 0$. We have

$$Q_{act}(t_0)=\tilde{R} \tag{6}$$

$$Q_{act}(t_i)=\left\{(q,\mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q',a,q)\in\Delta\right\},\ \forall i \geq 1 \tag{7}$$

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j will lead to the multi-membership resolution algorithm.

Algorithm1.6. (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to that

(1) Each transition weight $|\delta(q_i,a_k,q_j)|$ together with $\mu^t(q_i)$, will be processed by the Membership assignment function F_1, and will produce a membership value. Call this ν_i,

$$\nu_i = \left|\tilde{\delta}((q_i,\mu^t(q_i)),a_k,q_j)\right| = F_1(\mu^t(q_i),|\delta(q_i,a_k,q_j)|) \tag{8}$$

(2) These membership values are not necessarily equal. Hence, they will be processed by another function F_2, called the multi-membership resolution function.

(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = \overset{n}{\underset{i=1}{F_2}}[v_i] = \overset{n}{\underset{i=1}{F_2}}[F_1(\mu^t(q_i), |\delta(q_i, a_k, q_j)|)] \tag{9}$$

Where

- n: is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $|\delta(q_i, a_k, q_j)|$: is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$: is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$: is the final membership value of q_j at time $t+1$.

Definition 1.7. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general complex fuzzy automaton. We define max-min general complex fuzzy automata of the form $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$, where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), ...\}$ such that $\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to C^*$ and let for every i, $i \geq 0$

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1 & if \quad q = p \\ 0 & , otherwise \end{cases} \tag{10}$$

and for every i, $i \geq 1$

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p) = r\exp(i\theta) \tag{11}$$

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \underset{q' \in Q_{act}(t_i)}{\vee} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p)) \tag{12}$$

and recursively

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 ... u_n, p) = \vee \{ \tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \ldots$$
$$\wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) \mid p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \ldots, p_{n-1} \in Q_{act}(t_{n-1}) \} \tag{13}$$

in which $u_i \in \Sigma, \forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be u_i, for all $1 \leq i \leq n-1$.

Definition 1. 8. [7] Let $(X, *)$ be a semigroup and $\equiv$ be an equivalence relation on X. Then $\equiv$ is called a right (left) congruence relation on X if $\forall a, b, c \in X$, $a \equiv b$ implies

$$a * c \equiv b * c \ (c * a \equiv c * b) \tag{14}$$

A right and left congruence relation $\equiv$ on X is called a congruence relation.

Definition 1.9. [7] Let $\tilde{F}_1 = (Q_1, \Sigma_1, \tilde{R}, Z, \omega, \tilde{\delta}_1, F_1, F_2)$ and $\tilde{F}_2 = (Q_2, \Sigma_2, \tilde{R}, Z, \omega, \tilde{\delta}_2, F_1, F_2)$ be general fuzzy automata. Then a pair (f, g) of mappings, where $f : Q_1 \to Q_2$ and $g : \Sigma_1 \to \Sigma_2$, is said to be a homomorphism from $\tilde{F}_1$ to $\tilde{F}_2$ if for all $p, q \in Q$ and $x \in \Sigma_1^*$,

$$\tilde{\delta}_1((q, \mu^{t_i}(q)), x, p) \leq \tilde{\delta}_2((f(q), \mu^{t_i}(f(q))), g(x), f(p)) \tag{15}$$

The pair (f, g) is called a strong homomorphism of $\tilde{F}_1$ to $\tilde{F}_2$ if for all $p, q \in Q$ and $x \in \Sigma_1^*$,

$$\tilde{\delta}_2((f(q), \mu^{t_i}(f(q))), g(x), f(p)) = \vee\{\tilde{\delta}_1((q, \mu^{t_i}(q)), x, t) : t \in Q_1, f(t) = f(p)\} \tag{16}$$

Let (f, g) be a homomorphism of $\tilde{F}_1$ to $\tilde{F}_2$. Define $g^* : \Sigma_1^* \to \Sigma_2^*$ by $g^*(\Lambda) = \Lambda$ and $g^*(uv) = g^*(u)\ g^*(v), \forall u, v \in \Sigma_1^*$.

2. General Complex Fuzzy Transformation Semigroups in Automata

In this section, we define the concept of a general complex fuzzy transformation semigroup and we derive relationships between a max-min general complex fuzzy automaton and a general complex fuzzy transformation semigroup.

Theorem 2.1. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton. Define the relation $\equiv$ on $\sum^*$ by $x \equiv y$, for all $x, y \in \sum^*$ if and only if

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), x, p) = \tilde{\delta}^*((q, \mu^{t_i}(q)), y, p) \tag{17}$$

for all $p, q \in Q$. Then the relation $\equiv$ is a congruence relation on $\sum^*$.

Proof. Clearly $\equiv$ is an equivalence relation on $\sum^*$. Let $z \in \sum^*$ and $x \equiv y$. Then for all $p, q \in Q$, we have

$$\begin{aligned} \tilde{\delta}^*((q, \mu^{t_i}(q)), xz, p) &= \vee\{\tilde{\delta}^*((q, \mu^{t_i}(q)), x, s) \wedge \tilde{\delta}^*((s, \mu^{t_j}(s)), z, p) : s \in Q\} \\ &= \vee\{\tilde{\delta}^*((q, \mu^{t_i}(q)), y, s) \wedge \tilde{\delta}^*((s, \mu^{t_j}(s)), z, p) : s \in Q\} \\ &= \tilde{\delta}^*((q, \mu^{t_i}(q)), yz, p) \end{aligned} \tag{18}$$

Thus $xz \equiv yz$. Similarly $zx \equiv zy$. Thus $\equiv$ is a congruence relation on $\sum^*$. ■

Let $x \in \sum^*$, $[x] = \{y \in \sum^* : x \equiv y\}$ and $B(\tilde{F}^*) = \{[x] : x \in \sum^*\}$.

Theorem 2.2. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton. Define a binary operation $*$ on $B(\tilde{F}^*)$ by for all $[x], [y] \in B(\tilde{F}^*)$,

$$[x] * [y] = [xy] \tag{19}$$

Then $(B(\tilde{F}^*), *)$ is a monoid.

Proof. Clearly $*$ is associative. Now we have

$$[x] * [\Lambda] = [x\Lambda] = [x] = [\Lambda x] = [\Lambda] * [x] \tag{20}$$

Thus $[\Lambda]$ is the identity of $(B(\tilde{F}^*), *)$. Hence $(B(\tilde{F}^*), *)$ is a monoid. ■

Example 2.3. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton, where $Q = \{q\}$ is the set of states, $\Sigma = \{a\}$ is the set of input symbols, $\tilde{R} = \{(q,1)\}$, $F_1(\mu, |\delta|) = Min(\mu, |\delta|)$, $Z = \emptyset$, ω and F_2 are not applicable and $\delta(q, a, q) = 0.2 + 0.3i$. Now, we have

$$Q_{act}(t_0) = \tilde{R} = \{(q,1)\} \tag{21}$$

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q))\}, \forall i \geq 1 \tag{22}$$

$$\mu^{t_0}(q) = 1 \tag{23}$$

$$\mu^{t_1}(q) = \left|\tilde{\delta}((q, \mu^{t_0}(q)), a, q)\right| = F_1(\mu^{t_0}(q), |\delta(q, a, q)|) = F_1(1, 0.4) = 0.4 \tag{24}$$

$$\mu^{t_2}(q) = \left|\tilde{\delta}((q, \mu^{t_1}(q)), a, q)\right| = F_1(\mu^{t_1}(q), |\delta(q, a, q)|) = F_1(0.4, 0.4) = 0.4 \tag{25}$$

$$\mu^{t_3}(q) = \left|\tilde{\delta}((q, \mu^{t_2}(q)), a, q)\right| = F_1(\mu^{t_2}(q), |\delta(q, a, q)|) = F_1(0.4, 0.4) = 0.4 \tag{26}$$

$$\mu^{t_4}(q) = \left|\tilde{\delta}((q, \mu^{t_3}(q)), a, q)\right| = F_1(\mu^{t_3}(q), |\delta(q, a, q)|) = F_1(0.4, 0.4) = 0.4 \tag{27}$$

$$\mu^{t_i}(q) = 0.4, \forall i \geq 5 \tag{28}$$

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), a, q) = 0.4 \exp(56.3i) \tag{29}$$

$$\tilde{\delta}^*((q,\mu^{t_1}(q)),a,q) = 0.4\exp(56.3i) \tag{30}$$

$$\begin{aligned}\tilde{\delta}^*((q,\mu^{t_0}(q)),aa,q) &= \tilde{\delta}^*((q,\mu^{t_0}(q)),a,q) \wedge \tilde{\delta}^*((q,\mu^{t_1}(q)),a,q) \\ &= 0.4\exp(56.3i) \wedge 0.4\exp(56.3i) \\ &= 0.4\exp(56.3i)\end{aligned} \tag{31}$$

$$\tilde{\delta}^*((q,\mu^{t_0}(q)),a^n,q) = 0.4\exp(56.3i), \forall n \ge 3 \tag{32}$$

So for all $x, y \in \sum^* \setminus \{\Lambda\}$, we have

$$\tilde{\delta}^*((q,\mu^{t_0}(q)),x,q) = \tilde{\delta}^*((q,\mu^{t_0}(q)),y,q) = 0.4\exp(56.3i) \tag{33}$$

Thus $B(\tilde{F}^*) = \{[\Lambda],[x]\}$, where $x \in \sum^* \setminus \{\Lambda\}$.

Theorem 2.4. Let $\tilde{F}^* = (Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^*,F_1,F_2)$ be a max-min general complex fuzzy automaton. Define the relation $\approx$ on $\sum^*$ by $x \approx y$, for all $x, y \in \sum^*$ if and only if

$$\arg(\tilde{\delta}^*((q,\mu^{t_i}(q)),x,p)) > 0 \Leftrightarrow \arg(\tilde{\delta}^*((q,\mu^{t_i}(q)),y,p)) > 0 \tag{34}$$

for all $p, q \in Q$. Then the relation $\approx$ is a congruence relation on $\sum^*$.

Proof. Clearly $\approx$ is an equivalence relation on $\sum^*$. Let $z \in \sum^*$ and $x \approx y$.
Then for all $p, q \in Q$,

$$\arg(\tilde{\delta}^*((q,\mu^{t_i}(q)),xz,p)) > 0 \tag{35}$$

if and only if

$$\arg(\vee\{\tilde{\delta}^*((q,\mu^{t_i}(q)),x,s) \wedge \tilde{\delta}^*((s,\mu^{t_j}(s)),z,p) : s \in Q\}) > 0 \tag{36}$$

if and only if there exists $s \in Q$ such that

$$\arg(\tilde{\delta}^*((q,\mu^{t_i}(q)),x,s) \wedge \tilde{\delta}^*((s,\mu^{t_j}(s)),z,p)) > 0 \tag{37}$$

if and only if there exists $s \in Q$ such that

$$\arg(\tilde{\delta}^*((q,\mu^{t_i}(q)),y,s) \wedge \tilde{\delta}^*((s,\mu^{t_j}(s)),z,p)) > 0 \tag{38}$$

if and only if

$$\arg(\vee\{\tilde{\delta}^*((q,\mu^{t_i}(q)),y,s) \wedge \tilde{\delta}^*((s,\mu^{t_j}(s)),z,p) : s \in Q\}) > 0 \tag{39}$$

if and only if

$$\arg(\tilde{\delta}^*((q,\mu^{t_i}(q)),yz,p)) > 0 \tag{40}$$

Thus $xz \approx yz$. Similarly $zx \approx zy$. Thus $\approx$ is a congruence relation on $\sum^*$. ■

Let $x \in \sum^*$, $\prec x \succ = \{y \in \sum^* : x \approx y\}$ and $C(\tilde{F}^*) = \{\prec x \succ : x \in \sum^*\}$.

Theorem 2. 5. Let $\tilde{F}^* = (Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^*,F_1,F_2)$ be a max-min general complex fuzzy automaton. Define a binary operation $\tilde{*}$ on $C(\tilde{F}^*)$ by

$$\prec x \succ \tilde{*} \prec y \succ = \prec xy \succ \tag{41}$$

for all $\prec x \succ, \prec y \succ \in C(\tilde{F}^*)$. Then $(C(\tilde{F}^*),\tilde{*})$ is a monoid and the map

$$\begin{aligned}\varphi : B(\tilde{F}^*) &\to C(\tilde{F}^*) \\ [x] &\mapsto \prec x \succ\end{aligned} \tag{42}$$

is a homomorphism as semigroups.

Proof. Clearly $\tilde{*}$ is associative. Now we have

$$\prec x \succ \tilde{*} \prec \Lambda \succ = \prec x\Lambda \succ = \prec x \succ = \prec \Lambda x \succ = \prec \Lambda \succ \tilde{*} \prec x \succ \tag{43}$$

Thus $\prec \Lambda \succ$ is the identity of $(C(\tilde{F}^*),\tilde{*})$. Hence $(C(\tilde{F}^*),\tilde{*})$ is a monoid.

Now, let $x, y \in \sum^*$ and $[x] = [y]$. Then for all $p, q \in Q$,

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), x, p) = \tilde{\delta}^*((q, \mu^{t_i}(q)), y, p) \tag{44}$$

Thus for all $p, q \in Q$, we have

$$\arg(\tilde{\delta}^*((q, \mu^{t_i}(q)), x, p)) > 0 \Leftrightarrow \arg(\tilde{\delta}^*((q, \mu^{t_i}(q)), y, p)) > 0 \tag{45}$$

Hence $x \approx y$ or $\prec x \succ = \prec y \succ$. Thus φ is well defined and we have

$$\varphi([x] * [y]) = \varphi([xy]) = \prec xy \succ = \prec x \succ \tilde{*} \prec y \succ = \varphi([x]) \tilde{*} \varphi([y]) \tag{46}$$

Thus φ is a homomorphism as semigroups. ■

Example 2.6. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton as defined in Example 2.3. Then for all $x, y \in \sum^*$, we have

$$\arg(\tilde{\delta}^*((q, \mu^{t_0}(q)), x, q)) > 0 \Leftrightarrow \arg(\tilde{\delta}^*((q, \mu^{t_0}(q)), y, q)) > 0 \tag{47}$$

So $x \approx y$. Thus $C(\tilde{F}^*) = \{\prec x \succ\}$, where $x \in \sum^*$.

Theorem 2.7. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton. Define the complex fuzzy subset $x_i^{\tilde{F}^*}$ of $Q \times Q$ by

$$x_i^{\tilde{F}^*}(q, p) = \tilde{\delta}^*((q, \mu^{t_i}(q)), x, p) \tag{48}$$

for all $p, q \in Q$ and for all $x \in \sum^*$. Let $S_{\tilde{F}^*} = \{x_i^{\tilde{F}^*} : x \in \Sigma^*, i \geq 0\}$. Then

(1) $x_i^{\tilde{F}^*} \circ y_j^{\tilde{F}^*} = (xy)_i^{\tilde{F}^*}$, where $i \leq j$ and

$$(x_i^{\tilde{F}^*} \circ y_j^{\tilde{F}^*})(q, p) = \vee \{x_i^{\tilde{F}^*}(q, s) \wedge y_j^{\tilde{F}^*}(s, p) : s \in Q\} \tag{49}$$

(2) $(S_{\tilde{F}^*}, \circ)$ is a monoid.

Proof. (1) Let $p, q \in Q$. Then we have

$$\begin{aligned}(x_i^{\tilde{F}^*} \circ y_j^{\tilde{F}^*})(q, p) &= \vee \{x_i^{\tilde{F}^*}(q, s) \wedge y_j^{\tilde{F}^*}(s, p) : s \in Q\} \\ &= \vee \{\tilde{\delta}^*((q, \mu^{t_i}(q)), x, s) \wedge \tilde{\delta}^*((s, \mu^{t_j}(s)), y, p) : s \in Q\} \\ &= \tilde{\delta}^*((q, \mu^{t_i}(q)), xy, p) = (xy)_i^{\tilde{F}^*}\end{aligned} \tag{50}$$

(2) Clearly $(S_{\tilde{F}^*}, \circ)$ is a semigroup and we have

$$x_i^{\tilde{F}^*} \circ \Lambda_i^{\tilde{F}^*} = (x\Lambda)_i^{\tilde{F}^*} = x_i^{\tilde{F}^*} = (\Lambda x)_i^{\tilde{F}^*} = \Lambda_i^{\tilde{F}^*} \circ x_i^{\tilde{F}^*} \tag{51}$$

Thus $(S_{\tilde{F}^*}, \circ)$ is a monoid. ■

Example 2.8. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton as defined in Example 2.3. Then we have

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), a, q) = 0.4 \exp(56.3i) \tag{52}$$

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), aa, q) = 0.4 \exp(56.3i) \tag{53}$$

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), a^n, q) = 0.4 \exp(56.3i), \forall n \geq 3 \tag{54}$$

So we have

$$a_0^{\tilde{F}^*}(q, q) = \tilde{\delta}^*((q, \mu^{t_0}(q)), a, q) = 0.4 \exp(56.3i) \tag{55}$$

$$(a^2)_0^{\widetilde{F}^*}(q,q) = \widetilde{\delta}^*((q,\mu^{t_0}(q)), aa, q) = 0.4\exp(56.3i) \tag{56}$$

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$$(a^n)_0^{\widetilde{F}^*}(q,q) = \widetilde{\delta}^*((q,\mu^{t_0}(q)), a^n, q) = 0.4\exp(56.3i), \forall n \geq 3 \tag{57}$$

Thus $S_{\widetilde{F}^*} = \{x_i^{\widetilde{F}^*} : x \in \Sigma^*, i \geq 0\} = \{a_0^{\widetilde{F}^*}, (a^2)_0^{\widetilde{F}^*}, (a^3)_0^{\widetilde{F}^*}, \ldots\}$.

Theorem 2.9. Let $\widetilde{F}^* = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton. Then $(S_{\widetilde{F}^*}, \circ)$ and $(B(\widetilde{F}^*), *)$ are isomorphic as semigroups.

Proof. Define $f : S_{\widetilde{F}^*} \to B(\widetilde{F}^*)$ by $f(x_i^{\widetilde{F}^*}) = [x]$ for all $x_i^{\widetilde{F}^*} \in S_{\widetilde{F}^*}$.

Clearly f is onto. Now, let $x_i^{\widetilde{F}^*}$, $y_i^{\widetilde{F}^*} \in S_{\widetilde{F}^*}$. Then we have

$$\begin{aligned} x_i^{\widetilde{F}^*} = y_i^{\widetilde{F}^*} &\Leftrightarrow x_i^{\widetilde{F}^*}(q,p) = y_i^{\widetilde{F}^*}(q,p), \forall p,q \in Q \\ &\Leftrightarrow \widetilde{\delta}^*((q,\mu^{t_i}(q)), x, p) = \widetilde{\delta}^*((q,\mu^{t_i}(q)), y, p), \forall p,q \in Q \\ &\Leftrightarrow [x] = [y] \\ &\Leftrightarrow f(x_i^{\widetilde{F}^*}) = f(y_i^{\widetilde{F}^*}) \end{aligned} \tag{58}$$

Thus f is well defined and one to one. Also we have

$$f(x_i^{\widetilde{F}^*} \circ y_j^{\widetilde{F}^*}) = f((xy)_i^{\widetilde{F}^*}) = [xy] = [x] * [y] = f(x_i^{\widetilde{F}^*}) * f(y_j^{\widetilde{F}^*}) \tag{59}$$

Thus f is homomorphism. Hence $(S_{\widetilde{F}^*}, \circ)$ and $(B(\widetilde{F}^*), *)$ are isomorphic as semigroups.■

Definition 2.10. A general complex fuzzy transformation semigroup (GCFTS) is an eight-tuple machine denote as $\widetilde{T} = (Q, S, \widetilde{R}, Z, \omega, \widetilde{\rho}, F_1, F_2)$, where

(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,

(ii) S is a finite semigroup,

(iii) $\widetilde{R}$ is the set of fuzzy start states,

(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,

(v) $\omega : Q \to Z$ is the output function,

(vi) $\widetilde{\rho} : (Q \times [0,1]) \times \Sigma \times Q \to C^*$ is the augmented transition function such that

(1) For all $p, q \in Q$ and for all $a, b \in S$,

$$\widetilde{\rho}((q,\mu^{t_i}(q)), ab, p) = \vee\{\widetilde{\rho}((q,\mu^{t_i}(q)), a, s) \wedge \widetilde{\rho}((s,\mu^{t_j}(s)), b, p) : s \in Q\} \tag{60}$$

(2) If S contains the identity e, then for all $p, q \in Q$,

$$\widetilde{\rho}((q,\mu^{t_i}(q)), e, p) = \begin{cases} 1 & if \quad q = p \\ 0 & , otherwise \end{cases} \tag{61}$$

(vii) $F_1 : [0,1] \times [0,1] \to [0,1]$ is called membership assignment function,

(viii) $F_2 : [0,1]^* \to [0,1]$ is called multi-membership resolution function.

We note that the function $F_1(\mu, |\delta|)$ has two parameters μ and $|\delta|$, where μ is the membership value of a predecessor and $|\delta|$ is the weight of a transition and the set of all transitions of $\widetilde{T}$ is denoted as Δ.

In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as

$$\mu^{t+1}(q_j) = \left|\tilde{\rho}((q_i, \mu^t(q_i)), a_k, q_j)\right| = F_1(\mu^t(q_i), \left|\delta(q_i, a_k, q_j)\right|) \quad (62)$$

So $\tilde{\rho}((q_i, \mu^t(q_i)), a_k, q_j) = \mu^{t+1}(q_j)\exp(i\theta)$ such that θ is argument of $\delta(q_i, a_k, q_j)$.

This means that membership value of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Definition 2.11. Let $\tilde{T} = (Q, S, \tilde{R}, Z, \omega, \tilde{\rho}, F_1, F_2)$ be a general complex fuzzy transformation semigroup. Then $\tilde{T}$ is called faithful if

$$\tilde{\rho}((q, \mu^{t_i}(q)), a, p) = \tilde{\rho}((q, \mu^{t_i}(q)), b, p) \Rightarrow a = b \quad (63)$$

Theorem 2.12. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton. Then $(Q, B(\tilde{F}^*), \tilde{R}, Z, \omega, \tilde{\rho}, F_1, F_2)$ is a faithful general complex fuzzy transformation semigroup, where

$$\tilde{\rho}((q, \mu^{t_i}(q)), [x], p) = \tilde{\delta}^*((q, \mu^{t_i}(q)), x, p),\ \forall p, q \in Q,\ x \in \sum^* \quad (64)$$

Proof. By Theorem 2.2, $B(\tilde{F}^*)$ is a semigroup with identity $[\Lambda]$.

Let $p, q \in Q$ and $[x], [y] \in B(\tilde{F}^*)$. Then we have

$$\begin{aligned} &\tilde{\rho}((q, \mu^{t_i}(q)), [x]*[y], p) \\ &= \tilde{\rho}((q, \mu^{t_i}(q)), [xy], p) \\ &= \tilde{\delta}^*((q, \mu^{t_i}(q)), xy, p) \\ &= \vee\{\tilde{\delta}^*((q, \mu^{t_i}(q)), x, s) \wedge \tilde{\delta}^*((s, \mu^{t_j}(s)), y, p) : s \in Q\} \\ &= \vee\{\tilde{\rho}((q, \mu^{t_i}(q)), [x], s) \wedge \tilde{\rho}((s, \mu^{t_j}(s)), [y], p) : s \in Q\} \end{aligned} \quad (65)$$

Also, if $q = p$, we have

$$\tilde{\rho}((q, \mu^{t_i}(q)), [\Lambda], p) = \tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = 1 \quad (66)$$

and if $q \neq p$, then

$$\tilde{\rho}((q, \mu^{t_i}(q)), [\Lambda], p) = \tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = 0 \quad (67)$$

Suppose $\tilde{\rho}((q, \mu^{t_i}(q)), [x], p) = \tilde{\rho}((q, \mu^{t_i}(q)), [y], p)$, $\forall p, q \in Q$. Then

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), x, p) = \tilde{\delta}^*((q, \mu^{t_i}(q)), y, p),\ \forall p, q \in Q \quad (68)$$

Thus $x \equiv y$ or $[x] = [y]$.

So $(Q, B(\tilde{F}^*), \tilde{R}, Z, \omega, \tilde{\rho}, F_1, F_2)$ is a faithful general fuzzy transformation semigroup. ■

Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton. Then by Theorem 2.12 $(Q, B(\tilde{F}^*), \tilde{R}, Z, \omega, \tilde{\rho}, F_1, F_2)$ is a faithful general complex fuzzy transformation semigroup that we denote by $GCFTS(\tilde{F}^*)$.

We call $GCFTS(\tilde{F}^*)$ the general complex fuzzy transformation semigroup associated with $\tilde{F}^*$.

Definition 2.13. Let $\tilde{T}_1 = (Q_1, S_1, \tilde{R}_1, Z, \omega, \tilde{\rho}_1, F_1, F_2)$ and $\tilde{T}_2 = (Q_2, S_2, \tilde{R}_2, Z, \omega, \tilde{\rho}_2, F_1,$

F_2) be general complex fuzzy transformation semigroups. Then a pair (f,g) of mappings, where $f:Q_1\to Q_2$ and $g:S_1\to S_2$, is said to be a homomorphism from $\widetilde{T}_1$ to $\widetilde{T}_2$ if

(1) $g(xy)=g(x)\,g(y)$, $\forall x,y\in S_1$ (69)

(2) If e_1 is the identity of S_1 and e_2 is the identity of S_2, then $g(e_1)=e_2$,

(3) For all $p,q\in Q$ and $x\in S_1$,

$$\widetilde{\rho}_1((q,\mu^{t_i}(q)),x,p)\le\widetilde{\rho}_2((f(q),\mu^{t_i}(f(q))),g(x),f(p)) \tag{70}$$

The pair (f,g) is called a strong homomorphism from $\widetilde{T}_1$ to $\widetilde{T}_2$ if for all $p,q\in Q$ and $x\in S_1$,

$$\widetilde{\rho}_2((f(q),\mu^{t_i}(f(q))),g(x),f(p))=\vee\{\widetilde{\rho}_1((q,\mu^{t_i}(q)),x,t):t\in Q_1,\,f(t)=f(p)\} \tag{71}$$

Theorem 2.14. Let $\widetilde{F}_1^*=(Q_1,\Sigma_1,\widetilde{R},Z,\omega,\widetilde{\delta}_1^*,F_1,F_2)$ and $\widetilde{F}_2^*=(Q_2,\Sigma_2,\widetilde{R},Z,\omega,\widetilde{\delta}_2^*,F_1,F_2)$ be general complex fuzzy automata and let the pair (α,β^*) be a strong homomorphism from $\widetilde{F}_1^*$ to $\widetilde{F}_2^*$, where $\alpha:Q_1\to Q_2$ and $\beta^*:S_1\to S_2$, with α one to one and onto. Then there exists a strong homomorphism (f_α,g_β) from $GCFTS(\widetilde{F}_1^*)$ to $GCFTS(\widetilde{F}_2^*)$.

Proof. Define $f_\alpha:Q_1\to Q_2$ by $f_\alpha(q)=\alpha(q)$, for all $q\in Q_1$ and $g_\beta:B(\widetilde{F}_1^*)\to B(\widetilde{F}_2^*)$ by $g_\beta([x])=[\beta^*(x)]$, $\forall[x]\in B(\widetilde{F}_1^*)$. Let $[x],[y]\in B(\widetilde{F}_1^*)$ and $[x]=[y]$.

Then for all $p,q\in Q_1$,

$$\widetilde{\delta}_1^*((q,\mu^{t_i}(q)),x,p)=\widetilde{\delta}_1^*((q,\mu^{t_i}(q)),y,p) \tag{72}$$

Now, since α is one to one and onto, we have

$$\begin{aligned}\widetilde{\delta}_2^*((\alpha(q),\mu^{t_i}(\alpha(q))),\beta^*(x),\alpha(p))&=\widetilde{\delta}_1^*((q,\mu^{t_i}(q)),x,p)\\&=\widetilde{\delta}_1^*((q,\mu^{t_i}(q)),y,p)\\&=\widetilde{\delta}_2^*((\alpha(q),\mu^{t_i}(\alpha(q))),\beta^*(y),\alpha(p))\end{aligned} \tag{73}$$

So $[\beta^*(x)]=[\beta^*(y)]$. Hence g_β is well defined. Now we have

$$\begin{aligned}g_\beta([x]*[y])&=g_\beta([xy])\\&=[\beta^*(xy)]\\&=[\beta^*(x)\beta^*(y)]\\&=[\beta^*(x)]*[\beta^*(y)]\\&=g_\beta([x])*g_\beta([y])\end{aligned} \tag{74}$$

and $g_\beta([\Lambda])=[\beta^*(\Lambda)]=[\Lambda]$. Also we have

$$\begin{aligned}\widetilde{\rho}_1((q,\mu^{t_i}(q)),[x],p)&=\widetilde{\delta}_1^*((q,\mu^{t_i}(q)),x,p)\\&=\widetilde{\delta}_2^*((\alpha(q),\mu^{t_i}(\alpha(q))),\beta^*(x),\alpha(p))\\&=\widetilde{\rho}_2((f_\alpha(q),\mu^{t_i}(f_\alpha(q))),g_\beta([x]),f_\alpha(p))\end{aligned} \tag{75}$$

Hence (f_α,g_β) is a strong homomorphism from $GCFTS(\widetilde{F}_1^*)$ to $GCFTS(\widetilde{F}_2^*)$. ■

3. Conclusions

In this paper, we have constructed some equivalence relations and some congruence relations in a max-min general complex fuzzy automaton and obtained different types of monoids in a max-min general complex fuzzy automaton. Then we have defined the concept of a general complex fuzzy transformation semigroup and we have derived relationships between a max-min general complex fuzzy automaton and a general complex
fuzzy transformation semigroup, for example we have shown that if $\widetilde{F}^* = (Q, \Sigma, \widetilde{R}, Z, \omega, \widetilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton, then $(Q, B(\widetilde{F}^*), \widetilde{R}, Z, \omega, \widetilde{\rho}, F_1, F_2)$ is a faithful general complex fuzzy transformation semigroup.

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